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Linear-Quadratic-Gaussian Optimization of Urban Transportation Network with Application to Sofia Traffic Optimization

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Abstract: *The paper defines and solves a Linear-Quadratic-Gaussian (LQG) optimization problem addressing real time control policy of Urban Transportation Network (UTN). The paper presents UTN model definition, analysis and LQG optimization problem definition, resulting in special problem structure. A real application for UTN situated in Sofia, Bulgaria along Yosif Gurko street was provided for testing this control policy.*

Keywords: *Control of transportation systems, urban transportation network, linear-quadratic-Gaussian optimization, special case of discrete algebraic Riccati equation solution.*

0. Introduction and problem formulation

The reduction of urban traffic congestion is still an area for significant improvements. Due to the development of many cities the traffic congestion problems take more attention. Traffic congestion produces environment pollution, reduces the traveling efficiency, and thus causes economic losses [1, 2].

A solution of the problem may consist into development a high-level controller design technique to regulate the traffic demands of complex urban traffic networks. This solution may seriously reduce congestion problems through the better utilization of the existing infrastructure [1-3].

A plenty of more or less sophisticated urban control strategies have been developed and implemented in the last decades [1, 2, 4-9]. A contemporary overview of the proposed approaches and programs may be found in the papers [1, 5, 9, 10]. The more thorough overview may be found in the papers [1, 2, 6, 9, 11].

A ‘store-and-forward-based approach’ proposed by Gazis and Potts in [1, 6, 12-14, 18] is an important class of coordinated traffic responsive strategies. Combination of store-and-forward approach with classical control (the method is called traffic-responsive urban control [1-4, 6, 8, 11, 14, 15, 18]) is proposed in a quadratic programming form, and the controller is designed as a linear-quadratic regulator.

In the same time some specific aspects of linear-quadratic optimization are not considered in details, which influence negatively the control process. This paper presents a linear-quadratic optimization problem, taking into account the special structure of the model. The problem is extended for the case when random processes exist in the traffic behavior and when the real traffic information may be given by noisy sensors.

The paper outlines are the following. The first section discusses the UTN model when random events exist. The second section presents results of UTN analysis and behavior of the disturbances, which are included in the traffic model. The last section presents results of the application of the LQG control for a real UTN in Sofia. Particularly, the numerical implementation and solution of the LQG optimization problem is discussed.

1. UTN state-space model design

An urban transportation network is regarded as a set of linked junctions which are controlled by the traffic lights [1]. An UTN behavior in saturated mode may be described by the well known store-and-forward model [1, 12, 14], initially suggested by Gazis and Potts [12]. This model is presented as a set of discrete linear state-space equations and describes the flow process in an UTN in a simple way. The UTN may be schematically presented according to

Fig. 1.

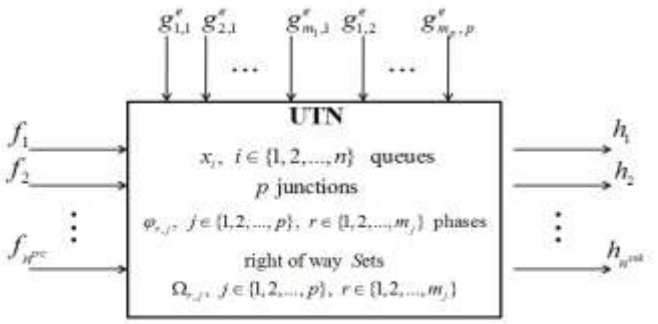

Fig. 1. Urban transportation network

Following Fig. 1, the UTN contains n^{src} external flows of vehicles sources $f_l,\ l \in \{1, 2, ..., n^{\text{src}}\}$ which form external network demand; n^{snk} outgoing flows of vehicles leaving the UTN $h_r,\ r \in \{1, 2, ..., n^{\text{snk}}\}$; p traffic light controlled junctions which are situated on a subset of the n links of the UTN; the notations x_i, $i = \{1, 2, \dots, n\}$, correspond to the number of vehicles on each link of the network.

