

## CONTROLLABILITY OF NONLINEAR STOCHASTIC SYSTEMS WITH MULTIPLE TIME-VARYING DELAYS IN CONTROL

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This paper is concerned with the problem of controllability of semi-linear stochastic systems with time varying multiple delays in control in finite dimensional spaces. Sufficient conditions are established for the relative controllability of semi-linear stochastic systems by using the Banach fixed point theorem. A numerical example is given to illustrate the application of the theoretical results. Some important comments are also presented on existing results for the stochastic controllability of fractional dynamical systems.

**Keywords:** relative controllability, stochastic control system, multiple delays in control, Banach fixed point theorem.

### 1. Introduction

Modelling and control of dynamical systems with input/output delays arise naturally in numerous engineering applications. Further, satisfactory modelling of time-varying delays is also important for the synthesis of effective control systems since they show significantly different characteristics from that of fixed time delays (Basin *et al.*, 2004; Klein and Ramirez, 2001; Li, 1970). In practical applications, time varying input delays always exist in a flexible spacecraft due to the physical structure and energy consumption of the actuators (Zhang *et al.*, 2013). It is essential that system models must take into account these time delays in order to predict the true system dynamics. The presence of time delays is often the main cause of substantial performance deterioration and even instability of the system. Moreover, a majority of processes in industrial practice have stochastic characteristics and systems have to be modelled in the form of stochastic differential equations (Oksendal, 2003). Thus, it is of theoretical and practical significance to address controllability problems

for such stochastic systems with delays in control input (Gu and Niculescu, 2003; Richard, 2003).

Controllability is one of the most important aspects of industrial process operability, because it can be used to assess the attainable operation of a given process and improve its dynamic performance. It refers to the ability of a controller to arbitrarily alter the functionality of the dynamical system. Controllability of nonlinear deterministic systems in a finite dimensional space was extensively studied (Klamka, 1991; 2000). Conditions for controllability of linear and nonlinear systems with delays in control were well studied as well (Klamka, 1976; 1978; 1980; 2009; Somasundaram and Balachandran, 1984; Balachandran, 1987; Balachandran and Dauer, 1996; Dauer *et al.*, 1998). Further, one can refer to the survey article by Klamka (2013) for recent developments in this topic.

The results on controllability of linear and nonlinear stochastic systems have been a subject of intense research over the past few years (Mahmudov, 2001; Mahmudov and Denker, 2000; Mahmudov and Zorlu, 2003; Zabczyk, 1981). However, the situation is less satisfactory for stochastic systems with state/control delays. In recent

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years, we have witnessed increasing interest in stochastic systems involving state or control delays (see the works of Balachandran and Karthikeyan (2009) as well as Karthikeyan and Balachandran (2013) and the references therein). Klamka (2008a) investigated the controllability of linear stochastic systems with single time-variable delay in control. Shen and Sun (2012) extended the above results to nonlinear stochastic systems via a fixed point technique. So far, there have been very few results for stochastic systems in which multiple delays in control input are involved (Klamka, 2008b; Sikora and Klamka, 2012). Recently, Balachandran et al. (2012) established global relative controllability of fractional dynamical systems with multiple delays in control. Inspired by the above recent works, this study focuses on the controllability problem for semi-linear stochastic systems involving multiple time varying delays in control input.

The outline of this paper is as follows. Section 2 formulates the problem and presents preliminary ideas. Section 3 investigates the controllability of linear stochastic systems with time delay in control inputs. Section 4 is entirely devoted to establishing sufficient controllability conditions for semi-linear stochastic systems via one of the fixed point methods, namely, the contraction mapping principle. An illustrative example to show the effectiveness of the obtained results is given in Section 5. Some important remarks on fractional systems driven by white noise processes are also discussed. In addition, the proposed result is applied to an example which illustrates that a time delay in the control input contributes to controllability of systems.

**Notation.** The notation used in this paper is fairly standard. Throughout the paper,  $(\Omega, \mathcal{F}, \mathbb{P})$  is a complete probability space with a probability measure  $\mathbb{P}$  on  $\Omega$  and a filtration  $\{\mathcal{F}_t | t \in [t_0, T]\}$  generated by an  $l$ -dimensional Wiener process  $\{w(s) : t_0 \leq s \leq t\}$ .  $L_2(\Omega, \mathcal{F}_t, \mathbb{R}^n)$  denotes the Hilbert space of all  $\mathcal{F}_t$ -measurable square-integrable random variables with values in  $\mathbb{R}^n$ .  $L_2^{\mathcal{F}}([t_0, T], \mathbb{R}^n)$  denotes the Hilbert space of all square-integrable and  $\mathcal{F}_t$ -measurable processes with values in  $\mathbb{R}^n$ .  $\mathcal{U}_{ad} := L_2^{\mathcal{F}}([t_0, T], \mathbb{R}^l)$  is the set of admissible controls,  $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$  denotes the space of all linear transformations from  $\mathbb{R}^n$  to  $\mathbb{R}^m$ ,  $\mathbb{E}$  denotes the mathematical expectation operator of a stochastic process with respect to the given probability measure  $\mathbb{P}$ .

