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Repdigits as Euler functions of Lucas numbers

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Abstract

We prove some results about the structure of all Lucas numbers whose Euler function is a repdigit in base 10. For example, we show that if L_n is such a Lucas number, then $n < 10^{111}$ is of the form p or p^2, where $p^3 \mid 10^{p-1} - 1$.

1 Introduction

Let $\phi(m)$ be the Euler function of the positive integer m. Let $\{F_n\}_{n\geq 0}$ and $\{L_n\}_{n\geq 0}$ be the sequence of Fibonacci and Lucas numbers given by $F_0 = 0,\ F_1 = 1$ and $L_0 = 2,\ L_1 = 1$ and recurrences

$$F_{n+2} = F_{n+1} + F_n \quad \text{and} \quad L_{n+2} = L_{n+1} + L_n \quad \text{for all} \quad n \geq 0.$$

Various Diophantine equations involving the Euler function of members of Fibonacci and Lucas numbers were investigated (see [6], [8], [9]). In [10], it was shown that $n = 11$ is the largest solution of the Diophantine equation

$$\phi(F_n) = d\left(\frac{10^m - 1}{9}\right) \qquad d \in \{1, \ldots, 9\}. \tag{1}$$

Numbers as the ones appearing in the right–hand side of equation (1) are called *rep-digits* in base 10, since their base 10 representation is the string $\underbrace{dd\cdots d}_{m \text{ times}}$. Here, we look at Diophantine equation (1) with F_n replaced by L_n:

$$\phi(L_n) = d\left(\frac{10^m - 1}{9}\right) \qquad d \in \{1, \ldots, 9\}. \tag{2}$$

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Theorem 1. *Assume that $n > 6$ is such that equation (2) holds with some d. Then:*

- $d = 8$;
- *m is even;*
- $n = p$ *or* p^2, *where* $p^3 \mid 10^{p-1} - 1$.
- $10^9 < p < 10^{111}$.

2 Preliminaries

We will use the property that $L_u \mid L_v$ whenever $u \mid v$ and v/u is odd. One important property that we will use over and over again is the existence of the primitive divisors for the sequence $\{L_n\}_{n\geq 0}$. To formulate it, a primitive divisor of L_n is a prime factor p of L_n which does not divide L_m for any $1 \leq m < n$.

Lemma 2.1 (Carmichael [5])**.** *L_n has a primitive divisor for all $n \neq 6$, while $L_6 = 2 \times 3^2$, and $2 \mid L_3,\ 3 \mid L_2$.*

A primitive prime factor p of L_n has the property that $p \equiv \left(\frac{p}{5}\right) \pmod{n}$. Here and in what follows, for an integer a and an odd prime p we use $\left(\frac{a}{p}\right)$ for the Legendre symbol of a with respect to p. In particular, if p is primitive for L_n, then $p \equiv 1 \pmod{n}$ if $p \equiv 1, 4 \pmod{5}$, and $p \equiv -1 \pmod{n}$ if $p \equiv 2, 3 \pmod{5}$.

Finally, we will use the fact that there are no perfect powers other than 1, 4 in the Lucas sequence $\{L_n\}_{n\geq 0}$. More precisely, we have the following result.

Lemma 2.2 (Bugeaud, Luca, Mignotte and Siksek, [3] and [4])**.** *The equation $L_n = y^k$ with some $k \geq 1$ implies that $n \in \{1, 3\}$. Furthermore, the only solutions of the equation $L_n = q^a y^k$ for some prime $q < 1087$ and integers $a > 0,\ k \geq 2$ have $n \in \{0, 2, 3, 4, 5, 6, 7, 8, 9, 11, 13, 17\}$.*

We will also need the following result about square-classes of members of Lucas sequences due to McDaniel and Ribenboim.

Lemma 2.3 (MacDaniel and Ribenboim [12])**.** *If $L_m L_n = \square$ with $n > m \geq 0$, then $(m, n) = (1, 3),\ (0, 6)$ or $(m, 3m)$ with $3 \nmid m$ odd.*

3 Linear forms in logarithms

Let η be an algebraic number of degree d over $\mathbb{Q}$ with minimal primitive polynomial over the integers

$$f(X) = a_0 \prod_{i=1}^{d} (X - \eta^{(i)}) \in \mathbb{Z}[X],$$

where the leading coefficient a_0 is positive. The logarithmic height of η is given by

$$h(\eta) = \frac{1}{d}\left(\log a_0 + \sum_{i=1}^{d} \log \max\{|\eta^{(i)}|, 1\}\right).$$

Later in the paper we use the following theorem of Matveev [11].

Theorem 2 (Matveev [11]). *Let $\mathbb{K}$ be a number field of degree D over $\mathbb{Q}$ $\eta_1, \ldots, \eta_t$ be positive real numbers of $\mathbb{K}$, and $b_1, \ldots, b_t$ rational integers. Put*

$$\Lambda = \eta_1^{b_1} \cdots \eta_t^{b_t} - 1 \qquad \text{and} \qquad B \geq \max\{|b_1|, \ldots, |b_t|\}.$$

Let $A_i \geq \max\{Dh(\eta_i), |\log \eta_i|, 0.16\}$ be real numbers, for $i = 1, \ldots, t$. Then, assuming that $\Lambda \neq 0$, we have

$$|\Lambda| > \exp(-1.4 \times 30^{t+3} \times t^{4.5} \times D^2 (1 + \log D)(1 + \log B) A_1 \cdots A_t).$$

4 The Baker–Davenport lemma

In 1998, Dujella and Pethő in [7, Lemma 5(a)] gave a version of the reduction method based on a lemma of Baker–Davenport lemma [1]. We next present the following lemma from [2], which is an immediate variation of the result due to Dujella and Pethő from [7], and will be the key tool used to reduce the upper bound on the variable n when we assume that $n \notin \{p, p^2\}$.

Lemma 4.1. *Let M be a positive integer, let p/q be a convergent of the continued fraction of the irrational γ such that $q > 6M$, and let A, B, μ be some real numbers with $A > 0$ and $B > 1$. Let $\epsilon := ||\mu q|| - M||\gamma q||$, where $||\cdot||$ denotes the distance from the nearest integer. If $\epsilon > 0$, then there is no solution to the inequality*

$$0 < u\gamma - v + \mu < AB^{-w},$$

in positive integers u, v and w with

$$u \leq M \quad \text{and} \quad w \geq \frac{\log(Aq/\epsilon)}{\log B}.$$

5 The proof of Theorem 1

5.1 The exponent of 2 in both sides of (2)

Write

$$L_n = 2^{\delta} p_1^{\alpha_1} \cdots p_r^{\alpha_r}, \tag{3}$$

where $\delta \geq 0$, $r \geq 0$, $p_1, \ldots, p_r$ are distinct odd primes and $\alpha_1, \ldots, \alpha_r$ are positive integers. Then

$$\phi(L_n) = 2^{\max\{0,\delta-1\}} p_1^{\alpha_1-1}(p_1-1) p_2^{\alpha_2-1}(p_2-1) \cdots p_r^{\alpha_r-1}(p_r-1). \tag{4}$$

For a nonzero integer m we write $\operatorname{ord}_2(m)$ for the exponent of 2 in the factorization of m. Applying the ord_2 function in both sides of (2) and using (4), we get

$$\begin{aligned} \max\{0,\delta-1\} + \sum_{i=1}^{r} \operatorname{ord}_2(p_i - 1) &= \operatorname{ord}_2(\phi(L_n)) \\ &= \operatorname{ord}_2\left(d\left(\frac{10^m-1}{9}\right)\right) = \operatorname{ord}_2(d). \end{aligned} \tag{5}$$

Note that $\operatorname{ord}_2(d) \in \{0,1,2,3\}$. Note also that $r \leq 3$ and since L_n is never a multiple of 5, we have that

$$\frac{\phi(L_n)}{L_n} \geq \left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{7}\right) > \frac{1}{4}, \tag{6}$$

so $\phi(L_n) > L_n/4$. This shows that if $n \geq 8$ satisfies equation (2), then $\phi(L_n) > L_8/4 > 10$, so $m \geq 2$.