In a saturated UTN in front of each junction, controlled by traffic lights, queues of waiting vehicles arise. For sake of simplicity, the paper considers the practical case when each link contains only one queue. This simplification can be easily avoided by increase of the model dimensions. The traffic lights change their phases periodically to provide each queue right of way. For the current developments it was assumed that for each junction the phase sequent is specified. This case concerns the existing traffic infrastructure. It is noted that the j-th junction applies m_j, $j \in \{1, 2, ..., p\}$, phases. The sequential traffic light phases repetition defines the duration of the traffic light cycle. This research assumes that the traffic cycle is constant value, equal to c seconds. The phase notation $\varphi_{r,j}$, $j \in \{1, 2, ..., p\}$, $r \in \{1, 2, ..., m_j\}$, means r-th phase of j-th junction. Each phase $\varphi_{r,j}$ provides control for queues set $\Omega_{r,j}$, giving right of way during the corresponding effective green time $g^e_{r,j}$. Each phase $\varphi_{r,j}$ is connected with the next one with a lost time, noted as $l_{r,j}$. During the lost time $l_{r,j}$ no traffic flows are possible to operate. In general the lost time $l_{r,j}$ includes the duration of the yellow light and pedestrian time. For each j-th junction the mentioned times are related as

(1) $$\sum_{r=1}^{m_j}\left(g^e_{r,j} + l_{r,j}\right) = c\,,\ j \in \{1,\ 2, ...,\ p\}.$$

The lost time $l_{r,j}$ may be assumed as constant value or proportional to the effective green time, $l_{r,j} = \mu_{r,j} g^e_{r,j}$, where $\mu_{r,j}$ are coefficients with values $0 \le \mu_{r,j} < 1$ and practically belong to $0 \le \mu_{r,j} \le 0.25$.

A junction control may be implemented by changing effective green time durations within the cycle c (it is so called splits control [1]). Following relation (1), for each j-th junction there are $m_j - 1$ independent control influences $g^e_{r,j}$. Let denotes the dependent one by index v_j and it duration is $g^e_{v_j,j}$, where $1 \le v_j \le m_j$,

(2) $$g^e_{v_j,j} = \frac{1}{\left(1+\mu_{v_j,j}\right)}\left(c - \sum_{r=1,\ r\ne v_j}^{m_j}\left(1+\mu_{r,j}\right) g^e_{r,j}\right),\ j \in \{1, 2, ..., p\}.$$

Let the value $\bar{m} = \sum_{j=1}^{p} m_j$ corresponds to the total number of traffic lights phases in the UTN. Generally, UTN has $m = \sum_{j=1}^{p}\left(m_j - 1\right) = \bar{m} - p$ independent variables that can be changed to implement a control strategy. Let $\bar{\boldsymbol{g}}^e = \left[g^e_{1,1},\ g^e_{2,1},\ \dots\ ,\ g^e_{m_1,1},\ g^e_{1,2},\ \dots\ ,\ g^e_{m_p,p}\right]^{\mathrm{T}}$ denote the effective green time vector $\bar{\mathbf{g}}^e \in R^{\bar{m}}$ and let $g^e(j)$ is a j-th element of the vector.

Let g_i means a total green time appointed to i-th queue during the cycle. Green time g_i consists from effective green times according to the set $\Omega_{r,j}$. For instance, when for i-th queue, the right of way is allowed on first and third phases on j-th junction, there is a relation $g_i = g^e_{1,j} + g^e_{3,j}$. Generally, $g_i = \sum_{j=1}^{\bar{m}} \gamma_{i,j} g^e(j)$, where elements $\gamma_{i,j} \in \{0,1\}$ define if the appropriate effective green time $g^e(j)$ contribute to the green time g_i. The last equality can be rewritten in matrix form $\mathbf{g} = \bar{\mathbf{G}}\bar{\mathbf{g}}^e$, where $\mathbf{g} = [g_1, g_2, \dots, g_n]^{\mathrm{T}}$, $\bar{\mathbf{G}} \in R^{n \times \bar{m}}$, $\bar{\mathbf{G}} = \{\gamma_{i,j}\}$, $i = 1, 2, \dots, n$; $j = 1, 2, \dots, \bar{m}$.