## 2. System description and preliminaries

Consider the linear time-varying stochastic system with time-varying delays in control of the form

$$\left. \begin{aligned} dx(t) &= \left[ A(t)x(t) + \sum_{i=0}^M B_i(t)u(\delta_i(t)) \right] dt \\ &\quad + \tilde{\sigma}(t) dw(t), \\ x(t_0) &= x_0, \end{aligned} \right\} \quad (1)$$

where  $x(t) \in \mathbb{R}^n$  is the instantaneous state of the system,  $A(t)$  and  $B_i(t)$  ( $i = 0, 1, \dots, M$ ) are respectively  $n \times n$  and  $n \times l$  time-varying matrices whose elements are bounded measurable functions on  $[t_0, T]$  and  $\tilde{\sigma} : [t_0, T] \rightarrow \mathbb{R}^{n \times n}$ . Further,  $u(t) \in \mathbb{R}^l$  is a vector input to the stochastic dynamical system. The functions  $\delta_i : [t_0, T] \rightarrow \mathbb{R}$ ,  $i = 0, 1, \dots, M$ , are twice continuously differentiable and strictly increasing in  $[t_0, T]$ , and

$$\delta_i(t) \leq t \quad \text{for } t \in [t_0, T], \quad i = 0, 1, \dots, M.$$

Here, the control function  $u(t)$  regulates the system state by fusing the values of  $u(t)$  at various time moments  $\delta_i(t)$ ,  $i = 1, \dots, M$ , where  $\delta_i(t)$  are time varying delays as well at the current time  $t$ , which assumes that the current system state depends not only on the current value of  $u(t)$  but also on its values after certain lags  $\delta_i(t)$ ,  $i = 1, \dots, M$ .

For a given initial condition (1) and any admissible control  $u \in \mathcal{U}_{ad}$ , there exists a unique solution  $x(t; x_0, u) \in L_2(\Omega, \mathcal{F}_t, \mathbb{R}^n)$  of the linear system (1) which can be represented in the following integral form (Enrhardt and Kliemann, 1982; Mahmudov and Denker, 2000):

$$\begin{aligned} x(t) &= \Phi(t, t_0)x_0 \\ &\quad + \int_{t_0}^t \Phi(t, s) \sum_{i=0}^M B_i(s)u(\delta_i(s)) ds \\ &\quad + \int_{t_0}^t \Phi(t, s)\tilde{\sigma}(s) dw(s), \end{aligned} \quad (2)$$

where  $\Phi(t, t_0)$  is the transition matrix of the linear system  $\dot{x}(t) = A(t)x(t)$  with  $\Phi(t_0, t_0) = I$  being the identity matrix.

Let us introduce the time lead functions  $r_i(t) : [\delta_i(t_0), \delta_i(T)] \rightarrow [t_0, T]$  such that

$$r_i(\delta_i(t)) = t, \quad i = 0, 1, \dots, M, \quad t \in [t_0, T].$$

We also introduce the so-called complete state of the system (1) at time  $t$  to be the set  $y(t) = \{x(t), u_t(s)\}$ , where  $u_t(s) = u(s)$  for  $s \in [\min_i \delta_i(t), t)$ .

Taking  $\delta_i(s) = \tau$  in (2) and using the time lead function  $r_i(t)$ , we have

$$s = r_i(\tau) \quad \text{and} \quad ds = \dot{r}_i(\tau) d\tau.$$

Thus (2) can be written as

$$\begin{aligned}
 x(t) &= \Phi(t, t_0)x_0 \\
 &+ \sum_{i=0}^M \int_{\delta_i(t_0)}^{\delta_i(t)} \Phi(t, r_i(s))B_i(r_i(s))\dot{r}_i(s)u(s) ds \\
 &+ \int_{t_0}^t \Phi(t, s)\tilde{\sigma}(s) dw(s).
 \end{aligned}
 \tag{3}$$

Without loss of generality, it can be assumed that

$$\delta_0(t) = t,$$

and the following inequalities hold for  $t = T$ :

$$\begin{aligned}
 \delta_M(T) &\leq \delta_{M-1}(T) \leq \dots \leq \delta_{m+1}(T) \\
 &\leq t_0 = \delta_m(T) < \delta_{m-1}(T) = \dots \\
 &= \delta_1(T) = \delta_0(T) = T.
 \end{aligned}
 \tag{4}$$