We will also use in the later stages of the paper the Binet formula

$$L_n = \alpha^n + \beta^n \qquad (n \geq 0), \tag{7}$$

where $(\alpha, \beta) = ((1+\sqrt{5})/2, (1-\sqrt{5})/2)$. In particular,

$$L_n - 1 = \alpha^n - (1-\beta^n) \leq \alpha^n \qquad \text{for all} \qquad n \geq 0. \tag{8}$$

Furthermore,

$$\alpha^{n-1} \leq L_n < \alpha^{n+1} \qquad \text{for all} \qquad n \geq 1. \tag{9}$$

5.2 The case of the digit $d \notin \{4,8\}$

If $\operatorname{ord}_2(d) = 0$, we get that $d \in \{1,3,5,7,9\}$, $\phi(L_n)$ is odd, so $L_n \in \{1,2\}$, therefore $n = 0,\ 1$. If $\operatorname{ord}_2(d) = 1$, we get that $d \in \{2,6\}$, and from (5) either

$\delta = 2$ and $r = 0$, so $L_n = 4$, therefore $n = 3$, or $\delta \in \{0, 1\}$, $r = 1$ and $p_1 \equiv 3 \pmod 4$. Thus, $L_n = p_1^{\alpha_1}$ or $L_n = 2p_1^{\alpha_1}$. Lemma 2.2 shows that $\alpha_1 = 1$ except for the case when $n = 6$ when $L_6 = 2 \times 3^2$. So, for $n \neq 6$, we get that $L_n = p_1$ or $2p_1$. Let us see that the second case is not possible. Assuming it is, we get $6 \mid n$. Write $n = 2^t \times 3 \times m$, where $t \geq 1$ and m is odd. Clearly, $n \neq 6$.

If $m > 1$, then $L_{2^t 3m}$ has a primitive divisor which does not divide the number $L_{2^t 3}$. Hence, $L_n = 2p_1$ is not possible in this case. However, if $m = 1$ then $t > 1$, and both L_{2^t} and $L_{2^t 3}$ have primitive divisors, so the equation $L_n = 2p_1$ is not possible in this case either. So, the only possible case is $L_n = p_1$. Thus, we get

$$\phi(L_n) = L_n - 1 = d\left(\frac{10^m - 1}{9}\right) \quad \text{and} \quad d \in \{2, 6\},$$

so

$$L_n = d\left(\frac{10^m - 1}{9}\right) + 1 \quad \text{and} \quad d \in \{2, 6\}.$$

When $d = 2$, we get that $L_n \equiv 3 \pmod 5$. The period of the Lucas sequence $\{L_n\}_{n \geq 0}$ modulo 5 is 4. Furthermore, from $L_n \equiv 3 \pmod 5$, we get that $n \equiv 2 \pmod 4$. Thus, $n = 2(2k + 1)$ for some $k \geq 0$. However, this is not possible for $k \geq 1$, since for $k = 1$, we get that $n = 6$ and $L_6 = 2 \times 3^2$, while for $k > 1$, we have that L_n is divisible by both the primes 3 and at least another prime, namely a primitive prime factor of L_n, so $L_n = p_1$ is not possible. Thus, $k = 0$, so $n = 2$.

When $d = 6$, we get that $L_n \equiv 2 \pmod 5$. This shows that $4 \mid n$. Write $n = 2^t(2k + 1)$ for some $t \geq 2$ and $k \geq 0$. As before, if $k \geq 1$, then L_n cannot be a prime since either $k = 1$, so $3 \mid n$, and then $L_n > 2$ is even, or $k \geq 2$, and then L_n is divisible by at least two primes, namely the primitive prime factors of L_{2^t} and of L_n. Thus, $n = 2^t$. Assuming $m \geq 2$, and reducing both sides of the above formula

$$L_{2^{t-1}}^2 - 2 = L_{2^t} = 6\left(\frac{10^m - 1}{9}\right) + 1$$

modulo 8, we get $7 \equiv -5 \pmod 8$, which is not possible. This shows that $m = 1$, so $t = 2$, therefore $n = 4$.

To summarize, we have proved the following result.

Lemma 5.1. *Equation* (2) *has no solutions with* $n > 6$ *if* $d \notin \{4, 8\}$.

5.3 The case of L_n even

Next we treat the case $\delta > 0$. It is well-known and easy to see by looking at the period of $\{L_n\}_{n \geq 0}$ modulo 8 that $8 \nmid L_n$ for any n. Hence, we only need to deal with the cases $\delta = 1$ or 2.

If $\delta = 2$, then $3 \mid n$ and n is odd. Furthermore, relation (5) shows that $r \leq 2$. Assume first that $n = 3^t$. We check that $t = 2, 3$ are not convenient. For $t \geq 4$, we have that L_9, L_{27} and L_{81} are divisors of L_n and all have odd primitive divisors which are prime factors of L_n, contradicting the fact that $r \leq 2$. Assume now that n is a multiple of some prime $p \geq 5$. Then L_p and L_{3p} already have primitive prime factors, so $n = 3p$, for if not, then $n > 3p$, and L_n would have (at least) one additional prime factor, namely a primitive prime factor of L_n. Thus, $n = 3p$. Write

$$L_n = L_{3p} = L_p(L_p^2 + 3).$$

The two factors above are coprime, so, up to relabeling the prime factors of L_n, we may assume that $L_p = p_1^{\alpha_1}$ and $L_p^2 + 3 = 4p_2^{\alpha_2}$. Lemma 2.2 shows that $\alpha_1 = 1$. Further, since p is odd, we get that $L_p \equiv 1, 4 \pmod 5$, therefore the second relation above implies that $p_2^{\alpha_2} \equiv 1 \pmod 5$. If α_2 is odd, we then get that $p_2 \equiv 1 \pmod 5$. This leads to $5 \mid (p_2 - 1) \mid \phi(L_n) = d(10^m - 1)/9$ with $d \in \{4, 8\}$, which is a contradiction. Thus, α_2 is even, showing that

$$L_p^2 + 3 = \square,$$

which is impossible.

If $\delta = 1$, then $6 \mid n$. Assume first that $p \mid n$ for some prime $p > 3$. Write $n = 2^t \times 3 \times m$. If $t \geq 2$, then $r \geq 4$, since L_n is then a multiple of a primitive prime factor of L_{2^t}, a primitive prime factor of $L_{2^t 3}$, a primitive prime factor of $L_{2^t p}$ and a primitive prime factor of $L_{2^t 3p}$. So, $t = 1$. Then L_n is a multiple of 3 and of the primitive prime factors of L_{2p} and L_{6p}, showing that $n = 6p$, for if not, then $n > 6p$ and L_n would have (at least) an additional prime factor, namely a primitive prime factor of L_n. Thus, with $n = 6p$, we may write

$$L_n = L_{6p} = L_{2p}(L_{2p}^2 - 3).$$

Further, it is easy to see that up to relabeling the prime factors of L_n, we may assume that $p_1 = 3$, $\alpha_1 = 2$, $L_{2p} = 3p_2^{\alpha_2}$ and $L_{2p}^2 - 3 = 6p_3^{\alpha_3}$. Furthermore, since $r = 3$, relation (5) tells us that $p_i \equiv 3 \pmod 4$ for $i = 2, 3$. Reducing equation

$$L_p^2 + 2 = L_{2p} = 3p_2^{\alpha_2}$$

modulo 4 we get $3 \equiv 3^{\alpha_2+1} \pmod 4$, so α_2 is even. We thus get $L_{2p} = 3\square$, an equation which has no solutions by Lemma 2.2.

