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REVERSE JENSEN'S TYPE TRACE INEQUALITIES FOR CONVEX FUNCTIONS OF SELFADJOINT OPERATORS IN HILBERT SPACES

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Abstract. Some reverse Jensen's type trace inequalities for convex functions of selfadjoint operators in Hilbert spaces are provided. Applications for some convex functions of interest and reverses of Hölder and Schwarz trace inequalities are also given.

1. Introduction

Let $(H, \langle \cdot, \cdot \rangle)$ be a complex Hilbert space and $\{e_i\}_{i \in I}$ an *orthonormal basis* of H. We say that $A \in \mathcal{B}(H)$ is a *Hilbert-Schmidt operator* if

$$\sum_{i \in I} \|Ae_i\|^2 < \infty. \tag{1.1}$$

It is well know that, if $\{e_i\}_{i \in I}$ and $\{f_j\}_{j \in J}$ are orthonormal bases for H and $A \in \mathcal{B}(H)$ then

$$\sum_{i \in I} \|Ae_i\|^2 = \sum_{j \in I} \|Af_j\|^2 = \sum_{j \in I} \|A^* f_j\|^2 \tag{1.2}$$

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showing that the definition (1.1) is independent of the orthonormal basis and A is a Hilbert-Schmidt operator iff A^* is a Hilbert-Schmidt operator.

Let $\mathcal{B}_2(H)$ the set of Hilbert-Schmidt operators in $\mathcal{B}(H)$. For $A \in \mathcal{B}_2(H)$ we define

$$\|A\|_2 := \left(\sum_{i \in I} \|Ae_i\|^2\right)^{1/2} \tag{1.3}$$

for $\{e_i\}_{i \in I}$ an orthonormal basis of H. This definition does not depend on the choice of the orthonormal basis.

Using the triangle inequality in $l^2(I)$, one checks that $\mathcal{B}_2(H)$ is a *vector space* and that $\|\cdot\|_2$ is a norm on $\mathcal{B}_2(H)$, which is usually called in the literature as the *Hilbert-Schmidt norm.*

Denote *the modulus* of an operator $A \in \mathcal{B}(H)$ by $|A| := (A^*A)^{1/2}$.

Because $\||A|x\| = \|Ax\|$ for all $x \in H$, A is Hilbert-Schmidt iff $|A|$ is Hilbert-Schmidt and $\|A\|_2 = \||A|\|_2$. From (1.2) we have that if $A \in \mathcal{B}_2(H)$, then $A^* \in \mathcal{B}_2(H)$ and $\|A\|_2 = \|A^*\|_2$.

The following theorem collects some of the most important properties of Hilbert-Schmidt operators.

Theorem 1.1. *We have*

(i) $(\mathcal{B}_2(H), \|\cdot\|_2)$ *is a Hilbert space with inner product*

$$\langle A, B\rangle_2 := \sum_{i \in I} \langle Ae_i, Be_i\rangle = \sum_{i \in I} \langle B^*Ae_i, e_i\rangle \tag{1.4}$$

and the definition does not depend on the choice of the orthonormal basis $\{e_i\}_{i \in I}$;

(ii) *We have the inequalities*

$$\|A\| \le \|A\|_2 \tag{1.5}$$

for any $A \in \mathcal{B}_2(H)$ *and*

$$\|AT\|_2, \|TA\|_2 \le \|T\| \|A\|_2 \tag{1.6}$$

for any $A \in \mathcal{B}_2(H)$ *and* $T \in \mathcal{B}(H)$;

(iii) $\mathcal{B}_2(H)$ *is an operator ideal in* $\mathcal{B}(H)$, *i.e.*

$$\mathcal{B}(H)\mathcal{B}_2(H)\mathcal{B}(H) \subseteq \mathcal{B}_2(H);$$

(iv) $\mathcal{B}_{fin}(H)$, *the space of operators of finite rank, is a dense subspace of* $\mathcal{B}_2(H)$;

(v) $\mathcal{B}_2(H) \subseteq \mathcal{K}(H)$, *where* $\mathcal{K}(H)$ *denotes the algebra of compact operators on* H.

If $\{e_i\}_{i\in I}$ an orthonormal basis of H, we say that $A \in \mathcal{B}(H)$ is *trace class* if

$$\|A\|_1 := \sum_{i\in I} \langle |A| e_i, e_i\rangle < \infty. \tag{1.7}$$

The definition of $\|A\|_1$ does not depend on the choice of the orthonormal basis $\{e_i\}_{i\in I}$. We denote by $\mathcal{B}_1(H)$ the set of trace class operators in $\mathcal{B}(H)$.

The following proposition holds.

Proposition 1.2. *If* $A \in \mathcal{B}(H)$, *then the following are equivalent:*

(i) $A \in \mathcal{B}_1(H)$;

(ii) $|A|^{1/2} \in \mathcal{B}_2(H)$;

(iii) A *(or* $|A|$*) is the product of two elements of* $\mathcal{B}_2(H)$.

The following properties are also well known.

Theorem 1.3. *With the above notations:*

(i) *We have*

$$\|A\|_1 = \|A^*\|_1 \text{ and } \|A\|_2 \le \|A\|_1 \tag{1.8}$$

for any $A \in \mathcal{B}_1(H)$;

(ii) $\mathcal{B}_1(H)$ *is an operator ideal in* $\mathcal{B}(H)$, *i.e.*

$$\mathcal{B}(H)\mathcal{B}_1(H)\mathcal{B}(H) \subseteq \mathcal{B}_1(H);$$

(iii) *We have*

$$\mathcal{B}_2(H)\mathcal{B}_2(H) = \mathcal{B}_1(H);$$

(iv) *We have*

$$\|A\|_1 = \sup\{\langle A, B\rangle_2 \mid B \in \mathcal{B}_2(H), \ \|B\| \le 1\};$$

(v) $(\mathcal{B}_1(H), \|\cdot\|_1)$ *is a Banach space.*

(vi) *We have the following isometric isomorphisms*

$$\mathcal{B}_1(H) \cong K(H)^* \text{ and } \mathcal{B}_1(H)^* \cong \mathcal{B}(H),$$

where $K(H)^*$ *is the dual space of* $K(H)$ *and* $\mathcal{B}_1(H)^*$ *is the dual space of* $\mathcal{B}_1(H)$.

We define the *trace* of a trace class operator $A \in \mathcal{B}_1(H)$ to be

$$\operatorname{tr}(A) := \sum_{i \in I} \langle Ae_i, e_i \rangle, \tag{1.9}$$

where $\{e_i\}_{i \in I}$ an orthonormal basis of H. Note that this coincides with the usual definition of the trace if H is finite-dimensional. We observe that the series (1.9) converges absolutely and it is independent from the choice of basis.

The following result collects some properties of the trace.