By using (4), Eqn. (3) for  $t = T$  can be expressed as

$$\begin{aligned}
 x(T) &= \Phi(T, t_0)x_0 \\
 &+ \sum_{i=0}^m \int_{\delta_i(t_0)}^{t_0} \Phi(T, r_i(s))B_i(r_i(s))\dot{r}_i(s)u_{t_0}(s) ds \\
 &+ \sum_{i=0}^m \int_{t_0}^{\delta_i(T)} \Phi(T, r_i(s))B_i(r_i(s))\dot{r}_i(s)u(s) ds \\
 &+ \sum_{i=m+1}^M \int_{\delta_i(t_0)}^{\delta_i(T)} \Phi(T, r_i(s)) \\
 &\times B_i(r_i(s))\dot{r}_i(s)u_{t_0}(s) ds \\
 &+ \int_{t_0}^T \Phi(T, s)\tilde{\sigma}(s) dw(s) \\
 &= \Phi(T, t_0)x_0 \\
 &+ \sum_{i=0}^m \int_{\delta_i(t_0)}^{t_0} \Phi(T, r_i(s))B_i(r_i(s))\dot{r}_i(s)u_{t_0}(s) ds \\
 &+ \sum_{i=0}^m \int_{t_0}^T \Phi(T, r_i(s))B_i(r_i(s))\dot{r}_i(s)u(s) ds \\
 &+ \sum_{i=m+1}^M \int_{\delta_i(t_0)}^{\delta_i(T)} \Phi(T, r_i(s)) \\
 &\times B_i(r_i(s))\dot{r}_i(s)u_{t_0}(s) ds \\
 &+ \int_{t_0}^T \Phi(T, s)\tilde{\sigma}(s) dw(s).
 \end{aligned}$$

It has to be noted that the last term of the third integral is zero by the definition of the time lead function  $r_m(t)$  which is a constant  $r_m(t_0)$  in the interval  $[t_0, T]$ .

For convenience, we introduce the following notation:

$$\begin{aligned}
 H(t, t_0) &= \sum_{i=0}^m \int_{\delta_i(t_0)}^{t_0} \Phi(t, r_i(s))B_i(r_i(s))\dot{r}_i(s)u_{t_0}(s) ds \\
 &+ \sum_{i=m+1}^M \int_{\delta_i(t_0)}^{\delta_i(t)} \Phi(t, r_i(s)) \\
 &\times B_i(r_i(s))\dot{r}_i(s)u_{t_0}(s) ds \\
 G_i(t, s) &= \sum_{j=0}^i \Phi(t, r_j(s))B_j(r_j(s))\dot{r}_j(s), \\
 & \quad i = 1, 2, \dots, M.
 \end{aligned}$$

We define the linear and bounded control operator

$$\mathbb{L} : L_2^{\mathcal{F}}([t_0, T], \mathbb{R}^l) \rightarrow L_2(\Omega, \mathcal{F}_T, \mathbb{R}^n)$$

as follows:

$$\mathbb{L}u = \int_{t_0}^T G_m(T, s)u(s) ds,$$

and its adjoint bounded linear operator

$$\mathbb{L}^* : L_2(\Omega, \mathcal{F}_T, \mathbb{R}^n) \rightarrow L_2^{\mathcal{F}}([t_0, T], \mathbb{R}^l)$$

as

$$(\mathbb{L}^*z)(t) = G_m^*(T, t)\mathbb{E}\{z \mid \mathcal{F}_t\}, \quad t \in [t_0, T],$$

where the star (\*) denotes the adjoint matrix.

From the above notation it follows that the set of all states reachable from the initial state  $x(t_0) = x_0 \in L_2(\Omega, \mathcal{F}_T, \mathbb{R}^n)$  in time  $T > 0$ , using admissible controls, has the form

$$\begin{aligned}
 \mathcal{R}_T(\mathcal{U}_{ad}) &= \{x(T; x_0, u) \in L_2(\Omega, \mathcal{F}_T, \mathbb{R}^n) : u(\cdot) \in \mathcal{U}_{ad}\} \\
 &= \Phi(T, t_0)x_0 + \text{Im } \mathbb{L} \\
 &+ \sum_{i=0}^m \int_{\delta_i(t_0)}^{t_0} \Phi(T, r_i(s))B_i(r_i(s))\dot{r}_i(s)u_{t_0}(s) ds \\
 &+ \sum_{i=m+1}^M \int_{\delta_i(t_0)}^{\delta_i(T)} \Phi(T, r_i(s))B_i(r_i(s))\dot{r}_i(s)u_{t_0}(s) ds \\
 &+ \int_{t_0}^T \Phi(T, s)\tilde{\sigma}(s) dw(s).
 \end{aligned}$$

The linear controllability operator  $\mathcal{W} : L_2(\Omega, \mathcal{F}_T, \mathbb{R}^n) \rightarrow L_2(\Omega, \mathcal{F}_T, \mathbb{R}^n)$  is associated with the system (1) and defined by

$$\mathcal{W} = \mathbb{L} \mathbb{L}^* \{ \cdot \} = \int_{t_0}^T G_m(T, s)G_m^*(T, s)\mathbb{E}\{ \cdot \mid \mathcal{F}_s \} ds,$$

and the deterministic controllability matrix  $\Gamma_s^T \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$  is

$$\Gamma_s^T = \int_s^T G_m(T, s)G_m^*(T, s) ds, \quad s \in [t_0, T].$$

**Definition 1.** (Klamka, 1976) The stochastic system (1) is said to be *relatively controllable* on  $[t_0, T]$  if, for every complete state  $y(t_0)$  and every  $x_1 \in \mathbb{R}^n$ , there exists a control  $u(t)$  defined on  $[t_0, T]$  such that the corresponding trajectory of the stochastic system (1) satisfies the condition  $x(T) = x_1$ .