Let's assume that by the linear transformation $\mathbf{T}_g \bar{\mathbf{g}}^e = \begin{bmatrix} \mathbf{g}^e \\ \tilde{\mathbf{g}}^e \end{bmatrix}$, the effective green time vector elements can be reordered such that $\mathbf{g}^e$, $\tilde{\mathbf{g}}^e$ are vectors of independent and dependent variables respectively. Representation (2) also may be rewritten in the matrix form as $\tilde{\mathbf{g}}^e = c\mathbf{M}_1 - \mathbf{M}_2\mathbf{g}^e$, $\mathbf{M}_1 \in R^p$, $\mathbf{M}_2 \in R^{p \times m}$. Then collecting above, the following expression exists:

(3) $$\mathbf{g} = \mathbf{G}\mathbf{g}^e + c\mathbf{L},$$

where $\mathbf{G} = \mathbf{G}_1 - \mathbf{G}_2\mathbf{M}_2 \in R^{n \times m}$, $\mathbf{L} = \mathbf{G}_2\mathbf{M}_1 \in R^n$ and matrices $\mathbf{G}_1 \in R^{n \times m}$, $\mathbf{G}_2 \in R^{n \times p}$ correspond to block representation $\bar{\mathbf{G}}\mathbf{T}_g^{-1} = [\mathbf{G}_1 \ \ \mathbf{G}_2]$. It is worth to remember for permutation matrices hold $\mathbf{T}_g^{-1} = \mathbf{T}_g^{\mathrm{T}}$.

Usually UTN control strategies apply the relative green times towards the cycle time or $\hat{g}_i = g_i / c$, $\hat{g}^e_{i,j} = g^e_{i,j} / c$, $i, j = 1, 2, \dots$. Then the equality (3) can be written as

(4) $$\hat{\mathbf{g}} = \mathbf{G}\hat{\mathbf{g}}^e + \mathbf{L},$$

where $\hat{\mathbf{g}} = [\hat{g}_1, \hat{g}_2, \dots, \hat{g}_n]^{\mathrm{T}}$, $\hat{\mathbf{g}}^e = [\hat{g}^e_{1,1}, \hat{g}^e_{2,1}, \dots, \hat{g}^e_{m_1,1}, \hat{g}^e_{1,2}, \dots, \hat{g}^e_{m_p,p}]^{\mathrm{T}}$. It is supposed here that dependent variables $\hat{g}^e_{i,j}$ are excluded from vector $\hat{\mathbf{g}}^e$.

The discrete time dynamics of vehicles for the i-th queue for one cycle is given on Fig. 2. The notations q_i^{in}, q_i^{out} are the input and output flows to the i-th vehicle queue; d_i^{in}, d_i^{out} are the input and output flows within i-th link; $s_{i,j}$ denotes vehicles exchange from j-th to i-th queue; $f_{i,j}$ means vehicles inflow from j-th source to i-th queue; $h_{i,j}$ means vehicles outflow from j-th queue to i-th sink. Since all flows are considered for one cycle time c, all flows are measured by the number of vehicles during the cycle c.

The dynamical change of i-th queue length (number of vehicles) in k-th cycle can be represented by the equation

(5) $$x_i(k+1) = x_i(k) + q_i^{\mathrm{in}}(k) - q_i^{\mathrm{out}}(k) + d_i^{\mathrm{in}}(k) - d_i^{\mathrm{out}}(k), \ i \in \{1, 2, \dots, n\},$$

where $x_i(k)$, $x_i(k+1)$ are the i-th queue lengths at the beginning of k-th and $(k+1)$-th cycle respectively. Following Fig. 2, q_i^{in} gives the vehicle income from external sources and other queues, $q_i^{\mathrm{in}} = \sum_{l=1}^{n^{\mathrm{src}}} f_{i,l} + \sum_{j=1, j \neq i}^{n} s_{i,j}$. The same holds for outflows $q_i^{\mathrm{out}} = \sum_{r=1}^{n^{\mathrm{snk}}} h_{r,i} + \sum_{e=1, e\neq i}^{n} s_{e,i}$. For simplification of (5) instead of values d_i^{in} , d_i^{out} the research will use the variance $d_i^{\mathrm{dif}} = d_i^{\mathrm{in}} - d_i^{\mathrm{out}}$.

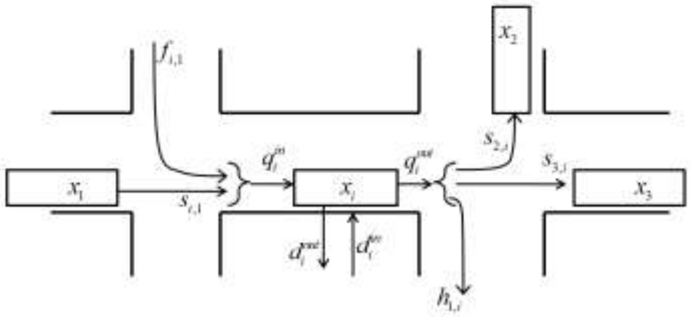

Fig. 2. The flows dynamics related with *i*-th queue

Let v is a vehicles flow, [veh/sec], then vehicles amount z appeared during time τ, [sec], can be expressed as $z = v\tau$. Let outgoing flow can be split to n directions