THEOREM 1.4. *We have*

(i) *If* $A \in \mathcal{B}_1(H)$ *then* $A^* \in \mathcal{B}_1(H)$ *and*

$$\operatorname{tr}(A^*) = \overline{\operatorname{tr}(A)}; \tag{1.10}$$

(ii) *If* $A \in \mathcal{B}_1(H)$ *and* $T \in \mathcal{B}(H)$, *then* AT, $TA \in \mathcal{B}_1(H)$ *and*

$$\operatorname{tr}(AT) = \operatorname{tr}(TA) \text{ and } |\operatorname{tr}(AT)| \le \|A\|_1 \|T\|; \tag{1.11}$$

(iii) $\operatorname{tr}(\cdot)$ *is a bounded linear functional on* $\mathcal{B}_1(H)$ *with* $\|\operatorname{tr}\| = 1$;

(iv) *If* A, $B \in \mathcal{B}_2(H)$ *then* AB, $BA \in \mathcal{B}_1(H)$ *and* $\operatorname{tr}(AB) = \operatorname{tr}(BA)$;

(v) $\mathcal{B}_{fin}(H)$ *is a dense subspace of* $\mathcal{B}_1(H)$.

Utilising the trace notation we obviously have that

$$\langle A, B \rangle_2 = \operatorname{tr}(B^*A) = \operatorname{tr}(AB^*) \text{ and } \|A\|_2^2 = \operatorname{tr}(A^*A) = \operatorname{tr}\left(|A|^2\right)$$

for any A, $B \in \mathcal{B}_2(H)$.

For the theory of trace functionals and their applications the reader is referred to [38].

For some classical trace inequalities see [5, 7, 35, 50], which are continuations of the work of Bellman [2]. For related works the reader can refer to [1, 3, 5, 29, 32–34, 36, 47].

Consider the orthonormal basis $\mathcal{E} := \{e_i\}_{i\in I}$ in the complex Hilbert space $(H, \langle\cdot,\cdot\rangle)$ and for a nonzero operator $B \in \mathcal{B}_2(H)$ let introduce the subset of indices from I defined by

$$I_{\mathcal{E},B} := \{i \in I : \ Be_i \neq 0\}.$$

We observe that $I_{\mathcal{E},B}$ is non-empty for any nonzero operator B and if $\ker(B) = 0$, i.e. B is injective, then $I_{\mathcal{E},B} = I$. We also have for $B \in \mathcal{B}_2(H)$ that

$$\operatorname{tr}\left(|B|^2\right) = \operatorname{tr}(B^*B) = \sum_{i\in I} \langle B^*Be_i, e_i\rangle = \sum_{i\in I} \|Be_i\|^2 = \sum_{i\in I_{\mathcal{E},B}} \|Be_i\|^2.$$

In the recent paper [26] we obtained among others the following result for convex functions.

THEOREM 1.5. *Let A be a selfadjoint operator on the Hilbert space H and assume that $\operatorname{Sp}(A) \subseteq [m, M]$ for some scalars m, M with $m < M$. If f is a continuous convex function on $[m, M]$, $\mathcal{E} := \{e_i\}_{i\in I}$ is an orthonormal basis in H and $B \in \mathcal{B}_2(H)\setminus\{0\}$, then $\frac{\operatorname{tr}\left(|B|^2 A\right)}{\operatorname{tr}\left(|B|^2\right)} \in [m, M]$ and*

$$(1.12)\quad f\left(\frac{\operatorname{tr}\left(|B|^2 A\right)}{\operatorname{tr}\left(|B|^2\right)}\right)\operatorname{tr}\left(|B|^2\right) \leq J_{\varepsilon}(f; A, B) \leq \operatorname{tr}\left(|B|^2 f(A)\right)$$

$$\leq \frac{1}{M-m}\left(f(m)\operatorname{tr}\left[|B|^2(M1_H - A)\right] + f(M)\operatorname{tr}\left[|B|^2(A - m1_H)\right]\right),$$

where

$$(1.13)\qquad J_{\varepsilon}(f; A, B) := \sum_{i\in I_{\mathcal{E},B}} f\left(\frac{\langle B^*ABe_i, e_i\rangle}{\|Be_i\|^2}\right)\|Be_i\|^2.$$

For related functionals and their superadditivity and monotonicity properties see [26].

In [27] we obtained the following reverse of Jensen's inequality.

THEOREM 1.6. *Let A be a selfadjoint operator on the Hilbert space H and assume that $\operatorname{Sp}(A) \subseteq [m, M]$ for some scalars m, M with $m < M$. If f is a* *continuously differentiable convex function on* $[m, M]$ *and* $P \in \mathcal{B}_1(H) \setminus \{0\}$, $P \geq 0$, *then we have*

$$0 \leq \frac{\operatorname{tr}(Pf(A))}{\operatorname{tr}(P)} - f\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right) \tag{1.14}$$
$$\leq \frac{\operatorname{tr}(Pf'(A)A)}{\operatorname{tr}(P)} - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} \cdot \frac{\operatorname{tr}(Pf'(A))}{\operatorname{tr}(P)}$$
$$\leq \begin{cases} \frac{1}{2}[f'(M) - f'(m)] \frac{\operatorname{tr}\left(P\left|A - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} 1_H\right|\right)}{\operatorname{tr}(P)} \\ \\ \frac{1}{2}(M - m) \frac{\operatorname{tr}\left(P\left|f'(A) - \frac{\operatorname{tr}(Pf'(A))}{\operatorname{tr}(P)} 1_H\right|\right)}{\operatorname{tr}(P)} \end{cases}$$
$$\leq \begin{cases} \frac{1}{2}[f'(M) - f'(m)] \left[\frac{\operatorname{tr}(PA^2)}{\operatorname{tr}(P)} - \left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)^2\right]^{1/2} \\ \\ \frac{1}{2}(M - m) \left[\frac{\operatorname{tr}\left(P[f'(A)]^2\right)}{\operatorname{tr}(P)} - \left(\frac{\operatorname{tr}(Pf'(A))}{\operatorname{tr}(P)}\right)^2\right]^{1/2} \end{cases}$$
$$\leq \frac{1}{4}[f'(M) - f'(m)](M - m).$$

For some inequalities for convex functions see [8–12, 28, 46]. For inequalities for functions of selfadjoint operators, see [14–23, 39, 41–44] and the books [24, 25, 30].

Motivated by the above results we establish in this paper other trace inequalities for convex functions of selfadjoint operators. Some examples for convex functions of interest are also given.

2. New Reverse Inequalities for Convex Functions

We recall the *gradient inequality* for the convex function $f\colon [m, M] \to \mathbb{R}$, namely

$$f(\varsigma) - f(\tau) \geq \delta_f(\tau)(\varsigma - \tau) \tag{2.1}$$

for any $\varsigma, \tau \in [m, M]$ where $\delta_f(\tau) \in \left[f'_-(\tau), f'_+(\tau)\right]$, (for $\tau = m$ we take $\delta_f(\tau) = f'_+(m)$ and for $\tau = M$ we take $\delta_f(\tau) = f'_-(M)$). Here $f'_+(m)$ and $f'_-(M)$ are the lateral derivatives of the convex function f.