So, it remains to assume that $n = 2^t \times 3^s$.

Assume $s \geq 2$. If also $t \geq 2$, then L_n is divisible by the primitive prime factors of L_{2^t}, $L_{2^t 3}$ and $L_{2^t 9}$. This shows that $n = 2^t \times 9$ and we have

$$L_n = L_{2^t 9} = L_{2^t}(L_{2^t}^2 - 3)(L_{2^t 3}^2 - 3).$$

Up to relabeling the prime factors of L_n, we get $L_{2^t} = p_1^{\alpha_1}$, $L_{2^t}^2 - 3 = 2p_2^{\alpha_2}$, $L_{2^t3}^2 - 3 = p_3^{\alpha_3}$ and $p_i \equiv 3 \pmod 4$ for $i = 1, 2, 3$. Reducing the last relation modulo 4, we get $1 \equiv 3^{\alpha_3} \pmod 4$, so α_3 is even. We thus get $L_{2^t3}^2 - 3 = \square$, and this is false. Thus, $t = 1$. By the existence of primitive divisors Lemma 2.1, $s \in \{2, 3\}$, so $n \in \{18, 54\}$ and none leads to a solution.

Assume next that $s = 1$. Then $n = 2^t \times 3$ and $t \geq 2$. We write

$$L_n = L_{2^t3} = L_{2^t}(L_{2^t}^2 - 3).$$

Assume first that there exist i such that $p_i \equiv 1 \pmod 4$. Then $r \leq 2$ by (5). It then follows that in fact $r = 2$ and up to relabeling the primes we have $L_{2^t} = p_1^{\alpha_1}$ and $L_{2^t}^2 - 3 = 2p_2^{\alpha_2}$. Since $L_{2^t} = L_{2^{t-1}}^2 - 2$, we get that $L_{2^{t-1}}^2 - 2 = p_1^{\alpha_1}$, which reduced modulo 4 gives $3 \equiv p_1^{\alpha_1} \pmod 4$, therefore $p_1 \equiv 3 \pmod 4$. As for the second relation, we get $(L_{2^t}^2 - 3)/2 = p_2^{\alpha_2}$, which reduced modulo 4 also gives $3 \equiv p_2^{\alpha_2} \pmod 4$, so also $p_2 \equiv 3 \pmod 4$. But this contradicts the fact that $p_i \equiv 1 \pmod 4$ for some $i \in \{1, \ldots, r\}$. Thus, $p_i \equiv 3 \pmod 4$ for all $i \in \{1, \ldots, r\}$. Reducing relation

$$L_{2^t3}^2 - 5F_{2^t3}^2 = 4$$

modulo p_i, we get that $\left(\frac{-5}{p_i}\right) = -1$, and since $p_i \equiv 3 \pmod 4$, we get that $\left(\frac{5}{p_i}\right) = -1$ for $i \in \{1, \ldots, r\}$. Since p_i are also primitive prime factors for L_{2^t} and/or L_{2^t3}, respectively, we get that $p_i \equiv -1 \pmod{2^t}$.

Suppose next that $r = 2$. We then get that $d = 4$,

$$L_{2^{t-1}}^2 - 2 = L_{2^t} = p_1^{\alpha_1} \qquad \text{and} \qquad L_{2^t}^2 - 3 = 2p_2^{\alpha_2}.$$

Reducing the above relations modulo 8, we get that α_1, α_2 are odd. Thus,

$$\begin{aligned} 4\left(\frac{10^m - 1}{9}\right) &= \phi(L_n) = p_1^{\alpha_1 - 1}(p_1 - 1)p_2^{\alpha_2 - 1}(p_2 - 1) \\ &\equiv (-1)^{\alpha_1 - 1}(-2)(-1)^{\alpha_2 - 1}(-2) \pmod{2^t} \equiv 4 \pmod{2^t}, \end{aligned}$$

giving

$$\frac{10^m - 1}{9} \equiv 1 \pmod{2^{t-2}} \qquad \text{therefore} \qquad 10^m \equiv 10 \pmod{2^{t-2}},$$

so $t \leq 3$ for $m \geq 2$. Thus, $n \in \{12, 24\}$, and none of these values leads to a solution of equation (2).

Assume next that $r = 3$. We then get that $d = 8$ and either

$$L_{2^{t-1}}^2 - 2 = L_{2^t} = p_1^{\alpha_1}p_2^{\alpha_2} \qquad \text{and} \qquad L_{2^t}^2 - 3 = 2p_3^{\alpha_3},$$

or

$$L_{2^{t-1}}^2 - 2 = L_{2^t} = p_1^{\alpha_1} \qquad \text{and} \qquad L_{2^t}^2 - 3 = 2p_2^{\alpha_2} p_3^{\alpha_3}.$$

Reducing the above relations modulo 8 as we did before, we get that exactly one of $\alpha_1, \alpha_2, \alpha_3$ is even and the other two are odd. Then

$$\begin{aligned} & 8\left(\frac{10^m - 1}{9}\right) = \phi(L_n) = p_1^{\alpha_1 - 1}(p_1 - 1)p_2^{\alpha_2 - 1}(p_2 - 1)p_3^{\alpha_3 - 1}(p_3 - 1) \\ \equiv \quad & (-1)^{\alpha_1 + \alpha_2 + \alpha_3 - 3}(-2)^3 \pmod{2^t} \equiv 8 \pmod{2^t} \end{aligned}$$

giving

$$\frac{10^m - 1}{9} \equiv 1 \pmod{2^{\max\{0, t-3\}}} \quad \text{therefore} \quad 10^m \equiv 10 \pmod{2^{\max\{0, t-3\}}},$$

which implies that $t \leq 4$ for $m \geq 2$. The only new possibility is $n = 48$, which does not fulfill (2).

So, we proved the following result.

Lemma 5.2. *There is no $n > 6$ with L_n even such that relation* (2) *holds.*

5.4 The case of n even

Next we look at solutions of (2) with n even. Write $n = 2^t m$, where $t \geq 1$, m is odd and coprime to 3.

Assume first that there exists i such that $p_i \equiv 1 \pmod 4$. Without loss of generality we assume that $p_1 \equiv 1 \pmod 4$. It then follows from (5) that $r \leq 2$, and that $r = 1$ if $d = 4$. So, if $d = 4$, then $r = 1$, $L_n = p_1^{\alpha_1}$, and by Lemma 2.2, we get that $\alpha_1 = 1$. In this case, by the existence of primitive divisors Lemma 2.1, we get that $m = 1$, otherwise L_n would be divisible both by a primitive prime factor of L_{2^t} as well as by a primitive prime factor of L_n. Hence, $L_{2^t} = p_1$, so

$$L_{2^t} - 1 = \phi(L_{2^t}) = 4\left(\frac{10^m - 1}{9}\right), \quad \text{therefore} \quad L_{2^t} \equiv 5 \pmod{10}.$$

Thus, $5 \mid L_n$ and this is not possible for any n. Suppose now that $d = 8$. If $t \geq 2$, then

$$L_{n/2}^2 - 2 = L_n$$

and reducing the above relation modulo p_1, we get that $\left(\frac{2}{p_1}\right) = 1$. Since $p_1 \equiv 1 \pmod 4$, we read that $p_1 \equiv 1 \pmod 8$. Relation (5) shows that $r = 1$

so $L_n = p_1^{\alpha_1}$. By Lemma 2.2, we get again that $\alpha_1 = 1$ and by the existence of primitive divisors Lemma 2.1, we get that $m = 1$. Thus,

$$L_{2^t} - 1 = \phi(L_{2^t}) = 8\left(\frac{10^m - 1}{9}\right), \quad \text{therefore} \quad L_{2^t} \equiv 4 \pmod 5,$$

which is impossible for $t \geq 2$, since $L_n \equiv 2 \pmod 5$ whenever n is a multiple of 4. This shows that $t = 1$, so $m > 1$. Let $p \geq 5$ be a prime factor of n. Then L_n is divisible by 3 and by the primitive prime factor of L_{2p}, and since $r \leq 2$, we get that $r = 2$, and $n = 2p$. Thus, $L_n = L_{2p} = 3p_2^{\alpha_2}$, and, by Lemma 2.2, we get that $\alpha_2 = 1$. Reducing the above relation modulo 5, we get that $3 \equiv 3p_2 \pmod 5$, so $p_2 \equiv 1 \pmod 5$, showing that $5 \mid (p_2 - 1) \mid \phi(L_n) = 8(10^m - 1)/9$, which is impossible.