(6) $$z = \sum_{i=1}^{n} z_i = \sum_{i=1}^{n} \lambda_i v\tau ,$$

where λ_i is a split (proportion) of vehicles going toward i-th direction. The outgoing proportions satisfy the relationships $0 \le \lambda_i \le 1, \sum_{i=1}^{n} \lambda_i = 1$. It is also assumed that values v, λ_i are normally distributed random numbers with known statistic characteristics. Let $v = \bar{v} + \Delta v$, $\lambda_i = \bar{\lambda}_i + \Delta\lambda_i$, where $\bar{v}$, $\bar{\lambda}_i$ are average values and Δv , $\Delta\lambda_i$ are the appropriate centered random numbers.

Then, the variables in (5) can be represented in form (6) as

(7) $$\begin{cases} f_{i,l} = \left(\bar{\lambda}_{i,l}^{f} + \Delta\lambda_{i,l}^{f}\right)\left(\bar{v}_l^{f} + \Delta v_l^{f}\right)c = \bar{\lambda}_{i,l}^{f}\bar{v}_l^{f}c + \psi_{i,l}^{f}c, \\ s_{i,j} = \left(\bar{\lambda}_{i,j}^{s} + \Delta\lambda_{i,j}^{s}\right)\left(\bar{v}_j^{s} + \Delta v_j^{s}\right)g_j = \bar{\lambda}_{i,j}^{s}\bar{v}_j^{s}g_j + \psi_{i,j}^{s}g_j , \\ h_{r,i} = \left(\bar{\lambda}_{r,i}^{h} + \Delta\lambda_{r,i}^{h}\right)\left(\bar{v}_i^{s} + \Delta v_i^{s}\right)g_i = \bar{\lambda}_{r,i}^{h}\bar{v}_i^{s}g_i + \psi_{r,i}^{h}g_i , \\ s_{e,i} = \left(\bar{\lambda}_{e,i}^{s} + \Delta\lambda_{e,i}^{s}\right)\left(\bar{v}_i^{s} + \Delta v_i^{s}\right)g_i = \bar{\lambda}_{e,i}^{s}\bar{v}_i^{s}g_i + \psi_{e,i}^{s}g_i , \\ d_i^{\mathrm{dif}} = \left(\bar{v}_i^{d} + \Delta v_i^{d}\right)c, \end{cases}$$

where $\bar{\lambda}_{i,l}^{f}$, $\bar{v}_l^{f}$ are average values of splits and flow from source f_l toward i-th queue, $\Delta\lambda_{i,l}^{f}$, Δv_l^{f} are deviations of these values. The similar meanings have the rest of values in (7). The additional notations in (7) are

$$\psi_{i,l}^{f} = \Delta\lambda_{i,l}^{f}\overline{v}_{l}^{f} + \overline{\lambda}_{i,l}^{f}\Delta v_{l}^{f} + \Delta\lambda_{i,l}^{f}\Delta v_{l}^{f},\ \psi_{i,j}^{s} = \Delta\lambda_{i,j}^{s}\overline{v}_{j}^{s} + \overline{\lambda}_{i,j}^{s}\Delta v_{j}^{s} + \Delta\lambda_{i,j}^{s}\Delta v_{j}^{s},$$
$$\psi_{r,i}^{h} = \Delta\lambda_{r,i}^{h}\overline{v}_{i}^{s} + \overline{\lambda}_{r,i}^{h}\Delta v_{i}^{s} + \Delta\lambda_{r,i}^{h}\Delta v_{i}^{s},\ \psi_{e,i}^{s} = \Delta\lambda_{e,i}^{s}\overline{v}_{i}^{s} + \overline{\lambda}_{e,i}^{s}\Delta v_{i}^{s} + \Delta\lambda_{e,i}^{s}\Delta v_{i}^{s}.$$

For flows f_l, q_i^{out} the split equalities should be held $\sum_{i=1}^{n^{\text{src}}}\overline{\lambda}_{i,l}^{f} = 1, \sum_{i=1}^{n^{\text{src}}}\Delta\lambda_{i,l}^{f} = 0$,

(8) $$\sum_{r=1}^{n^{\text{snk}}}\overline{\lambda}_{r,i}^{h} + \sum_{e=1,e\neq i}^{n}\overline{\lambda}_{e,i}^{s} = 1,\ \sum_{r=1}^{n^{\text{snk}}}\Delta\lambda_{r,i}^{h} + \sum_{e=1,e\neq i}^{n}\Delta\lambda_{e,i}^{s} = 0.$$

The sign of the value $\overline{v}_i^d$ defines the flow type. If $\overline{v}_i^d$ sign is positive it matches to a source otherwise it matches to a sink. Practically, the value $\overline{v}_i^d$ is hard to be estimated and usually considered as zero.