The following result holds.

Theorem 2.1. *Let A be a selfadjoint operator on the Hilbert space H and assume that $\operatorname{Sp}(A) \subseteq [m, M]$ for some scalars m, M with $m < M$. If f is a continuos convex function on $[m, M]$ and $P \in \mathcal{B}_1(H) \setminus \{0\}$, $P \geq 0$ is such that $\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} \in (m, M)$ then we have*

$$
\begin{aligned}
(2.2)\qquad 0 &\leq \frac{\operatorname{tr}(Pf(A))}{\operatorname{tr}(P)} - f\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right) \\
&\leq \frac{\left(M - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} - m\right)}{M-m} \Psi_f\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}; m, M\right) \\
&\leq \frac{\left(M - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} - m\right)}{M-m} \sup_{t\in(m,M)} \Psi_f(t; m, M) \\
&\leq \left(M - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} - m\right) \frac{f'_-(M) - f'_+(m)}{M-m} \\
&\leq \frac{1}{4}(M-m)\left[f'_-(M) - f'_+(m)\right],
\end{aligned}
$$

where $\Psi_f(\cdot; m, M) : (m, M) \to \mathbb{R}$ is defined by

$$
\Psi_f(t; m, M) = \frac{f(M) - f(t)}{M-t} - \frac{f(t) - f(m)}{t-m}.
$$

We also have

$$
\begin{aligned}
(2.3)\qquad 0 &\leq \frac{\operatorname{tr}(Pf(A))}{\operatorname{tr}(P)} - f\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right) \\
&\leq \frac{\left(M - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} - m\right)}{M-m} \Psi_f\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}; m, M\right) \\
&\leq \frac{1}{4}(M-m)\Psi_f\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}; m, M\right) \\
&\leq \frac{1}{4}(M-m) \sup_{t\in(m,M)} \Psi_f(t; m, M) \\
&\leq \frac{1}{4}(M-m)\left[f'_-(M) - f'_+(m)\right]
\end{aligned}
$$

for any $P \in \mathcal{B}_1(H) \setminus \{0\}$, $P \geq 0$ such that $\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} \in (m, M)$.

Proof. Since f is convex, then we have

$$f(t)=f\left(\frac{m(M-t)+M(t-m)}{M-m}\right)\leq\frac{(M-t)f(m)+(t-m)f(M)}{M-m}$$

for any $t\in[m,M]$.

This scalar inequality implies, by utilizing the spectral representation of continuous functions of selfadjoint operators, the following inequality

$$f(A)\leq\frac{f(m)(M1_M-A)+f(M)(A-m1_H)}{M-m} \tag{2.4}$$

in the operator order of $\mathcal{B}(H)$.

Utilising the properties of the trace and the inequality (2.4), we have

$$\begin{aligned}
&\frac{\operatorname{tr}(Pf(A))}{\operatorname{tr}(P)}-f\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right) \\
&=\frac{\operatorname{tr}(Pf(A))}{\operatorname{tr}(P)}-f\left(\frac{\operatorname{tr}\left(P\frac{m(M1_H-A)+M(A-1_Hm)}{M-m}\right)}{\operatorname{tr}(P)}\right) \\
&\leq\frac{\operatorname{tr}\left(P\frac{f(m)(M1_M-A)+f(M)(A-m1_H)}{M-m}\right)}{\operatorname{tr}(P)} \\
&\quad-f\left(\frac{\operatorname{tr}\left(P\frac{m(M1_H-A)+M(A-1_Hm)}{M-m}\right)}{\operatorname{tr}(P)}\right) \\
&=\frac{\left(M-\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)f(m)+\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}-m\right)f(M)}{M-m} \\
&\quad-f\left(\frac{\left(M-\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)m+\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}-m\right)M}{M-m}\right) \\
&=:B(f,P,A,m,M)
\end{aligned} \tag{2.5}$$

for any $P\in\mathcal{B}_1(H)\setminus\{0\}$, $P\geq0$.

By denoting

$$\Delta_f(t;m,M):=\frac{(t-m)f(M)+(M-t)f(m)}{M-m}-f(t),\quad t\in[m,M],$$

we have

$$
\begin{aligned}
(2.6)\qquad & \Delta_f(t;m,M) \\
&= \frac{(t-m)f(M)+(M-t)f(m)-(M-m)f(t)}{M-m} \\
&= \frac{(t-m)f(M)+(M-t)f(m)-(M-t+t-m)f(t)}{M-m} \\
&= \frac{(t-m)[f(M)-f(t)]-(M-t)[f(t)-f(m)]}{M-m} \\
&= \frac{(M-t)(t-m)}{M-m}\Psi_f(t;m,M)
\end{aligned}
$$

for any $t\in(m,M)$. Therefore

$$
\begin{aligned}
(2.7)\quad & B(f,P,A,m,M) \\
&= \frac{\left(M-\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}-m\right)}{M-m}\Psi_f\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)};m,M\right),
\end{aligned}
$$

provided that $\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\in(m,M)$. If $\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\in(m,M)$, then

$$
\begin{aligned}
(2.8)\quad & \Psi_f\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)};m,M\right) \\
&\le \sup_{t\in(m,M)}\Psi_f(t;m,M) \\
&= \sup_{t\in(m,M)}\left[\frac{f(M)-f(t)}{M-t}-\frac{f(t)-f(m)}{t-m}\right] \\
&\le \sup_{t\in(m,M)}\left[\frac{f(M)-f(t)}{M-t}\right]+\sup_{t\in(m,M)}\left[-\frac{f(t)-f(m)}{t-m}\right] \\
&= \sup_{t\in(m,M)}\left[\frac{f(M)-f(t)}{M-t}\right]-\inf_{t\in(m,M)}\left[\frac{f(t)-f(m)}{t-m}\right] \\
&= f'_-(M)-f'_+(m),
\end{aligned}
$$

which by (2.5) and (2.7) produces the second, third and fourth inequalities in (2.2). Since, obviously

$$\frac{1}{M-m}\left(M-\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}-m\right)\leq\frac{1}{4}(M-m),$$

then the last part of (2.2) also holds.

The second part of the theorem is clear and the details are omitted. □

The following result also holds.