This shows that in fact we have $p_i \equiv 3 \pmod 4$ for $i = 1, \ldots, r$. Reducing relation $L_n^2 - 5F_n^2 = 4$ modulo p_i, we get that $\left(\frac{-5}{p_i}\right) = 1$ for $i = 1, \ldots, r$. Since we already know that $\left(\frac{-1}{p_i}\right) = -1$, we get that $\left(\frac{5}{p_i}\right) = -1$ for all $i = 1, \ldots, r$. Since in fact p_i is always a primitive divisor for $L_{2^t d_i}$ for some divisor d_i of m, we get that $p_i \equiv -1 \pmod{2^t}$. Reducing relation

$$L_n = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$$

modulo 4, we get $3 \equiv 3^{\alpha_1 + \cdots + \alpha_r} \pmod 4$, therefore $\alpha_1 + \cdots + \alpha_r$ is odd. Next, reducing the relation

$$\phi(L_n) = p_1^{\alpha_1 - 1}(p_1 - 1) \cdots p_r^{\alpha_r - 1}(p_r - 1)$$

modulo 2^t, we get

$$d\left(\frac{10^m - 1}{9}\right) = \phi(L_n) \equiv (-1)^{\alpha_1 + \cdots + \alpha_r - r}(-2)^r \pmod{2^t} \equiv -2^r \pmod{2^t}.$$

Since $r \in \{2, 3\}$ and $d = 2^r$, we get that

$$\frac{10^m - 1}{9} \equiv -1 \pmod{2^{\max\{0, t-r\}}}, \quad \text{so} \quad 10^m \equiv 8 \pmod{2^{\max\{0, t-r\}}}.$$

Thus, if $m \geq 4$, then $t \leq 6$. Suppose that $m \geq 4$. Computing L_{2^t} for $t \in \{5, 6\}$, we get that each of them has a prime factor p such that $p \equiv 1 \pmod 5$. Thus, $5 \mid (p - 1) \mid \phi(L_n) = d(10^m - 1)/9$, which is impossible. Hence, $t \in \{1, 2, 3, 4\}$. We get the relations

$$L_{2^t m} = L_{2^t} p_1^{\alpha_1}, \quad \text{or} \quad L_{2^t m} = L_{2^t} p_2^{\alpha_2} p_3^{\alpha_3} \quad \text{and} \quad t \in \{1, 2, 3, 4\}. \tag{10}$$

Assume that the left relation (10) holds for some $t \in \{1,2,3,4\}$. Reducing the left equation (10) modulo 5, we get that $L_{2^t} \equiv L_{2^t} p_1^{\alpha_1} \pmod 5$, therefore $p_1^{\alpha_1} \equiv 1 \pmod 5$. If α_1 is odd, we then get that $p_1 \equiv 1 \pmod 5$; hence, $5 \mid (p_1 - 1) \mid \phi(L_n) = d(10^m - 1)/9$ with $d \in \{4,8\}$, which is impossible. If α_1 is even, we then get that $L_n/L_{2^t} = p_1^{\alpha_1} = \square$, and this is impossible since $n \neq 2^t \times 3$ by Lemma 2.3. Assume now that the right relation (10) holds for some $t \in \{2,3,4\}$. Reducing it modulo 5, we get $L_{2^t} \equiv L_{2^t} p_2^{\alpha_2} p_3^{\alpha_3} \pmod 5$. Hence, $p_2^{\alpha_2} p_3^{\alpha_3} \equiv 1 \pmod 5$. Now

$$\begin{aligned} 8\left(\frac{10^m - 1}{9}\right) &= \phi(L_n) = (L_{2^t} - 1) p_2^{\alpha_2 - 1} p_3^{\alpha_3 - 1} (p_2 - 1)(p_3 - 1) \\ &\equiv \left(\frac{p_2 - 1}{p_2}\right)\left(\frac{p_3 - 1}{p_3}\right) \pmod 5, \end{aligned}$$

so

$$\left(\frac{p_2 - 1}{p_2}\right)\left(\frac{p_3 - 1}{p_3}\right) \equiv 3 \pmod 5.$$

The above relation shows that p_2 and p_3 are distinct modulo 5, because otherwise the left–hand side above is a quadratic residue modulo 5 while 3 is not a quadratic residue modulo 5. Thus, $\{p_2, p_3\} \equiv \{2,3\} \pmod 5$, and we get

$$\left(\frac{2-1}{2}\right)\left(\frac{3-1}{3}\right) \equiv 3 \pmod 5 \quad \text{or} \quad 1 \equiv 3^2 \pmod 5,$$

a contradiction. Finally, assume that $t = 1$ and that the right relation (10) holds. Reducing it modulo 4, we get $3 \equiv 3^{\alpha_2 + \alpha_3} \pmod 4$, therefore $\alpha_2 + \alpha_3$ is even. If α_2 is even, then so is α_3, so we get that $L_{2m} = 3\square$, which is false by Lemma 2.3. Hence, α_2 and α_3 are both odd. Furthermore, since m is odd and not a multiple of 3, we get that $2m \equiv 2 \pmod 4$ and $2m \equiv 2, 4 \pmod 6$, giving $2m \equiv 2, 10 \pmod{12}$. The period of $\{L_n\}_{n \geq 1}$ modulo 8 is 12, and $L_2 \equiv L_{10} \equiv 3 \pmod 8$, showing that $L_{2m} \equiv 3 \pmod 8$. This shows that $p_2^{\alpha_2} p_3^{\alpha_3} \equiv 1 \pmod 8$, and since α_2 and α_3 are odd, we get the congruence $p_2 p_3 \equiv 1 \pmod 8$. This together with the fact that $p_i \equiv 3 \pmod 4$ for $i = 1, 2$, implies that $p_2 \equiv p_3 \pmod 8$. Thus, $(p_2 - 1)/2$ and $(p_3 - 1)/2$ are congruent modulo 4 so their product is 1 modulo 4. Now we write

$$\begin{aligned} \phi(L_n) &= (3-1)(p_2 - 1) p_2^{\alpha_2 - 1} (p_3 - 1) p_3^{\alpha_3 - 1} \\ &= 8\left(\frac{(p_2 - 1)}{2}\frac{(p_3 - 1)}{2}\right) p_2^{\alpha_2 - 1} p_3^{\alpha_3 - 1} = 8M, \end{aligned}$$

where $M \equiv 1 \pmod 4$. However, since in fact $M = (10^m - 1)/9$, we get that $M \equiv 3 \pmod 4$ for $m \geq 2$, a contradiction. So, we must have $m \leq 3$, therefore $L_n < 4000$, so $n \leq 17$, and such values can be dealt with by hand.