**Definition 2.** (Klamka, 2007b) The stochastic system (1) is said to be *relatively exact controllable* on  $[t_0, T]$  if

$$\mathcal{R}_T(\mathcal{U}_{ad}) = L_2(\Omega, \mathcal{F}_T, \mathbb{R}^n),$$

that is, if all the points in  $L_2(\Omega, \mathcal{F}_T, \mathbb{R}^n)$  can be exactly reached at time  $T$  from any arbitrary initial point  $x_0 \in L_2(\Omega, \mathcal{F}_T, \mathbb{R}^n)$  at time  $T > 0$ .

**Definition 3.** (Klamka, 2007b) The stochastic system (1) is said to be *relatively approximate controllable* on  $[t_0, T]$  if

$$\overline{\mathcal{R}_T(\mathcal{U}_{ad})} = L_2(\Omega, \mathcal{F}_T, \mathbb{R}^n),$$

that is, if all the points in  $L_2(\Omega, \mathcal{F}_T, \mathbb{R}^n)$  can be approximately reached at time  $T$  from any arbitrary initial point  $x_0 \in L_2(\Omega, \mathcal{F}_T, \mathbb{R}^n)$  at time  $T > 0$ .

### 3. Linear stochastic systems

In this section, we recall some important results to establish the relative controllability of the linear stochastic system (1).

Consider the corresponding deterministic system of the following form:

$$z'(t) = A(t)z(t) + \sum_{i=0}^M B_i(t)v(\delta_i(t)), \quad (5)$$

where the admissible controls  $v \in L_2([t_0, T], \mathbb{R}^l)$ .

For the deterministic system (5) let us denote by  $R_T$  the set of all states reachable from the initial state  $z(t_0) = z_0$  in time  $T > 0$  using admissible controls.

**Definition 4.** (Klamka, 1991) The deterministic system (5) is said to be *relatively controllable* on  $[t_0, T]$  if  $R_T = \mathbb{R}^n$ .

**Lemma 1.** (Klamka, 1991) *The following conditions are equivalent:*

- (i) *The deterministic system (5) is relatively controllable on  $[t_0, T]$ .*
- (ii) *The controllability matrix  $\mathcal{W}$  is nonsingular.*

The following lemma shows that the relative controllability of the associated deterministic linear system (5) is equivalent to the relative exact controllability and the relative approximate controllability of the linear stochastic system (1).

**Lemma 2.** (Klamka, 2008a) *The following conditions are equivalent:*

- (i) *The deterministic system (5) is relatively controllable on  $[t_0, T]$ .*
- (ii) *The stochastic system (1) is relatively exact controllable on  $[t_0, T]$ .*
- (iii) *The stochastic system (1) is relatively approximate controllable on  $[t_0, T]$ .*

Note that, from the work of Klamka (2007a), we see that if the linear stochastic system (1) is relatively exact controllable then the operator  $\mathcal{W}$  is strictly positive definite and thus the inverse linear operator  $\mathcal{W}^{-1}$  is bounded. Using the fact that the operator  $\mathcal{W}^{-1}$  is bounded, we shall construct a control  $u^0(t), t \in [t_0, T]$  that steers the system from the initial state  $x_0$  to a desired final state  $x_1$  at time  $T$ .

**Lemma 3.** *Assume that the stochastic system (1) is relatively exact controllable on  $[t_0, T]$ . Then, for an arbitrary target  $x_1 \in L_2(\Omega, \mathcal{F}_t, \mathbb{R}^n)$  and  $\tilde{\sigma}(\cdot) \in L_2^{\mathcal{F}}([t_0, T], \mathbb{R}^{n \times n})$ , the control*

$$\begin{aligned} u^0(t) &= G_m^*(t, T)\mathbb{E}\left\{\mathcal{W}^{-1}\left(x_1 - \Phi(T, t_0)x_0\right.\right. \\ &\quad - \sum_{i=0}^m \int_{\delta_i(t_0)}^{t_0} \Phi(T, r_i(s))B_i(r_i(s))\dot{r}_i(s)u_{t_0}(s) ds \\ &\quad - \sum_{i=m+1}^M \int_{\delta_i(t_0)}^{\delta_i(T)} \Phi(T, r_i(s)) \\ &\quad \times B_i(r_i(s))\dot{r}_i(s)u_{t_0}(s) ds \\ &\quad \left.\left. - \int_{t_0}^T \Phi(T, s)\tilde{\sigma}(s) dw(s)\right) \middle| \mathcal{F}_t\right\} \end{aligned} \quad (6)$$

*transfers the system*

$$\begin{aligned} x(t) &= \Phi(t, t_0)x_0 \\ &\quad + \sum_{i=0}^m \int_{\delta_i(t_0)}^{t_0} \Phi(t, r_i(s))B_i(r_i(s))\dot{r}_i(s)u_{t_0}(s) ds \\ &\quad + \sum_{i=m+1}^M \int_{\delta_i(t_0)}^{\delta_i(t)} \Phi(t, r_i(s))B_i(r_i(s))\dot{r}_i(s)u_{t_0}(s) ds \\ &\quad + \int_{t_0}^t G_m(t, s)u(s) ds + \int_{t_0}^t \Phi(t, s)\tilde{\sigma}(s) dw(s) \end{aligned}$$

from  $x_0 \in \mathbb{R}^n$  to  $x_1 \in \mathbb{R}^n$  at time  $T$ .