THEOREM 2.2. *Let A be a selfadjoint operator on the Hilbert space H and assume that $\operatorname{Sp}(A)\subseteq[m,M]$ for some scalars m, M with $m<M$. If f is a continuos convex function on $[m,M]$ then for all $P\in\mathcal{B}_1(H)\setminus\{0\}$, $P\geq 0$ we have that $\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\in[m,M]$ and*

$$\begin{aligned}0&\leq\frac{\operatorname{tr}(Pf(A))}{\operatorname{tr}(P)}-f\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)\\&\leq 2\max\left\{\frac{M-\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}}{M-m},\frac{\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}-m}{M-m}\right\}\\&\quad\times\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right]\\&\leq 2\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right].\end{aligned}\tag{2.9}$$

PROOF. Since $m1_H\leq A\leq M1_H$, it follows that $m\operatorname{tr}(P)\leq\operatorname{tr}(PA)\leq M\operatorname{tr}(P)$ for any $P\in\mathcal{B}_1(H)\setminus\{0\}$, $P\geq 0$, which shows that $\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\in[m,M]$. Further on, we recall the following result (see for instance [11]) that provides a refinement and a reverse for the weighted Jensen's discrete inequality

$$\begin{aligned}&n\min_{i\in\{1,\dots,n\}}\{p_i\}\left[\frac{1}{n}\sum_{i=1}^n f(x_i)-f\left(\frac{1}{n}\sum_{i=1}^n x_i\right)\right]\\&\leq\frac{1}{P_n}\sum_{i=1}^n p_i f(x_i)-f\left(\frac{1}{P_n}\sum_{i=1}^n p_i x_i\right)\\&\leq n\max_{i\in\{1,\dots,n\}}\{p_i\}\left[\frac{1}{n}\sum_{i=1}^n f(x_i)-f\left(\frac{1}{n}\sum_{i=1}^n x_i\right)\right],\end{aligned}\tag{2.10}$$

where $f\colon C \to \mathbb{R}$ is a convex function defined on the convex subset C of the linear space X, $\{x_i\}_{i\in\{1,\ldots,n\}} \subset C$ are vectors and $\{p_i\}_{i\in\{1,\ldots,n\}}$ are nonnegative numbers with $P_n := \sum_{i=1}^{n} p_i > 0$.

For $n = 2$ we deduce from (2.10) that

$$\begin{aligned}
(2.11) \qquad 2\min\{t, 1-t\} &\left[\frac{f(x)+f(y)}{2} - f\left(\frac{x+y}{2}\right)\right] \\
&\le tf(x) + (1-t)f(y) - f(tx + (1-t)y) \\
&\le 2\max\{t, 1-t\}\left[\frac{f(x)+f(y)}{2} - f\left(\frac{x+y}{2}\right)\right]
\end{aligned}$$

for any $x, y \in C$ and $t \in [0,1]$. If we use the second inequality in (2.11) for the convex function $f\colon I \to \mathbb{R}$ where m, $M \in \mathbb{R}$, $m < M$ with $[m, M] = I$, we have for $x = m$, $y = M$ and $t = \frac{M - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}}{M-m}$ that

$$\begin{aligned}
B(f, P, A, m, M) &= \frac{\left(M - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right) f(m) + \left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} - m\right) f(M)}{M-m} \\
&\quad - f\left(\frac{m\left(M - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right) + M\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} - m\right)}{M-m}\right) \\
&\le 2\max\left\{\frac{M - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}}{M-m}, \frac{\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} - m}{M-m}\right\} \\
&\quad \times \left[\frac{f(m)+f(M)}{2} - f\left(\frac{m+M}{2}\right)\right].
\end{aligned}$$

Making use of (2.5) we deduce the first inequality in (2.9).

Since

$$\max\left\{\frac{M - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}}{M-m}, \frac{\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} - m}{M-m}\right\} \le 1,$$

the last part of (2.9) is also proved. $\square$

3. Some Examples

For $p>1$ and $0<m<M<\infty$ consider the convex function $f(t)=t^p$ defined on $[m,M]$. Then $\Psi_p(\cdot;m,M):(m,M)\to\mathbb{R}$ is defined by

$$\begin{aligned}\Psi_p(t;m,M) &= \frac{M^p-t^p}{M-t}-\frac{t^p-m^p}{t-m}\\ &= \frac{t(M^p-m^p)-t^p(M-m)-mM\left(M^{p-1}-m^{p-1}\right)}{(M-t)(t-m)}.\end{aligned}$$

Let A be a nonnegative selfadjoint operator on the Hilbert space H and assume that $\operatorname{Sp}(A)\subseteq[m,M]$ for some scalars m, M with $0\le m<M$. If $P\in\mathcal{B}_1(H)\setminus\{0\}$, $P\ge 0$ such that $\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\in(m,M)$, then we have from (2.2) that

$$\begin{aligned}0 &\le \frac{\operatorname{tr}(PA^p)}{\operatorname{tr}(P)}-\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)^p\\ &\le \frac{\left(M-\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}-m\right)}{M-m}\Psi_p\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)};m,M\right)\\ &\le \frac{\left(M-\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}-m\right)}{M-m}\sup_{t\in(m,M)}\Psi_p(t;m,M)\\ &\le p\left(M-\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}-m\right)\frac{M^{p-1}-m^{p-1}}{M-m}\\ &\le \frac{1}{4}p(M-m)\left(M^{p-1}-m^{p-1}\right)\end{aligned}\tag{3.1}$$

and from (2.3) that

$$\begin{aligned}0 &\le \frac{\operatorname{tr}(PA^p)}{\operatorname{tr}(P)}-\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)^p\\ &\le \frac{\left(M-\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}-m\right)}{M-m}\Psi_p\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)};m,M\right)\\ &\le \frac{1}{4}(M-m)\Psi_p\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)};m,M\right)\\ &\le \frac{1}{4}(M-m)\sup_{t\in(m,M)}\Psi_p(t;m,M)\le\frac{1}{4}p(M-m)\left(M^{p-1}-m^{p-1}\right).\end{aligned}\tag{3.2}$$

For $p=2$, we have

$$\Psi_2(t;m,M)=\frac{M^2-t^2}{M-t}-\frac{t^2-m^2}{t-m}=M-m$$

and by (3.1) we get

$$\begin{aligned}(3.3)\qquad 0&\le\frac{\operatorname{tr}\left(PA^2\right)}{\operatorname{tr}(P)}-\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)^2\le\left(M-\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}-m\right)\\&\le\frac{1}{4}(M-m)^2\end{aligned}$$

for any $P\in\mathcal{B}_1(H)\setminus\{0\}$, $P\ge 0$. Making use of the inequality (2.9) we have

$$\begin{aligned}(3.4)\qquad 0&\le\frac{\operatorname{tr}(PA^p)}{\operatorname{tr}(P)}-\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)^p\\&\le 2\max\left\{\frac{M-\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}}{M-m},\frac{\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}-m}{M-m}\right\}\left[\frac{m^p+M^p}{2}-\left(\frac{m+M}{2}\right)^p\right]\\&\le 2\left[\frac{m^p+M^p}{2}-\left(\frac{m+M}{2}\right)^p\right]\end{aligned}$$

for any positive operator A with $\operatorname{Sp}(A)\subseteq[m,M]$ and for any $P\in\mathcal{B}_1(H)\setminus\{0\}$, $P\ge 0$.