Thus, we have proved the following result.

Lemma 5.3. *There is no $n > 6$ even such that relation* (2) *holds.*

5.5 $r = 3,\ d = 8$ and m is even

From now on, $n > 6$ is odd and L_n is also odd. If p is any prime factor of L_n, then reducing the equation $L_n^2 - 5F_n^2 = -4$ modulo p we get that $\left(\frac{5}{p}\right) = 1$. Thus, $p \equiv 1, 4 \pmod 5$. If $p \equiv 1 \pmod 5$, then $5 \mid (p-1) \mid \phi(L_n) = d(10^m - 1)/9$ with $d \in \{4, 8\}$, a contradiction. Thus, $p_i \equiv 4 \pmod 5$ for all $i = 1, \ldots, r$.

We next show that $p_i \equiv 3 \pmod 4$ for all $i = 1, \ldots, r$. Assume that this is not so and suppose that $p_1 \equiv 1 \pmod 4$. If $r = 1$, then $L_n = p_1^{\alpha_1}$ and by Lemma 2.2, we have $\alpha_1 = 1$. So,

$$L_n - 1 = \phi(L_n) = d\left(\frac{10^m - 1}{9}\right) \quad \text{so} \quad L_n = d\left(\frac{10^m - 1}{9}\right) + 1.$$

If $d = 4$, then $L_n \equiv 5 \pmod{10}$, so $5 \mid L_n$, which is false. When $d = 8$, we get that $L_n \equiv 4 \pmod 5$, showing that $n \equiv 3 \pmod 4$. However, we also have that $L_n \equiv 1 \pmod 8$, showing that $n \equiv 1 \pmod{12}$; in particular, $n \equiv 1 \pmod 4$, a contradiction.

Assume now that $r = 2$. Then $L_n = p_1^{\alpha_1} p_2^{\alpha_2}$ and $d = 8$. Then

$$\phi(L_n) = (p_1 - 1)p_1^{\alpha_1 - 1}(p_2 - 1)p_2^{\alpha_2 - 1} = 8\left(\frac{10^m - 1}{9}\right). \tag{11}$$

Reducing the above relation (11) modulo 5 we get $4^{\alpha_1 + \alpha_2 - 2} \times 3^2 \equiv 3 \pmod 5$, which is impossible since the left–hand side of it is a quadratic residue modulo 5 while the right–hand side of it is not.

Thus, $p_i \equiv 3 \pmod 4$ for $i = 1, \ldots, r$. Assume next that $r = 2$. Then $L_n = p_1^{\alpha_1} p_2^{\alpha_2}$ and $d = 4$. Then

$$\phi(L_n) = (p_1 - 1)p_1^{\alpha_1 - 1}(p_2 - 1)p_2^{\alpha_2 - 1} = 4\left(\frac{10^m - 1}{9}\right). \tag{12}$$

Reducing the above relation (12) modulo 5, we get $4^{\alpha_1 + \alpha_2 - 2} \times 3^2 \equiv 4 \pmod 5$, therefore $4^{\alpha_1 + \alpha_2 - 2} \equiv 1 \pmod 5$. Thus, $\alpha_1 + \alpha_2$ is even. If α_1 is even, so is α_2, so $L_n = \square$, and this is false by Lemma 2.2. Hence, α_2 and α_3 are both odd. It now follows that $L_n \equiv 3^{\alpha_1 + \alpha_2} \pmod 4$, so $L_n \equiv 1 \pmod 4$, therefore $n \equiv 1 \pmod 6$, and also $L_n \equiv 4^{\alpha_1 + \alpha_2} \pmod 5$, so $L_n \equiv 1 \pmod 5$, showing that $n \equiv 1 \pmod 4$. Hence, $n \equiv 1 \pmod{12}$, showing that $L_n \equiv 1 \pmod 8$. Thus, $p_1^{\alpha_1} p_2^{\alpha_2} \equiv 1 \pmod 8$, and since α_1 and α_2 are odd and $p_1^{\alpha_1 - 1}$ and $p_2^{\alpha_2 - 1}$ are congruent to 1 modulo 8 (as perfect squares), we therefore get that $p_1 p_2 \equiv 1$

(mod 8). Since also $p_1 \equiv p_2 \equiv 3 \pmod 4$, we get that in fact $p_1 \equiv p_2 \pmod 8$. Thus, $(p_1-1)/2$ and $(p_2-1)/2$ are congruent modulo 4 so their product is 1 modulo 4. Thus,

$$\phi(L_n) = 4\left(\frac{(p_1-1)}{2}\frac{(p_2-1)}{2}\right)p_1^{\alpha_1-1}p_2^{\alpha_2-1} = 4M,$$

where $M \equiv 1 \pmod 4$. Since in fact we have $M = (10^m-1)/9$, we get that $M \equiv 3 \pmod 4$ for $m \geq 2$, a contradiction.

Thus, $r=3$ and $d=8$. To get that m is even, we write $L_n = p_1^{\alpha_1}p_2^{\alpha_2}p_3^{\alpha_3}$. So,

$$\phi(L_n) = (p_1-1)p_1^{\alpha_1-1}(p_2-1)p_2^{\alpha_2-1}(p_3-1)p_3^{\alpha_3-1} = 8\left(\frac{10^m-1}{9}\right), \quad (13)$$

Reducing equation (13) modulo 5 we get $4^{\alpha_1+\alpha_2+\alpha_3-3}\times 3^3 \equiv 3 \pmod 5$, giving $4^{\alpha_1+\alpha_2+\alpha_3} \equiv 1 \pmod 5$. Hence, $\alpha_1+\alpha_2+\alpha_3$ is even. It is not possible that all α_i are even for $i=1,2,3$, since then we would get $L_n = \square$, which is not possible by Lemma 2.2. Hence, exactly one of them is even, say α_3 and the other two are odd. Then $L_n \equiv 3^{\alpha_1+\alpha_2+\alpha_3} \equiv 1 \pmod 4$ and $L_n \equiv 4^{\alpha_1+\alpha_2+\alpha_3} \equiv 1 \pmod 5$. Thus, $n \equiv 1 \pmod 6$ and $n \equiv 1 \pmod 4$, so $n \equiv 1 \pmod{12}$. This shows that $L_n \equiv 1 \pmod 8$. Since $p_1^{\alpha_1-1}p_2^{\alpha_2-1}p_3^{\alpha_3}$ is congruent to 1 modulo 8 (as a perfect square), we get that $p_1p_2 \equiv 1 \pmod 8$. Thus, $p_1 \equiv p_2 \pmod 8$, so $(p_1-1)/2$ and $(p_2-1)/2$ are congruent modulo 4 so their product is 1. Then

$$\phi(L_n) = 8\left(\frac{(p_1-1)}{2}\frac{(p_2-1)}{2}\right)\left(\frac{p_3(p_3-1)}{2}\right)p_1^{\alpha_1-1}p_2^{\alpha_2-1}p_3^{\alpha_3-2} = 8M, \quad (14)$$

where $M = (10^m-1)/9 \equiv 3 \pmod 4$. In the above product, all odd factors are congruent to 1 modulo 4 except possibly for $p_3(p_3-1)/2$. This shows that $p_3(p_3-1)/2 \equiv 3 \pmod 4$, which shows that $p_3 \equiv 3 \pmod 8$. Now since $p_3^2 \mid L_n$, we get that $p_3 \mid \phi(L_n) = 8(10^m-1)/9$. So, $10^m \equiv 1 \pmod{p_3}$. Assuming that m is odd, we would get

$$1 = \left(\frac{10}{p_3}\right) = \left(\frac{2}{p_3}\right)\left(\frac{5}{p_3}\right) = -1,$$

a contradiction. In the above, we used that $p_3 \equiv 3 \pmod 8$ and $p_3 \equiv 4 \pmod 5$ and quadratic reciprocity to conclude that $\left(\frac{2}{p_3}\right) = -1$ as well as $\left(\frac{5}{p_3}\right) = \left(\frac{p_3}{5}\right) = 1$.