Moreover, among all the admissible controls  $u(t)$  transferring the initial state  $x_0$  to the final state  $x_1$  at time  $T > 0$ , the control  $u^0(t)$  minimizes the integral performance index

$$\mathcal{J}(u) = \mathbb{E} \int_{t_0}^T \|u(t)\|^2 dt.$$

*Proof.* Since the stochastic dynamical system (1) is relatively exact controllable on  $[t_0, T]$ , the controllability operator  $\mathcal{W}$  is invertible and its inverse  $\mathcal{W}^{-1}$  is a linear and bounded operator, that is,

$$\mathcal{W}^{-1} \in \mathcal{L}(L_2(\Omega, \mathcal{F}_t, \mathbb{R}^n), L_2(\Omega, \mathcal{F}_t, \mathbb{R}^n)).$$

Substituting the control  $u^0(t)$  into the solution formula of the differential state equation and substituting  $t = T$ , one can easily verify that the control (6) steers the linear system from  $x_0$  to  $x_1$ . The second part of the proof is similar to that of Theorem 2 of Klamka (2007a). ■

#### 4. Nonlinear systems

Taking into account the above notation and results, we shall derive sufficient controllability conditions for the semi-linear stochastic system with multiple delays in control of the form

$$\left. \begin{aligned} dx(t) &= \left[ A(t)x(t) + \sum_{i=0}^M B_i(t)u(\delta_i(t)) \right] dt \\ &\quad + \sigma(t, x(t)) dw(t), \\ x(t_0) &= x_0, \end{aligned} \right\} \quad (7)$$

where  $\sigma : [t_0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}^{n \times n}$  and  $A(t), B_i(t), \delta_i(t), i = 1, 2, \dots, M$ , are defined as before.

Then the solution of the system (7) can be expressed in the following form:

$$\begin{aligned} x(t) &= \Phi(t, t_0)x_0 \\ &\quad + \int_{t_0}^t \Phi(t, s) \sum_{i=0}^M B_i(s)u(\delta_i(s)) ds \\ &\quad + \int_{t_0}^t \Phi(t, s)\sigma(s, x(s)) dw(s). \end{aligned}$$

Now, using the time lead function, we have

$$\begin{aligned} x(t) &= \Phi(t, t_0)x_0 \\ &\quad + \sum_{i=0}^M \int_{\delta_i(t_0)}^{\delta_i(t)} \Phi(t, r_i(s))B_i(r_i(s))\dot{r}_i(s)u(s) ds \\ &\quad + \int_{t_0}^t \Phi(t, s)\sigma(s, x(s)) dw(s) \end{aligned} \quad (8)$$

and, using (4), the above equation for  $t = T$  can be expressed as

$$\begin{aligned} x(T) &= \Phi(T, t_0)x_0 \\ &\quad + \sum_{i=0}^m \int_{\delta_i(t_0)}^{t_0} \Phi(T, r_i(s))B_i(r_i(s))\dot{r}_i(s)u_{t_0}(s) ds \\ &\quad + \sum_{i=0}^m \int_{t_0}^T \Phi(T, r_i(s))B_i(r_i(s))\dot{r}_i(s)u(s) ds \\ &\quad + \sum_{i=m+1}^M \int_{\delta_i(t_0)}^{\delta_i(T)} \Phi(T, r_i(s)) \\ &\quad \times B_i(r_i(s))\dot{r}_i(s)u_{t_0}(s) ds \\ &\quad + \int_{t_0}^T \Phi(T, s)\sigma(s, x(s)) dw(s). \end{aligned}$$

Now let us define the controllability operator and the control function associated with the system (7) as follows:

$$\begin{aligned} \mathcal{W} &= \mathcal{W}(t_0, T) = \int_{t_0}^T G_m(T, s)G_m^*(T, s)\mathbb{E}\{\cdot | \mathcal{F}_s\} ds, \\ u(t) &= G_m^*(T, t)\mathbb{E}\left\{ \mathcal{W}^{-1} \left( x_1 - \Phi(T, t_0)x_0 \right. \right. \\ &\quad - \sum_{i=0}^m \int_{\delta_i(t_0)}^{t_0} \Phi(T, r_i(s))B_i(r_i(s))\dot{r}_i(s)u_{t_0}(s) ds \\ &\quad - \sum_{i=m+1}^M \int_{\delta_i(t_0)}^{\delta_i(T)} \Phi(T, r_i(s)) \\ &\quad \times B_i(r_i(s))\dot{r}_i(s)u_{t_0}(s) ds \\ &\quad \left. \left. - \int_{t_0}^T \Phi(T, s)\sigma(s, x(s)) dw(s) \right) \middle| \mathcal{F}_t \right\}, \end{aligned} \quad (9)$$

where  $G_m$  is defined as in the linear case.