In particular, for $p=2$ we get

$$\begin{aligned}(3.5)\qquad 0&\le\frac{\operatorname{tr}\left(PA^2\right)}{\operatorname{tr}(P)}-\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)^2\\&\le\frac{1}{2}(M-m)\max\left\{M-\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)},\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}-m\right\}\\&\le\frac{1}{2}(M-m)^2.\end{aligned}$$

Since

$$\max\left\{M-\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)},\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}-m\right\}=\frac{1}{2}(M-m)+\left|\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}-\frac{1}{2}(m+M)\right|,$$

then the second inequality in (3.5) is not as good as the second inequality in (3.3).

For $p=-1$ and $0<m<M<\infty$ consider the convex function $f(t)=t^{-1}$ defined on $[m,M]$. Then $\Psi_{-1}(\cdot;m,M):\ (m,M)\to\mathbb{R}$ is defined by

$$\Psi_{-1}(t;m,M)=\frac{M^{-1}-t^{-1}}{M-t}-\frac{t^{-1}-m^{-1}}{t-m}=\frac{M-m}{mMt}.$$

The definition of $\Psi_{-1}(\cdot;m,M)$ can be extended to the closed interval $[m,M]$. We also have that

$$\sup_{t\in(m,M)}\Psi_{-1}(t;m,M)=\frac{M-m}{m^2M}.$$

From the inequality (2.2) we get

$$\begin{aligned}
0&\le\frac{\operatorname{tr}\left(PA^{-1}\right)}{\operatorname{tr}(P)}-\frac{\operatorname{tr}(P)}{\operatorname{tr}(PA)} \\
&\le\frac{\left(M-\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}-m\right)}{mM}\frac{\operatorname{tr}(P)}{\operatorname{tr}(PA)} \\
&\le\frac{1}{m^2M}\left(M-\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}-m\right) \\
&\le\frac{1}{4}\frac{(M-m)^2(M+m)}{m^2M^2},
\end{aligned}\tag{3.6}$$

while from (2.3) we get

$$\begin{aligned}
0&\le\frac{\operatorname{tr}\left(PA^{-1}\right)}{\operatorname{tr}(P)}-\frac{\operatorname{tr}(P)}{\operatorname{tr}(PA)} \\
&\le\frac{\left(M-\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}-m\right)}{mM}\frac{\operatorname{tr}(P)}{\operatorname{tr}(PA)} \\
&\le\frac{1}{4}\frac{(M-m)^2}{mM}\frac{\operatorname{tr}(P)}{\operatorname{tr}(PA)} \\
&\le\frac{1}{4}\frac{(M-m)^2}{m^2M}
\end{aligned}\tag{3.7}$$

for any positive definite operator A with $\operatorname{Sp}(A)\subseteq[m,M]$ and $P\in\mathcal{B}_1(H)\setminus\{0\}$, $P\ge0$. Since $m>0$, then $\operatorname{tr}(PA)\ge m\operatorname{tr}(P)>0$.

From the inequality (2.9) we have

$$0 \le \frac{\operatorname{tr}\left(PA^{-1}\right)}{\operatorname{tr}(P)} - \frac{\operatorname{tr}(P)}{\operatorname{tr}(PA)} \tag{3.8}$$

$$\le \frac{(M-m)^2}{mM(m+M)} \max\left\{\frac{M-\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}}{M-m}, \frac{\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}-m}{M-m}\right\}$$

$$\le \frac{(M-m)^2}{mM(m+M)}$$

for any positive definite operator A with $\operatorname{Sp}(A) \subseteq [m, M]$ and any $P \in \mathcal{B}_1(H) \setminus \{0\}$, $P \ge 0$.

In order to compare the upper bounds provided by (3.7) and (3.8) consider the difference

$$\Delta(m, M) := \frac{1}{4}\frac{(M-m)^2}{m^2 M} - \frac{(M-m)^2}{mM(m+M)}$$

$$= \frac{(M-m)^2}{mM}\left(\frac{1}{4m} - \frac{1}{m+M}\right)$$

$$= \frac{(M-m)^2(M-3m)}{4m^2M(m+M)},$$

where $0 < m < M$.

We observe that if $M < 3m$, then the upper bound provided by (3.7) is better than the bound provided by (3.8). The conclusion is the other way around if $M \ge 3m$.

If we consider the convex function $f(t) = -\ln t$ defined on $[m, M] \subset (0, \infty)$, then $\Psi_{-\ln}(\cdot; m, M) : (m, M) \to \mathbb{R}$ is defined by

$$\Psi_{-\ln}(t; m, M) = \frac{-\ln M + \ln t}{M-t} - \frac{-\ln t + \ln m}{t-m}$$

$$= \frac{(M-m)\ln t - (M-t)\ln m - (t-m)\ln M}{(M-t)(t-m)}$$

$$= \ln\left(\frac{t^{M-m}}{m^{M-t}M^{t-m}}\right)^{\frac{1}{(M-t)(t-m)}}.$$

Utilising the inequality (2.2) we have

$$
\begin{aligned}
(3.9)\qquad 0 &\le \ln\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)-\frac{\operatorname{tr}(P\ln A)}{\operatorname{tr}(P)}\\
&\le \frac{1}{M-m}\ln\left(\frac{\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)^{M-m}}{m^{M-\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}}M^{\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}-m}}\right)\\
&\le \frac{\left(M-\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}-m\right)}{M-m}\sup_{t\in(m,M)}\Psi_{-\ln}(t;m,M)\\
&\le \frac{1}{Mm}\left(M-\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}-m\right)\\
&\le \frac{(M-m)^2}{4mM}
\end{aligned}
$$

for any positive definite operator A with $\operatorname{Sp}(A)\subseteq[m,M]$ and $P\in\mathcal{B}_1(H)\setminus\{0\}$, $P\ge 0$.

From (2.3) we have

$$
\begin{aligned}
(3.10)\qquad 0 &\le \ln\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)-\frac{\operatorname{tr}(P\ln A)}{\operatorname{tr}(P)}\\
&\le \frac{1}{M-m}\ln\left(\frac{\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)^{M-m}}{m^{M-\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}}M^{\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}-m}}\right)\\
&\le \frac{1}{4}\frac{(M-m)}{\left(M-\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}-m\right)}\ln\left(\frac{\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)^{M-m}}{m^{M-\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}}M^{\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}-m}}\right)\\
&\le \frac{1}{4}(M-m)\sup_{t\in(m,M)}\Psi_{-\ln}(t;m,M)\\
&\le \frac{(M-m)^2}{4mM}
\end{aligned}
$$

for any positive definite operator A with $\operatorname{Sp}(A)\subseteq[m,M]$ and $P\in\mathcal{B}_1(H)\setminus\{0\}$, $P\ge 0$.