So, we have showed the following result.

Lemma 5.4. *If $n > 6$ is a solution of (2), then n is odd, L_n is odd, $r = 3$, $d = 8$ and m is even. Further, $L_n = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3}$, where $p_i \equiv 3 \pmod 4$ and $p_i \equiv 4 \pmod 5$ for $i = 1, 2, 3$, $p_1 \equiv p_2 \pmod 8$, $p_3 \equiv 3 \pmod 8$, α_1 and α_2 are odd and α_3 is even.*

5.6 $n \in \{p, p^2\}$ for some prime p with $p^3 \mid 10^{p-1} - 1$

The factorizations of all Lucas numbers L_n for $n \leq 1000$ are known. We used them and Lemma 5.4 and found no solution to equation (2) with $n \in [7, 1000]$.

Let p be a prime factor of n. Suppose first that $n = p^t$ for some positive integer t. If $t \geq 4$, then L_n is divisible by at least four primes, namely primitive prime factors of L_p, L_{p^2}, L_{p^3} and L_{p^4}, respectively, which is false. Suppose that $t = 3$. Write

$$L_n = L_p \left(\frac{L_{p^2}}{L_p} \right) \left(\frac{L_{p^3}}{L_{p^2}} \right).$$

The three factors above are coprime, so they are $p_1^{\alpha_1}$, $p_2^{\alpha_2}$, $p_3^{\alpha_3}$ in some order. Since α_3 is even, we get that one of L_p, L_{p^2}/L_p or L_{p^3}/L_{p^2} is a square, which is false by Lemmas 2.2 and 2.3. Hence, $n \in \{p, p^2\}$. All primes p_1, p_2, p_3 are quadratic residues modulo 5. When $n = p$, they are primitive prime factors of L_p. When $n = p^2$, all of them are primitive prime factors of L_p or L_{p^2} with at least one of them being a primitive prime factor of L_{p^2}. Thus, $p_i \equiv 1 \pmod p$ holds for all $i = 1, 2, 3$ both in the cace $n = p$ and $n = p^2$, and when $n = p^2$ at least one of the the above congruences holds modulo p^2. This shows that $p^3 \mid (p_1 - 1)(p_2 - 1)(p_3 - 1) \mid \phi(L_n) = 8(10^m - 1)/9$, so $p^3 \mid 10^m - 1$. When $n = p^2$, we in fact have $p^4 \mid 10^m - 1$. Assume now that $p^3 \nmid 10^{p-1} - 1$. Then the congruence $p^3 \mid 10^m - 1$ implies $p \mid m$, while the congruence $p^4 \mid 10^m - 1$ implies $p^2 \mid m$. Hence, when $n = p$, we have

$$2^p > L_p > \phi(L_n) = 8(10^m - 1)/9 > (10^p - 1)/9 > 10^{p-1}$$

which is false for any $p \geq 3$. Similarly, if $n = p^2$, then

$$2^{p^2} > L_{p^2} > \phi(L_n) = 8(10^m - 1)/9 > (10^{p^2} - 1)/9 > 10^{p^2-1}$$

which is false for any $p \geq 3$. So, indeed when n is a power of a prime p, then the congruence $p^3 \mid 10^{p-1} - 1$ must hold. We record this as follows.

Lemma 5.5. *If $n > 6$ and $n = p^t$ is solution of (2) with some $t \geq 1$ and p prime, then $t \in \{1, 2\}$ and $p^3 \mid 10^{p-1} - 1$.*

Suppose now that n is divisible by two distinct primes p and q. By Lemma 2.1, L_p, L_q and L_{pq} each have primitive prime factors. This shows that $n = pq$, for if $n > pq$, then L_n would have (at least) one additional prime factor, which is a contradiction. Assume $p < q$ and

$$L_n = L_p L_q \left(\frac{L_{pq}}{L_p L_q}\right).$$

Unless $q = L_p$, the three factors above are coprime. Say $q \neq L_p$. Then the three factors above are $p_1^{\alpha_1}$, $p_2^{\alpha_2}$ and $p_3^{\alpha_3}$ in some order. By Lemmas 2.2 and up to relabeling the primes p_1 and p_2, we may assume that $\alpha_1 = \alpha_2 = 1$, so $L_p = p_1$, $L_q = p_2$ and $L_{pq}/(L_pL_q) = p_3^{\alpha_3}$. On the other hand, if $q = L_p$, then $q^2 \| L_{pq}$. This shows then that up to relabeling the primes we may assume that $\alpha_2 = 1$, $\alpha_3 = 2$, $L_p = p_3$, $L_q = p_2$, $L_{pq}/(L_pL_q) = p_3 p_1^{\alpha_1}$. However, in this case $p_3 \equiv 3 \pmod 8$, showing that $p \equiv 5 \pmod 8$. In particular, we also have $p \equiv 1 \pmod 4$, so $p_3 = L_p \equiv 1 \pmod 5$, and this is not possible. So, this case cannot appear.

Write $m = 2m_0$. Then

$$(p_1 - 1)(p_2 - 1)(p_3 - 1)p_3^{\alpha_3 - 1} = \phi(L_n) = \frac{8(10^{m_0} - 1)(10^{m_0} + 1)}{9}.$$

If m_0 is even, then $p_3^{\alpha_3-1} \mid 10^{m_0} - 1$ because $p_3 \equiv 3 \pmod 4$, so p_3 cannot divide $10^{m_0}+1 = (10^{m_0/2})^2+1$. If m_0 is odd, then $p_3^{\alpha_3-1} \mid 10^{m_0}+1$, because if not we would have that $p_3 \mid 10^{m_0} - 1$, so $10^{m_0} \equiv 1 \pmod{p_3}$, and since m_0 is odd we would get $\left(\frac{10}{p_3}\right) = 1$, which is false since $\left(\frac{2}{p_3}\right) = -1$ and $\left(\frac{5}{p_3}\right) = 1$. Thus, we get, using (8), that

$$\begin{aligned} \alpha^{p+q} p_3 &> (L_p - 1)(L_q - 1)p_3 = p_1 p_2 p_3 > (p_1 - 1)(p_2 - 1)(p_3 - 1) \\ &\geq \frac{8(10^{m_0} - 1)}{9} > \frac{8}{10} \times 10^{m_0}. \end{aligned} \tag{15}$$

On the other hand, by inequality (6), we have

$$10^m > \frac{8(10^m - 1)}{9} = \phi(L_n) > \frac{L_n}{4},$$

so that

$$10^{m_0} > \frac{\sqrt{L_n}}{2} > \frac{\alpha^{pq/2 - 0.5}}{2}, \tag{16}$$

where we used the inequality (9). From (15) and (16), we get

$$p_3 > \frac{8}{20\sqrt{\alpha}} \alpha^{pq/2 - p - q} = \frac{8}{20\alpha^{4.5}} \alpha^{(p-2)(q-2)} > \frac{\alpha^{(p-2)(q-2)}}{25}.$$

Once checks that the inequality

$$\frac{\alpha^{(p-2)(q-2)/2}}{25} > \alpha^{q+1} \tag{17}$$

is valid for all pairs of primes $5 \le p < q$ with $pq > 100$. Indeed, the above inequality (17) is implied by

$$(p-2)(q-2)/2 - (q+1) - 7 > 0, \quad \text{or} \quad (q-2)(p-4) > 20. \tag{18}$$

If $p \ge 7$, then $q > p \ge 11$ and the above inequality (18) is clear, whereas if $p = 5$, then $q \ge 23$ and the inequality (18) is again clear.