Inserting (9) in (8), it is easy to verify that the control  $u(t)$  transfers  $x_0$  to the desired vector  $x_1$  at time  $T$ .

For the proof of the main result, we impose the following assumptions on the data of the problem:

(H1) The function  $\sigma$  is Lipschitz continuous, that is, for  $x, y \in \mathbb{R}^n$  and  $t_0 \leq t \leq T$  there exists a constant  $L_1 > 0$  such that

$$\|\sigma(t, x) - \sigma(t, y)\|^2 \leq L_1 \|x - y\|^2.$$

(H2) The function  $\sigma$  satisfies the usual linear growth condition, that is, there exists a constant  $L_2 > 0$  such that for all  $t \in [t_0, T]$  and all  $x \in \mathbb{R}^n$

$$\|\sigma(t, x)\|^2 \leq L_2(1 + \|x\|^2).$$

Let  $\mathcal{B}_2$  denote the Banach space of all square integrable and  $\mathcal{F}_t$ -adapted processes  $\varphi(t)$  with the norm

$$\|x\|^2 := \sup_{t \in [t_0, T]} \mathbb{E}\|x(t)\|^2.$$

Define the nonlinear operator  $\mathcal{P}$  from  $\mathcal{B}_2$  to  $\mathcal{B}_2$  by

$$\begin{aligned} (\mathcal{P}x)(t) &= \Phi(t, t_0)x_0 \\ &+ \sum_{i=0}^m \int_{\delta_i(t_0)}^{t_0} \Phi(t, r_i(s))B_i(r_i(s))\dot{r}_i(s)u_{t_0}(s) ds \\ &+ \sum_{i=m+1}^M \int_{\delta_i(t_0)}^{\delta_i(t)} \Phi(t, r_i(s)) \\ &\times B_i(r_i(s))\dot{r}_i(s)u_{t_0}(s) ds \\ &+ \int_{t_0}^t G_m(t, s)u(s) ds \\ &+ \int_{t_0}^t \Phi(t, s)\sigma(s, x(s)) dw(s), \end{aligned} \tag{10}$$

From Lemma 3, it follows that if the operator  $\mathcal{P}$  defined in (10) has a fixed point, then the system (7) has a solution  $x(t)$  defined in (8) with respect to  $u(\cdot)$ , and  $(\mathcal{P}x)(T) = x(T) = x_1$ , which implies that the system (7) is relatively controllable. Thus, the problem of the controllability of the semi-linear system (7) can be reduced to the existence of a unique fixed point of the operator  $\mathcal{P}$ .

Now, for our convenience, let us introduce the following notation:

$$\begin{aligned} M &= \max\{\|\Gamma_s^T\|^2 : s \in [t_0, T]\}, \\ k_1 &= \max\{\|\Phi(t, s)\|^2 : t_0 \leq s < t \leq T\}, \\ k_2 &= \max\{\|H(t, t_0)\|^2 : t_0 \leq t \leq T\}. \end{aligned}$$

Note that if the linear system (1) is relatively exact controllable, then for some  $\gamma > 0$  (Klamka, 2008b)

$$\langle \mathcal{W}z, z \rangle \geq \gamma \mathbb{E}\|z\|^2 \quad \text{for all } z \in L_2(\Omega, \mathcal{F}_T, \mathbb{R}^n),$$

and so

$$\|\mathcal{W}^{-1}\|^2 \leq \frac{1}{\gamma} = k_3.$$

**Theorem 1.** Assume that the conditions (H1) and (H2) hold and suppose that the linear stochastic system (1) is relatively exact controllable. Further, if the inequality

$$2k_1 L_1(1 + Mk_3)T < 1 \tag{11}$$

is satisfied, then the semi-linear stochastic system (7) is relatively exact controllable.

*Proof.* In order to prove the relative controllability of the system (7), it is enough to show that the operator  $\mathcal{P}$  has a fixed point in  $\mathcal{B}_2$ . To do this, we can employ the

contraction mapping principle. To apply the principle, first we show that  $\mathcal{P}$  maps  $\mathcal{B}_2$  into itself. Now, by Lemma 3, we have

$$\begin{aligned} &\mathbb{E}\|(\mathcal{P}x)(t)\|^2 \\ &= \mathbb{E}\left\| \Phi(t, t_0)x_0 \right. \\ &\quad + \sum_{i=0}^m \int_{\delta_i(t_0)}^{t_0} \Phi(t, r_i(s))B_i(r_i(s))\dot{r}_i(s)u_{t_0}(s) ds \\ &\quad + \sum_{i=m+1}^M \int_{\delta_i(t_0)}^{\delta_i(t)} \Phi(t, r_i(s))B_i(r_i(s))\dot{r}_i(s)u_{t_0}(s) ds \\ &\quad + \int_{t_0}^t G_m(t, s)u(s) ds \\ &\quad \left. + \int_{t_0}^t \Phi(t, s)\sigma(s, x(s)) dw(s) \right\|^2 \\ &\leq 4\mathbb{E}\|\Phi(t, t_0)\|^2 \|x_0\|^2 + 4\mathbb{E}\|H(t, t_0)\|^2 \\ &\quad + 4\mathbb{E}\left\| \int_{t_0}^t G_m(t, s)u(s) ds \right\|^2 \\ &\quad + 4\mathbb{E}\left\| \int_{t_0}^t \Phi(t, s)\sigma(s, x(s)) dw(s) \right\|^2. \end{aligned} \tag{12}$$