From the inequality (2.9) we get

$$0 \leq \ln\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right) - \frac{\operatorname{tr}(P\ln A)}{\operatorname{tr}(P)} \tag{3.11}$$
$$\leq \max\left\{\frac{M - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}}{M-m}, \frac{\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} - m}{M-m}\right\} \ln\left(\frac{\left(\frac{m+M}{2}\right)^2}{mM}\right)$$
$$\leq \ln\left(\frac{\left(\frac{m+M}{2}\right)^2}{mM}\right)$$

for any positive definite operator A with $\operatorname{Sp}(A) \subseteq [m, M]$ and $P \in \mathcal{B}_1(H) \setminus \{0\}$, $P \geq 0$.

We observe that, since $\ln x \leq x - 1$ for any $x > 0$, then

$$\ln\left(\frac{\left(\frac{m+M}{2}\right)^2}{mM}\right) \leq \frac{\left(\frac{m+M}{2}\right)^2}{mM} - 1 = \frac{(M-m)^2}{4mM},$$

which shows that the absolute upper bound for

$$\ln\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right) - \frac{\operatorname{tr}(P\ln A)}{\operatorname{tr}(P)}$$

provided by the inequality (3.11) is better than the one provided by (3.10).

4. Reverses of Hölder's Inequality

We have the following result.

THEOREM 4.1. *Assume that p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. Let S be a positive operator that commutes with Q, a positive invertible operator and such that there exists the constants k, $K > 0$ with*

$$k1_H \leq SQ^{1-q} \leq K1_H. \tag{4.1}$$

If S^p, $Q^q \in \mathcal{B}_1(H)$, then we have

$$0 \leq [\operatorname{tr}(S^p)]^{1/p} [\operatorname{tr}(Q^q)]^{1/q} - \operatorname{tr}(SQ) \leq B_p(k, K) \operatorname{tr}(Q^q), \tag{4.2}$$

where

$$B_p(k,K)=\begin{cases}\frac{1}{4^{1/p}}p^{1/p}(K-k)^{1/p}\left(K^{p-1}-k^{p-1}\right)^{1/p}\\ \\ 2^{1/p}\left[\frac{k^p+K^p}{2}-\left(\frac{k+K}{2}\right)^p\right]^{1/p}.\end{cases}\tag{4.3}$$

Proof. If we write the inequality

$$0\le\frac{\operatorname{tr}(PA^p)}{\operatorname{tr}(P)}-\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)^p\le\frac{1}{4}p(M-m)\left(M^{p-1}-m^{p-1}\right)\tag{4.4}$$

for the operators $P=Q^q$ and $A=SQ^{1-q}$ then we get

$$\begin{aligned}0&\le\frac{\operatorname{tr}\left(Q^q\left(SQ^{1-q}\right)^p\right)}{\operatorname{tr}(Q^q)}-\left(\frac{\operatorname{tr}\left(Q^qSQ^{1-q}\right)}{\operatorname{tr}(Q^q)}\right)^p\\&\le\frac{1}{4}p(K-k)\left(K^{p-1}-k^{p-1}\right).\end{aligned}\tag{4.5}$$

Observe that, by the properties of trace we have

$$\operatorname{tr}\left(Q^qSQ^{1-q}\right)=\operatorname{tr}\left(SQ^{1-q}Q^q\right)=\operatorname{tr}(SQ).$$

It is known, see for instance [45, p. 356-358], that if A and B are two *commuting bounded selfadjoint operators* on the complex Hilbert space H, then there exists a bounded selfadjoint operator T on H and two bounded functions φ and ψ such that $A=\varphi(T)$ and $B=\psi(T)$. Moreover, if $\{E_\lambda\}$ is the spectral family over the closed interval $[0,1]$ for the selfadjoint operator T, then $T=\int_{0-}^{1}\lambda dE_\lambda$, where the integral is taken in the Riemann-Stieltjes sense, the functions φ and ψ are summable with respect with $\{E_\lambda\}$ on $[0,1]$ and

$$A=\varphi(T)=\int_{0-}^{1}\varphi(\lambda)\,dE_\lambda\text{ and }B=\psi(T)=\int_{0-}^{1}\psi(\lambda)\,dE_\lambda.$$

Now, if A and B are as above with $\operatorname{Sp}(A),\operatorname{Sp}(B)\subseteq J$ an interval of real numbers, then for any continuous functions f, $g\colon J\to\mathbb{C}$ we have the representations

$$f(A)=\int_{0-}^{1}(f\circ\varphi)(\lambda)\,dE_\lambda\text{ and }g(B)=\int_{0-}^{1}(g\circ\psi)(\lambda)\,dE_\lambda.$$

If we apply the above property to the commuting selfadjoint operators S and Q, then we have two positive functions φ and ψ such that $S=\varphi(T)$

and $Q = \psi(T)$. Moreover, using the integral representation for functions of selfadjoint operators, we have

$$\begin{aligned}
Q^q \left(SQ^{1-q}\right)^p &= [\psi(T)]^q \left(\varphi(T)[\psi(T)]^{1-q}\right)^p \\
&= \int_{0-}^{1} [\psi(\lambda)]^q \left(\varphi(\lambda)[\psi(\lambda)]^{1-q}\right)^p dE_\lambda \\
&= \int_{0-}^{1} [\psi(\lambda)]^q [\varphi(\lambda)]^p [\psi(\lambda)]^{(1-q)p} dE_\lambda \\
&= \int_{0-}^{1} [\varphi(\lambda)]^p [\psi(\lambda)]^{q+p-qp} dE_\lambda = \int_{0-}^{1} [\varphi(\lambda)]^p dE_\lambda = S^p.
\end{aligned}$$

Therefore, the inequality (4.5) is equivalent to

$$0 \le \frac{\operatorname{tr}(S^p)}{\operatorname{tr}(Q^q)} - \left(\frac{\operatorname{tr}(SQ)}{\operatorname{tr}(Q^q)}\right)^p \le \frac{1}{4} p (K-k)\left(K^{p-1} - k^{p-1}\right), \tag{4.6}$$

which is of interest in itself. From this inequality we have

$$\operatorname{tr}(S^p)[\operatorname{tr}(Q^q)]^{p-1} \le (\operatorname{tr}(SQ))^p + \frac{1}{4} p (K-k)\left(K^{p-1} - k^{p-1}\right)[\operatorname{tr}(Q^q)]^p.$$