We thus get that

$$p_3 > \frac{\alpha^{(p-2)(q-2)}}{25} > \alpha^{q+1} > L_q = p_2 > L_p = p_1.$$

We exploit the two relations

$$\begin{aligned} 0 \quad < \quad 1 - \frac{\phi(L_n)}{L_n} &= 1 - \left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\left(1 - \frac{1}{p_3}\right) < \frac{3}{p_1} < \frac{5}{\alpha^p}; \\ 1 - \frac{(L_p - 1)\phi(L_n)}{L_p L_n} &= 1 - \left(1 - \frac{1}{p_2}\right)\left(1 - \frac{1}{p_3}\right) < \frac{2}{p_2} < \frac{4}{\alpha^q}. \end{aligned} \tag{19}$$

In the above, we used the inequality

$$1 - (1 - x_1) \cdots (1 - x_r) \le x_1 + \cdots + x_r$$

valid for all real numbers $x_i \in (0, 1)$ for $i = 1, \ldots, r$, which can be easily proved by induction on r. Since n is odd, we have $L_n = \alpha^n - \alpha^{-n}$. Then

$$1 + \frac{2}{\alpha^{2n}} > \frac{1}{1 - \alpha^{-2n}} > 1,$$

so

$$\frac{1}{\alpha^n} + \frac{2}{\alpha^{3n}} > \frac{1}{L_n} > \frac{1}{\alpha^n},$$

or

$$\frac{8 \times 10^m}{9\alpha^n} + \frac{16 \times 10^m}{9\alpha^{3n}} - \frac{8}{9L_n} > \frac{8(10^m - 1)}{9L_n} = \frac{\phi(L_n)}{L_n} > \frac{8 \times 10^m}{9\alpha^n} - \frac{8}{9L_n}. \tag{20}$$

The first inequality (19) and (20) show that

$$\left|1 - (8/9) \times 10^m \times \alpha^{-n}\right| < \frac{3}{p_1} + \frac{8}{9L_n} + \frac{16 \times 10^m}{9\alpha^{3n}}. \tag{21}$$

Now

$$8\times 10^{m-1} < \frac{8(10^m-1)}{9} = \phi(L_n) < L_n < \alpha^{n+1}, \quad \text{so} \quad 10^m < \frac{10\alpha}{8}\alpha^n,$$

showing that

$$\frac{16\times 10^m}{9\alpha^{3n}} < \frac{20\alpha}{9\alpha^{2n}} < \frac{0.5}{\alpha^n} \quad \text{for} \quad n > 1000.$$

Since also

$$\frac{8}{9L_n} < \frac{8\alpha}{9\alpha^n} < \frac{1.5}{\alpha^n},$$

we get that

$$\frac{16\times 10^m}{9\alpha^{3n}} + \frac{8}{9L_n} < \frac{0.5}{\alpha^n} + \frac{1.5}{\alpha^n} < \frac{2}{\alpha^n}.$$

Since also $p_1 < L_n^{1/3} < \alpha^{(n+1)/3}$, we get that (21) becomes

$$\left|1-(8/9)\times 10^m\times\alpha^{-n}\right| < \frac{3}{p_1} + \frac{2}{\alpha^n} < \frac{4}{p_1} = \frac{4}{L_p} < \frac{4\alpha}{\alpha^p} < \frac{7}{\alpha^p}, \tag{22}$$

where the middle inequality is implied by $\alpha^n > 2\alpha^{(n+1)/3} > 13p_1$, which holds for $n > 1000$.

The same argument based on (20) shows that

$$\left|1-\left(\frac{8(L_p-1)}{9L_p}\right)\times 10^m\times\alpha^{-n}\right| < \frac{4}{\alpha^q} + \frac{2}{\alpha^n} < \frac{5}{\alpha^q}. \tag{23}$$

We are in a situation to apply Theorem 2 to the left–hand sides of (22) and (23). The expressions there are nonzero, since any one of these expressions being zero means $\alpha^n \in \mathbb{Q}$ for some positive integer n, which is false. We always take $\mathbb{K} = \mathbb{Q}(\sqrt{5})$ for which $D = 2$. We take $t = 3$, $\alpha_1 = \alpha,\ \alpha_2 = 10$, so we can take $A_1 = \log\alpha = 2h(\alpha_1)$ and $A_2 = 2\log 10$. For (22), we take $\alpha_3 = 8/9$, and $A_3 = 2\log 9 = 2h(\alpha_3)$. For (23), we take $\alpha_3 = 8(L_p-1)/9L_p$, so we can take $A_3 = 2p > h(\alpha_3)$. This last inequality holds because $h(\alpha_3) \le \log(9L_p) < (p+1)\log\alpha + \log 9 < p$ for all $p \ge 7$, while for $p = 5$ we have $h(\alpha_3) = \log 99 < 5$. We take $\alpha_1 = -n,\ \alpha_2 = m,\ \alpha_3 = 1$. Since

$$2^n > L_n > \phi(L_n) > 8\times 10^{m-1}$$

it follows that $n > m$. So, $B = n$. Now Theorem 2 implies that a lower bound on the left–hand side of (22) is

$$\exp\left(-1.4\times 30^6\times 3^{4.5}\times 2^2\times(1+\log 2)(1+\log n)(\log\alpha)(2\log 10)(2\log 9)\right),$$

so inequality (22) implies

$$p\log\alpha - \log 7 < 9.5\times 10^{12}(1+\log n),$$

which implies

$$p < 2\times 10^{13}(1+\log n). \tag{24}$$

Now Theorem 2 implies that the right–hand side of inequality (23) is at least as large as

$$\exp\left(-1.4\times 30^6\times 3^{4.5}\times 2^2\times(1+\log 2)(1+\log n)(\log\alpha)(2\log 10)(2p)\right)$$

leading to

$$q\log\alpha - \log 4 < 4.3\times 10^{12}(1+\log n)p.$$

Using (24), we get

$$q < 9\times 10^{12}(1+\log n)p < 2\times 10^{26}(1+\log n)^2.$$

Using again (24), we get

$$n = pq < 4\times 10^{39}(1+\log n)^2,$$

leading to

$$n < 5\times 10^{43}. \tag{25}$$

Now we need to reduce the bound. We return to (22). Put

$$\Lambda = m\log 10 - n\log\alpha + \log(8/9).$$

Then (22) implies that

$$|e^{\Lambda}-1| < \frac{7}{\alpha^p}. \tag{26}$$

Assuming $p\ge 7$, we get that the right–hand side of (26) is $< 1/2$. Analyzing the cases $\Lambda > 0$ and $\Lambda < 0$ and using the fact that $1+x<e^x$ holds for all positive real numbers x, we get that

$$|\Lambda| < \frac{14}{\alpha^p}.$$

Assume say that $\Lambda > 0$. Dividing across by $\log\alpha$, we get

$$0 < m\left(\frac{\log 10}{\log\alpha}\right) - n + \left(\frac{\log(8/9)}{\log\alpha}\right) < \frac{30}{\alpha^p}.$$

We are now ready to apply Lemma 4.1 with the obvious parameters

$$\gamma = \frac{\log 10}{\log\alpha}, \quad \mu = \frac{\log(8/9)}{\log\alpha}, \quad A = 30, \quad B = \alpha.$$

Since $m < n$, we can take $M = 10^{45}$ by (25). Applying Lemma 4.1, performing the calculations and treating also the case when $\Lambda < 0$, we get that $p < 250$. Now we go to inequality (23) and for $p \in [5, 250]$, we consider

$$\Lambda_p = m \log 10 - n \log \alpha + \log \left(\frac{8(L_p - 1)}{9L_p} \right).$$