Using relations (7) and (8) Equation (5) becomes in the form

$$x_i(k+1) = x_i(k) + c\left(\sum_{l=1}^{n^{\text{src}}}\left(\overline{\lambda}_{i,l}^{f}\overline{v}_{l}^{f} + \psi_{i,l}^{f}\right) + \sum_{j=1,\,j\neq i}^{n}\left(\overline{\lambda}_{i,j}^{s}\overline{v}_{j}^{s}\hat{g}_j + \psi_{i,j}^{s}\hat{g}_j\right) - \right.$$
$$\left. - \sum_{r=1}^{n^{\text{snk}}}\left(\overline{\lambda}_{r,i}^{h}\overline{v}_{i}^{s}\hat{g}_i + \psi_{r,i}^{h}\hat{g}_i\right) - \sum_{e=1,e\neq i}^{n}\left(\overline{\lambda}_{e,i}^{s}\overline{v}_{i}^{s}\hat{g}_i + \psi_{e,i}^{s}\hat{g}_i\right) + \left(\overline{v}_i^{d} + \Delta v_i^{d}\right)\right).$$

Applying the notations $\phi_i = \sum_{l=1}^{n^{\text{src}}}\overline{\lambda}_{i,l}^{f}\overline{v}_{l}^{f}$, $\sigma_i = \left(\sum_{r=1}^{n^{\text{snk}}}\overline{\lambda}_{r,i}^{h}\overline{v}_{i}^{s} + \sum_{e=1,e\neq i}^{n}\overline{\lambda}_{e,i}^{s}\overline{v}_{i}^{s}\right) = \overline{v}_i^{s}$,

(see (8)),

$$w_i(k,\hat{\boldsymbol{g}}(k)) = \sum_{l=1}^{n^{\text{src}}}\psi_{i,l}^{f} + \sum_{j=1,\,j\neq i}^{n}\psi_{i,j}^{s}\hat{g}_j - \sum_{r=1}^{n^{\text{snk}}}\psi_{r,i}^{h}\hat{g}_i - \sum_{e=1,e\neq i}^{n}\psi_{e,i}^{s}\hat{g}_i + \Delta v_i^{d}(k),$$

$i \in \{1, 2, ..., n\}$ the final form of (5) becomes

(9) $$x_i(k+1) = x_i(k) + c\left(\sum_{j=1,\,j\neq i}^{n}\overline{\lambda}_{i,j}^{s}\overline{v}_{i,j}^{s}\hat{g}_j - \sigma_i\hat{g}_i + \phi_i + \overline{v}_i^{d}(k) + w_i(k)\right).$$

In (9) the value c may be interpreted as external disturbance. The structure of random values $w_i(k,\hat{\boldsymbol{g}}(k))$ is not trivial. As a kind of approximation, it has been assumed that $w_i(k)$ are normally distributed random numbers, which don't depend from the green time vector $\hat{\mathbf{g}}(k)$.

Equations (9) can be presented in matrix form as

(10) $$\mathbf{x}(k+1) = \mathbf{x}(k) + c\mathbf{S}\hat{\mathbf{g}}(k) + c\mathbf{T} + c\mathbf{D} + c\mathbf{w}(k),$$

where $\mathbf{x}(k) = \left[x_1(k),\ x_2(k),\ ...,\ x_n(k)\right]^{\text{T}}$, and $\mathbf{w}(k) = \left[w_1(k),\ w_2(k),\ ...,\ w_n(k)\right]^{\text{T}}$ are the state vector, and the disturbance vector respectively, $\mathbf{D} = \left[\overline{v}_1^{d},\ \overline{v}_2^{d},\ \dots,\ \overline{v}_n^{d}\right]^{\text{T}}$, $\mathbf{T} = \left[\phi_1,\ \phi_2,\ \dots,\ \phi_n\right]^{\text{T}}$, matrix $\mathbf{S} \in R^{n\times n}$ is defined by the elements $\mathbf{S}_{i,j} = \overline{\lambda}_{i,j}^{s}\overline{v}_{j}^{s}$ when $i \neq j$, and $\mathbf{S}_{i,i} = -\sigma_i$.