Taking the power $1/p \in (0,1)$ and using the property that

$$(\alpha+\beta)^r \le \alpha^r + \beta^r, \quad \text{where } \alpha, \beta \ge 0 \text{ and } r \in (0,1),$$

we get

$$\begin{aligned}
&[\operatorname{tr}(S^p)]^{1/p} [\operatorname{tr}(Q^q)]^{(p-1)/p} \\
&\qquad \le \left[(\operatorname{tr}(SQ))^p + \frac{1}{4} p (K-k)\left(K^{p-1} - k^{p-1}\right)[\operatorname{tr}(Q^q)]^p\right]^{1/p} \\
&\qquad \le \operatorname{tr}(SQ) + \frac{1}{4^{1/p}} p^{1/p} (K-k)^{1/p} \left(K^{p-1} - k^{p-1}\right)^{1/p} [\operatorname{tr}(Q^q)],
\end{aligned}$$

i.e.

$$\begin{aligned}
&[\operatorname{tr}(S^p)]^{1/p} [\operatorname{tr}(Q^q)]^{1/q} - \operatorname{tr}(SQ) \\
&\qquad \le \frac{1}{4^{1/p}} p^{1/p} (K-k)^{1/p} \left(K^{p-1} - k^{p-1}\right)^{1/p} [\operatorname{tr}(Q^q)].
\end{aligned}$$

The second part follows from the inequality

$$0 \le \frac{\operatorname{tr}(PA^p)}{\operatorname{tr}(P)} - \left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)^p \le 2\left[\frac{m^p + M^p}{2} - \left(\frac{m+M}{2}\right)^p\right],$$

and the details are omitted. □

Remark 4.2. We observe that under the previous assumptions, from any upper bound for the difference

$$0 \le \frac{\operatorname{tr}(PA^p)}{\operatorname{tr}(P)} - \left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)^p$$

we can deduce in a similar way an upper bound for the Hölder's difference

$$0 \le [\operatorname{tr}(S^p)]^{1/p} [\operatorname{tr}(Q^q)]^{1/q} - \operatorname{tr}(SQ).$$

Also, if the commutativity property of the operators S and Q is dropped, then we can prove that

$$\begin{aligned} 0 &\le \left[\operatorname{tr}\left(Q^q \left(SQ^{1-q}\right)^p\right)\right]^{1/p} [\operatorname{tr}(Q^q)]^{1/q} - \operatorname{tr}(SQ) \\ &\le B_p(k,K) \operatorname{tr}(Q^q) \end{aligned} \tag{4.7}$$

with the same $B_p(k,K)$. However, the noncommutative case of the second inequality in (4.2) is an open question for the author.

The following reverse of Schwarz inequality holds.

Corollary 4.3. *Let S be a positive operator that commutes with Q, a positive invertible operator and such that there exists the constants k, $K > 0$ with*

$$k1_H \le SQ^{-1} \le K1_H. \tag{4.8}$$

If S^2, $Q^2 \in \mathcal{B}_1(H)$, then we have

$$0 \le \left[\operatorname{tr}\left(S^2\right)\right]^{1/2} \left[\operatorname{tr}\left(Q^2\right)\right]^{1/2} - \operatorname{tr}(SQ) \le \frac{\sqrt{2}}{2}(K-k)\operatorname{tr}\left(Q^2\right). \tag{4.9}$$

Remark 4.4. If we take $p = q = 2$ in (4.7) and drop the commutativity assumption, then we get

$$0 \leq \left[\operatorname{tr}\left(QSQ^{-1}S\right)\right]^{1/2}\left[\operatorname{tr}\left(Q^2\right)\right]^{1/2} - \operatorname{tr}(SQ) \leq \frac{\sqrt{2}}{2}(K-k)\operatorname{tr}\left(Q^2\right),$$

provided that (4.8) holds true.

Also, if we use the inequality (3.3), then we have

$$\begin{aligned} 0 &\leq \operatorname{tr}\left(QSQ^{-1}S\right)\operatorname{tr}\left(Q^2\right) - \left[\operatorname{tr}(SQ)\right]^2 \\ &\leq \left(K\operatorname{tr}\left(Q^2\right) - \operatorname{tr}(SQ)\right)\left(\operatorname{tr}(SQ) - k\operatorname{tr}\left(Q^2\right)\right) \\ &\leq \frac{1}{4}(K-k)^2\left[\operatorname{tr}\left(Q^2\right)\right]^2 \end{aligned} \tag{4.10}$$

provided that (4.8) holds true.

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References

[1] Ando T., *Matrix Young inequalities*, Oper. Theory Adv. Appl. **75** (1995), 33–38.

[2] Bellman R., *Some inequalities for positive definite matrices*, in: E.F. Beckenbach (Ed.), *General Inequalities 2*, Proceedings of the 2nd International Conference on General Inequalities, Birkhäuser, Basel, 1980, pp. 89–90.

[3] Belmega E.V., Jungers M., Lasaulce S., *A generalization of a trace inequality for positive definite matrices*, Aust. J. Math. Anal. Appl. **7** (2010), no. 2, Art. 26, 5 pp.

[4] Carlen E.A., *Trace inequalities and quantum entropy: an introductory course*, in: *Entropy and the quantum*, Contemp. Math. 529, Amer. Math. Soc., Providence, RI, 2010, pp. 73–140.

[5] Chang D., *A matrix trace inequality for products of Hermitian matrices*, J. Math. Anal. Appl. **237** (1999), 721–725.

[6] Chen L., Wong C., *Inequalities for singular values and traces*, Linear Algebra Appl. **171** (1992), 109–120.

[7] Coop I.D., *On matrix trace inequalities and related topics for products of Hermitian matrix*, J. Math. Anal. Appl. **188** (1994), 999–1001.

[8] Dragomir S.S., *A converse result for Jensen's discrete inequality via Gruss' inequality and applications in information theory*, An. Univ. Oradea Fasc. Mat. **7** (1999/2000), 178–189.

[9] Dragomir S.S., *On a reverse of Jessen's inequality for isotonic linear functionals*, J. Ineqal. Pure Appl. Math. **2** (2001), No. 3, Art. 36.

[10] Dragomir S.S., *A Grüss type inequality for isotonic linear functionals and applications*, Demonstratio Math. **36** (2003), no. 3, 551–562. Preprint RGMIA Res. Rep. Coll. **5** (2002), Suplement, Art. 12. Available at http://rgmia.org/v5(E).php.

[11] Dragomir S.S., *Bounds for the normalized Jensen functional*, Bull. Austral. Math. Soc. **74** (2006), no. 3, 471–476.

[12] Dragomir S.S., *Bounds for the deviation of a function from the chord generated by its extremities*, Bull. Aust. Math. Soc. **78** (2008), no. 2, 225–248.

[13] Dragomir S.S., *Grüss' type inequalities for functions of selfadjoint operators in Hilbert spaces*. Preprint RGMIA Res. Rep. Coll. **11**(e) (2008), Art. 11. Available at http://rgmia.org/v11(E).php].

[14] Dragomir S.S., *Some inequalities for convex functions of selfadjoint operators in Hilbert spaces*, Filomat **23** (2009), no. 3, 81–92. Preprint RGMIA Res. Rep. Coll. **11**(e) (2008), Art. 10.