Then inequality (23) becomes

$$\left| e^{\Lambda_p} - 1 \right| < \frac{5}{\alpha^q}. \tag{27}$$

Since $q \geq 7$, the right–hand side is smaller than $1/2$. We thus get that

$$|\Lambda_p| < \frac{10}{\alpha^q}.$$

We proceed in the same way as we proceeded with Λ by applying Lemma 4.1 to Λ_p and distinguishing the cases in which $\Lambda_p > 0$ and $\Lambda_p < 0$, respectively. In all cases, we get that $q < 250$. Thus, $5 \leq p < q < 250$. Note however that we must have either $p^2 \mid 10^{p-1} - 1$ or $q^2 \mid 10^{q-1} - 1$. Indeed, the point is that since all three prime factors of L_n are quadratic residues modulo 5, and they are primitive prime factors of L_p, L_q and L_{pq}, respectively, it follows that $p_1 \equiv 1 \pmod p$, $p_2 \equiv 1 \pmod q$ and $p_3 \equiv 1 \pmod{pq}$. Thus, $(pq)^2 \mid (p_1 - 1)(p_2 - 1)(p_3 - 1) \mid \phi(L_n) = 8(10^m - 1)/9$, which in turn shows that $(pq)^2 \mid 10^m - 1$. Assume that neither $p^2 \mid 10^{p-1} - 1$ nor $q^2 \mid 10^{q-1} - 1$. Then relation $(pq)^2 \mid 10^m - 1$ implies that $pq \mid m$. Thus, $m \geq pq$, leading to

$$2^{pq} > L_n > \phi(L_n) = \frac{8(10^m - 1)}{9} > 10^{m-1} \geq 10^{pq-1},$$

a contradiction. So, indeed either $p^2 \mid 10^{p-1} - 1$ or $q^2 \mid 10^{q-1} - 1$. However, a computation with Mathematica revealed that there is no prime r such that $r^2 \mid 10^{r-1} - 1$ in the interval $[5, 250]$. In fact, the first such $r > 3$ is $r = 487$, but L_{487} is not prime!

This contradiction shows that indeed when $n > 6$, we cannot have $n = pq$. Hence, $n \in \{p, p^2\}$ and $p^3 \mid 10^{p-1} - 1$. We record this as follows.

Lemma 5.6. *Equation (2) has no solution $n > 6$ which is not of the form $n = p$ or p^2 for some prime p such that $p^3 \mid 10^{p-1} - 1$.*

5.7 Bounding n

Finally, we bound n. We assume again that $n > 1000$. Equation (3) becomes

$$L_n = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3}.$$

Throughout this last section, we assume that $p_1 < p_2 < p_3$. First, we bound p_1, p_2 and p_3 in terms of n. Using the first relation of (19), we have that

$$0 < 1 - \frac{\phi(L_n)}{L_n} = 1 - \left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\left(1 - \frac{1}{p_3}\right) < \frac{3}{p_1}. \tag{28}$$

By the argument used at estimates (20)–(22), we get that

$$|1 - (8/9) \times 10^m \times \alpha^{-n}| < \frac{3}{p_1} + \frac{2}{\alpha^n} < \frac{4}{p_1}, \tag{29}$$

where the last inequality holds because $p_1 \le L_n/(p_2p_3) < L_n/(7\times 11) < \alpha^n/2$.

We apply Theorem 2 to the left-hand side of (29) The expression there is nonzero by a previous argument. We take again $\mathbb{K} = \mathbb{Q}(\sqrt{5})$ for which $D = 2$. We take $t = 3$, $\alpha_1 = 8/9$, $\alpha_2 = 10$ and $\alpha_3 = \alpha$. Thus, we can take $A_1 = \log 9 = 2h(\alpha_1)$, $A_2 = 2\log 10$ and $A_3 = 2\log\alpha = 2h(\alpha_3)$. We also take $b_1 = 1$, $b_2 = m$, $b_3 = -n$. We already saw that $B = n$. Now Theorem 2 implies that a lower bound on the left–hand side of (29) is at least

$$\exp\left(-1.4 \times 30^6 \times 3^{4.5} \times 2^2 \times (1 + \log 2)(1 + \log n)2^3(\log\alpha)(\log 10)(\log 9)\right),$$

so inequality (22) implies

$$\log p_1 - \log 4 < 1.89 \times 10^{13}(1 + \log n),$$

Then we get

$$\log p_1 < 1.9 \times 10^{13}(1 + \log n). \tag{30}$$

We use the same argument to bound p_2. We have

$$0 < 1 - \left(\frac{p_1 - 1}{p_1}\right)\frac{\phi(L_n)}{L_n} = \left(1 - \frac{1}{p_2}\right)\left(1 - \frac{1}{p_3}\right) < \frac{2}{p_2}.$$

Thus, we get that:

$$\left|1 - \left(\frac{8(p_1 - 1)}{9p_1}\right) \times 10^m\alpha^{-n}\right| < \frac{2}{p_2} + \frac{2}{\alpha^n} < \frac{3}{p_2}, \tag{31}$$

where the last inequality follows again because $p_2 \le L_n/(p_1p_3) < \alpha^n/2$.

We apply Theorem 2 to the left–hand side of (31). We take $t = 3$, $\alpha_1 = 8(p_1 - 1)/(9p_1)$, $\alpha_2 = 10$ and $\alpha_3 = \alpha$, so we take $A_1 = 2\log(9p_1) \ge 2h(\alpha_1)$, $A_2 = 2\log 10$ and $A_3 = 2\log\alpha$. Again $b_1 = -1$, $b_2 = m$, $b_3 = -n$ and $B = n$. Now Theorem 2 implies that a lower bound on the left–hand side of (31) is

$$\exp\left(-1.4 \times 30^6 \times 3^{4.5} \times 2^2 \times (1 + \log 2)(1 + \log n)2^3(\log\alpha)\log 10\log(9p_1)\right).$$

Using estimate (30), inequality (32) implies

$$\log p_2 - \log 2 < 1.8 \times 10^{26}(1 + \log n)^2. \tag{32}$$

Using a similar argument, we get

$$\log p_3 - \log 2 < 1.8 \times 10^{39}(1 + \log n)^3. \tag{33}$$

Now can bound n. Equation (3), gives that :

$$\alpha^n + \beta^n = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3}.$$

Thus,

$$|p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \alpha^{-n} - 1| = \frac{1}{\alpha^{2n}} \tag{34}$$

We can apply Theorem 2, with $t = 4$, $\alpha_1 = p_1$, $\alpha_2 = p_2$, $\alpha_3 = p_3$, and $\alpha_4 = \alpha$. We take $A_1 = 2\log p_1 = 2h(\alpha_1)$, $A_2 = 2\log p_2$, $A_3 = 2\log p_3 = 2h(\alpha_3)$ and $A_4 = 2\log \alpha$. We take $B = n$. Then Theorem 2 implies that a lower bound on the left–hand side of (34) is

$$\exp\left(-1.4 \times 30^7 \times 4^{4.5} \times 2^2 \times (1 + \log 2)(1 + \log n)2^4(\log \alpha) \prod_{i=1}^{3} (\log p_i)\right).$$

Using (34) and inequalities (29), (32), (33), we get

$$n < 8 \times 10^{93}(1 + \log n)^7, \quad \text{so} \quad n < 10^{111}.$$

This gives the upper bound. As for the lower bound, a quick check with Mathematica revealed that the only primes $p < 2 \times 10^9$ such that $p^2 \mid 10^{p-1} - 1$ are $p \in \{3, 487, 56598313\}$ and none of these has in fact the stronger property that $p^3 \mid 10^{p-1} - 1$.

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