■

For simplification, first consider the third term in the above inequality,

$$\begin{aligned} &\mathbb{E}\left\| \int_{t_0}^t G_m(t, \tau)u(\tau) d\tau \right\|^2 \\ &= \mathbb{E}\left\| \int_{t_0}^t G_m(t, \tau)G_m^*(T, \tau) \right. \\ &\quad \times \mathbb{E}\left\{ \mathcal{W}^{-1}\left(x_1 - \Phi(T, t_0)x_0 H(T, t_0) \right. \right. \\ &\quad \left. \left. - \int_{t_0}^T \Phi(T, s)\sigma(s, x(s)) dw(s)\right) \middle| \mathcal{F}_\tau \right\} d\tau \right\|^2 \\ &\leq 4Mk_3 [\|x_1\|^2 + k_1\|x_0\|^2 + k_2 \\ &\quad + k_1 L_2 \int_{t_0}^T (1 + \mathbb{E}\|x(s)\|^2) ds]. \end{aligned} \tag{13}$$

Using (13) in (12), we have

$$\begin{aligned} &\mathbb{E}\|(\mathcal{P}x)(t)\|^2 \\ &\leq 4k_1\|x_0\|^2 + 4k_2 + 16Mk_3 [\|x_1\|^2 \\ &\quad + k_1\|x_0\|^2 + k_2 + k_1 L_2 \int_{t_0}^t (1 + \mathbb{E}\|x(s)\|^2) ds] \\ &\quad + 4k_1 L_2 \int_{t_0}^t (1 + \mathbb{E}\|x(s)\|^2) ds \end{aligned}$$

$$\begin{aligned} &\leq 4k_1\|x_0\|^2 + 4k_2 \\ &\quad + 16Mk_3(\|x_1\|^2 + k_1\|x_0\|^2 + k_2) \\ &\quad + (4k_1 + 16Mk_3k_1)L_2 \\ &\quad \times \int_{t_0}^T (1 + \mathbb{E}\|x(s)\|^2) ds. \end{aligned} \tag{14}$$

From (14) and (H2) it follows that there exists  $C > 0$  depending on  $x_0, T, L_2, M, k_1, k_2$  and  $k_3$  such that

$$\mathbb{E}\|(\mathcal{P}x)(t)\|^2 \leq C \left(1 + \int_{t_0}^T \mathbb{E}\|x(r)\|^2 dr\right).$$

Thus we have

$$\mathbb{E}\|(\mathcal{P}x)(t)\|^2 \leq C \left(1 + T \sup_{r \in [t_0, T]} \mathbb{E}\|x(r)\|^2\right).$$

Therefore  $\mathcal{P}$  maps  $\mathcal{B}_2$  into itself.

Secondly, we claim that  $\mathcal{P}$  is a contraction on  $\mathcal{B}_2$ . For  $x, y \in \mathcal{B}_2$ ,

$$\begin{aligned} &\mathbb{E}\|(\mathcal{P}x_1)(t) - (\mathcal{P}x_2)(t)\|^2 \\ &\leq \mathbb{E}\left\| \int_{t_0}^t \Phi(t, s) [\sigma(s, x_1(s)) - \sigma(s, x_2(s))] dw(s) \right. \\ &\quad \left. + \Gamma_{t_0}^T \mathcal{W}^{-1} \left( \int_{t_0}^T \Phi(T, s) [\sigma(s, x_2(s)) \right. \right. \\ &\quad \left. \left. - \sigma(s, x_1(s))] dw(s) \right) \right\|^2 \\ &\leq 2k_1L_1 \int_{t_0}^T \mathbb{E}\|x_1(s) - x_2(s)\|^2 ds \\ &\quad + 2Mk_1k_3L_1 \int_{t_0}^T \mathbb{E}\|x_1(s) - x_2(s)\|^2 ds \\ &\leq 2k_1(1 + Mk_3)L_1 \int_{t_0}^T \mathbb{E}\|x_1(s) - x_2(s)\|^2 ds. \end{aligned}$$

Accordingly,

$$\begin{aligned} &\sup_{t \in [t_0, T]} \mathbb{E}\|(\mathcal{P}x_1)(t) - (\mathcal{P}x_2)(t)\|^2 \\ &\leq 2k_1L_1(1 + Mk_3)T \sup_{t \in [t_0, T]} \mathbb{E}\|x_1(t) - x_2(t)\|^2. \end{aligned}$$

Therefore we conclude from (11) that  $\mathcal{P}$  is a contraction mapping on  $\mathcal{B}_2$ . Then the mapping  $\mathcal{P}$  has a unique fixed point  $x(\cdot) \in \mathcal{B}_2$ , which is the solution of Eqn. (8). Thus the system is relatively exact controllable on  $[t_0, T]$ .