[15] Dragomir S.S., *Some Jensen's type inequalities for twice differentiable functions of selfadjoint operators in Hilbert spaces*, Filomat **23** (2009), no. 3, 211–222. Preprint RGMIA Res. Rep. Coll. **11**(e) (2008), Art. 13.

[16] Dragomir S.S., *Some new Grüss' type inequalities for functions of selfadjoint operators in Hilbert spaces*, Sarajevo J. Math. **6(18)** (2010), no. 1, 89–107. Preprint RGMIA Res. Rep. Coll. **11**(e) (2008), Art. 12. Available at http://rgmia.org/v11(E).php.

[17] Dragomir S.S., *New bounds for the Čebyšev functional of two functions of selfadjoint operators in Hilbert spaces*, Filomat **24** (2010), no. 2, 27–39.

[18] Dragomir S.S., *Some Jensen's type inequalities for log-convex functions of selfadjoint operators in Hilbert spaces*, Bull. Malays. Math. Sci. Soc. **34** (2011), no. 3, 445–454. Preprint RGMIA Res. Rep. Coll. **13** (2010), Suplement, Art. 2.

[19] Dragomir S.S., *Some reverses of the Jensen inequality for functions of selfadjoint operators in Hilbert spaces*, J. Ineq. & Appl. (2010), Art. ID 496821. Preprint RGMIA Res. Rep. Coll. **11**(e) (2008), Art. 15. Available at http://rgmia.org/v11(E).php.

[20] Dragomir S.S., *Some Slater's type inequalities for convex functions of selfadjoint operators in Hilbert spaces*, Rev. Un. Mat. Argentina **52**(2011), no. 1, 109–120. Preprint RGMIA Res. Rep. Coll. **11**(e) (2008), Art. 7.

[21] Dragomir S.S., *Hermite-Hadamard's type inequalities for operator convex functions*, Appl. Math. Comp. **218** (2011), 766–772. Preprint RGMIA Res. Rep. Coll. **13** (2010), no. 1, Art. 7.

[22] Dragomir S.S., *Hermite-Hadamard's type inequalities for convex functions of selfadjoint operators in Hilbert spaces*, Preprint RGMIA Res. Rep. Coll. **13** (2010), no. 2, Art 1.

[23] Dragomir S.S., *New Jensen's type inequalities for differentiable log-convex functions of selfadjoint operators in Hilbert spaces*, Sarajevo J. Math. **19** (2011), no. 1, 67–80. Preprint RGMIA Res. Rep. Coll. **13** (2010), Suplement, Art. 2.

[24] Dragomir S.S., *Operator Inequalities of the Jensen, Čebyšev and Grüss Type*, Springer Briefs in Mathematics, Springer, New York, 2012.

[25] Dragomir S.S., *Operator Inequalities of Ostrowski and Trapezoidal Type*, Springer Briefs in Mathematics, Springer, New York, 2012.

[26] Dragomir S.S., *Some trace inequalities for convex functions of selfadjoint operators in Hilbert spaces*. Preprint RGMIA Res. Rep. Coll. **17** (2014), Art. 115. Available at http://rgmia.org/papers/v17/v17a115.pdf.

[27] Dragomir S.S., *Jensen's type trace inequalities for convex functions of selfadjoint operators in Hilbert spaces*. Preprint RGMIA Res. Rep. Coll. **17** (2014), Art. 116. Available at http://rgmia.org/papers/v17/v17a116.pdf.

[28] Dragomir S.S., Ionescu N.M., *Some converse of Jensen's inequality and applications*, Rev. Anal. Numér. Théor. Approx. **23** (1994), no. 1, 71–78.

[29] Furuichi S., Lin M., *Refinements of the trace inequality of Belmega, Lasaulce and Debbah*, Aust. J. Math. Anal. Appl. **7** (2010), no. 2, Art. 23, 4 pp.

[30] Furuta T., Mićić Hot J., Pečarić J., Seo Y., *Mond-Pečarić Method in Operator Inequalities. Inequalities for Bounded Selfadjoint Operators on a Hilbert Space*, Element, Zagreb, 2005.

[31] Helmberg G., *Introduction to Spectral Theory in Hilbert Space*, John Wiley, New York, 1969.

[32] Lee H.D., *On some matrix inequalities*, Korean J. Math. **16** (2008), no. 4, 565–571.

[33] Liu L., *A trace class operator inequality*, J. Math. Anal. Appl. **328** (2007), 1484–1486.

[34] Manjegani S., *Hölder and Young inequalities for the trace of operators*, Positivity **11** (2007), 239–250.

[35] Neudecker H., *A matrix trace inequality*, J. Math. Anal. Appl. **166** (1992), 302–303.

[36] Shebrawi K., Albadawi H., *Operator norm inequalities of Minkowski type*, J. Inequal. Pure Appl. Math. **9** (2008), Art. 26, 1–10.

[37] Shebrawi K., Albadawi H., *Trace inequalities for matrices*, Bull. Aust. Math. Soc. **87** (2013), 139–148.

[38] Simon B., *Trace Ideals and Their Applications*, Cambridge University Press, Cambridge, 1979.

[39] Matković A., Pečarić J., Perić I., *A variant of Jensen's inequality of Mercer's type for operators with applications*, Linear Algebra Appl. **418** (2006), no. 2-3, 551–564.

[40] McCarthy C.A., c_p, Israel J. Math. **5** (1967), 249–271.

[41] Mićić J., Seo Y., Takahasi S.-E., Tominaga M., *Inequalities of Furuta and Mond-Pečarić*, Math. Ineq. Appl. **2** (1999), 83–111.

[42] Mond B., Pečarić J., *Convex inequalities in Hilbert space*, Houston J. Math. **19** (1993), 405–420.

[43] Mond B., Pečarić J., *On some operator inequalities*, Indian J. Math. **35** (1993), 221–232.

[44] B. Mond and J. Pečarić, *Classical inequalities for matrix functions*, Utilitas Math. **46** (1994), 155–166.

[45] Riesz F., Sz-Nagy B., *Functional Analysis*, Dover Publications, New York, 1990.

[46] Simić S., *On a global upper bound for Jensen's inequality*, J. Math. Anal. Appl. **343** (2008), 414–419.

[47] Ulukök Z., Türkmen R., *On some matrix trace inequalities*, J. Inequal. Appl. **2010**, Art. ID 201486, 8 pp.

[48] Yang X., *A matrix trace inequality*, J. Math. Anal. Appl. **250** (2000), 372–374.

[49] Yang X.M., Yang X.Q., Teo K.L., *A matrix trace inequality*, J. Math. Anal. Appl. **263** (2001), 327–331.

[50] Yang Y., *A matrix trace inequality*, J. Math. Anal. Appl. **133** (1988), 573–574.

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