**Remark 1.** Obviously, the hypothesis (11) is fulfilled if  $L_1$  is sufficiently small.

### 5. Numerical example

To illustrate the applicability of the above results, in this section we consider the following semi-linear stochastic

system:

$$\begin{aligned} dx_1(t) &= [-0.5x_1(t) + u_1(t) + e^{-0.5t}u_2(t) \\ &\quad + 0.05u_1(0.75t) + e^{-0.4t}u_1(0.5t) \\ &\quad + 0.01t^2u_2(0.5t) + e^{-5t}u_2(0.25t)] dt \\ &\quad + \frac{x_1(t) \cos x_2(t)}{3} dw_1(t), \tag{15} \\ dx_2(t) &= [-0.1x_2(t) + tu_1(t)] dt \\ &\quad + \frac{x_2(t) \sin x_1(t)}{4} dw_2(t), \end{aligned}$$

which can be reformulated in the form of (7) with  $M = 3$ :

$$\begin{aligned} x(t) &= \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, & u(t) &= \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}, \\ A(t) &= \begin{bmatrix} -0.5 & 0 \\ 0 & -0.1 \end{bmatrix}, \\ B_0 &= \begin{bmatrix} 1 & e^{-0.5t} \\ t & 0 \end{bmatrix}, \\ B_1 &= \begin{bmatrix} 0.05t & 0 \\ 0 & 0 \end{bmatrix}, \\ B_2 &= \begin{bmatrix} e^{-0.4t} & 0.01t^2 \\ 0 & 0 \end{bmatrix}, \\ B_3 &= \begin{bmatrix} 0 & e^{-5t} \\ 0 & 0 \end{bmatrix}, \\ w(t) &= \begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix}, \\ \sigma(t, x(t)) &= \begin{bmatrix} \frac{1}{3}x_1(t) \cos x_2(t) & 0 \\ 0 & \frac{1}{4}x_2(t) \cos x_1(t) \end{bmatrix}. \end{aligned}$$

Moreover,

$$\begin{aligned} \delta_0(t) &= t, & \delta_1(t) &= 0.75t, \\ \delta_2(t) &= 0.5t, & \delta_3(t) &= 0.25t \end{aligned}$$

for  $t \in [0, 2]$  and

$$\begin{aligned} \delta_m(t) &< \delta_{m-1}(t) < \dots < \delta_k(t) \\ &< \dots < \delta_1(t) < \delta_0(t) = t \end{aligned}$$

for  $t \in [t_0, t_1]$ .

Consider the following lead functions

$$\begin{aligned} r_0(t) &= t, & r_1(t) &= \frac{4}{3}t, \\ r_2(t) &= 2t, & r_3(t) &= 4t. \end{aligned}$$

Moreover, for  $t_1 = 2$  we have

$$\delta_3(2) < \delta_2(2) < \delta_1(2) < \delta_0(2) = 2.$$

Taking into account the form of the matrices  $A(t), B_0(t), B_1(t), B_2(t), B_3(t)$  and the formula for

the computation of the exponent matrix function, we have the transition matrix

$$\Phi(t, t_0) = \begin{bmatrix} e^{-0.5t^2} & 0 \\ 0 & e^{-0.1t} \end{bmatrix},$$

and the controllability Grammian

$$\begin{aligned} \mathcal{W}(0, 2) &= \int_0^2 G_m(t, s)G_m^*(t, s) ds \\ &= \begin{bmatrix} 6.34 & 3.44 \\ 3.44 & 2.42 \end{bmatrix}. \end{aligned}$$

Hence rank  $\mathcal{W}(0, 2) = 2$ . Take the final point as  $x_T \in \mathbb{R}^2$ . It is easy to show that, for all  $x \in \mathbb{R}^2$ ,

$$\|\sigma(t, x(t))\|^2 \leq \frac{1}{9}(1 + \|x\|^2).$$

One can see that the inequality (11) holds and all other conditions stated in Theorem 1 are satisfied. Hence, the system (15) is relatively exact controllable on  $[0, 2]$ , that is, the system (15) can be steered from  $x_0$  to  $x_1$ .

**Remark 2.** It is important to note that the results discussed in the papers by Guendouzi and Hamada (2013; 2014) are not valid. In these papers, sufficient conditions for the controllability of nonlinear stochastic systems involving fractional derivatives are established. Since the integral representation of the fractional dynamical system considered completely relies on the Laplace transform, the solution representation is not valid, as the Laplace transform of the diffusion term involving the white noise term is not well defined.

## 6. Concluding remarks

In the paper, the relative controllability of semi-linear stochastic systems with time varying multiple delays in the control function is addressed. Sufficient conditions are established by the application of the Banach fixed point technique. It should be pointed out that the results obtained here generalize those by Klamka (2008a) as well as Shen and Sun (2012) from stochastic systems with single control delay to multiple time-varying delays. Further, they also generalize the results of Klamka (2008b) from stochastic systems with constant delays to time varying delays.

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