Substituting (4) in (10) gives the relation

$$\mathbf{x}(k+1)=\mathbf{x}(k)+c\mathbf{SG}\hat{\mathbf{g}}^e(k)+c\mathbf{SL}+c\mathbf{T}+c\mathbf{D}+c\mathbf{w}(k).$$

The analysis of this formal model will start with lack disturbance or $\mathbf{w}(k)=0$. Thus following [1, 9], there may exist a nominal situation when the input of each queue equals to its output, $\mathbf{x}(k+1)=\mathbf{x}(k)$. Obviously, this situation exists when the nominal green time vector $\hat{\mathbf{g}}^N$ is a unique solution of linear equation

$$c\mathbf{SG}\hat{\mathbf{g}}^N+c\mathbf{N}=0, \tag{11}$$

where $\mathbf{N}=\mathbf{SL}+\mathbf{T}+\mathbf{D}$ comprises network demand vector. Vector $\hat{\mathbf{g}}^N$ is usually associated with fixed control plan.

The unique solution of (11) exists when $\operatorname{rank}(\mathbf{SG})=\operatorname{rank}([\mathbf{SG}\quad \mathbf{N}])\le m$. Practically the last condition is hard to reach. Fortunately, the usual situation is when the vector $\hat{\mathbf{g}}^N$ may be chosen as solution of optimization problem

$$\sum_{i=1,\ \Delta_i>0}^{n}\Delta_i\to\min, \tag{12}$$

where Δ_i is a *i*-th component of vector $\mathbf{\Delta}=\mathbf{SG}\hat{\mathbf{g}}^N+\mathbf{N}$. Components Δ_i may be interpreted as fiction flows which are needed to reach nominal system state. The goal function (12) is a sum of non negative elements (this is an attempt to avoid existence of flows which can lead to links overflow). Then, under the solution $\hat{\mathbf{g}}^N$, which satisfies (12), it is possible to define new additional optimization problem, which will try to reach other targets, for instance to perform additional distribution of the traffic flows on non congested links. The solution $\hat{\mathbf{g}}^N$ is a unique one for the equation $\mathbf{SG}\hat{\mathbf{g}}^N+(\mathbf{N}-\mathbf{\Delta})=\mathbf{0}$ and, according to the existing relation $\hat{\mathbf{g}}^e(k)=\hat{\mathbf{g}}^N+\mathbf{u}(k)$, the general state equation can be rewritten in the form $\mathbf{x}(k+1)=\mathbf{x}(k)+\mathbf{Bu}(k)+\mathbf{Fw}(k)$. The vector $\mathbf{u}(k)=\hat{\mathbf{g}}^e(k)-\hat{\mathbf{g}}^N$, $\mathbf{u}\in R^m$ is a new control variable (green time deviation from the nominal one). To assess this UTN control policy it is necessary to estimate the characteristic of the system about it controllability and observability of the output variables.

Let all vehicle queues are observable and their measurements give the vectors $\mathbf{y}(k), \mathbf{v}(k)$, where $\mathbf{y}(k)$ mean queue lengths and $\mathbf{v}(k)$ mean of the measurement noise. Then measurement equation may be written as $\mathbf{y}(k)=\mathbf{x}(k)+\mathbf{v}(k)$.

Then, as controllable output variables may be chosen $\mathbf{z}(k)=\mathbf{x}(k)$. Under lack of disturbances a value $\tilde{\mathbf{z}}(k)=\mathbf{x}(k+1)-\mathbf{x}(k)=\mathbf{Bu}(k)$ may be interpreted as expected junction flows. This value can be included also as controllable output variable for the UTN but for the sake of simplicity it is omitted here.

Taking the considerations for system observability the final UTN model is represented as a discrete-time linear state-space model in the form

$$\begin{cases}\mathbf{x}(k+1)=\mathbf{Ax}(k)+\mathbf{Bu}(k)+\mathbf{Fw}(k),\ \mathbf{x}(0)=\bar{\mathbf{x}}_0,\\ \mathbf{y}(k)=\mathbf{Cx}(k)+\boldsymbol{v}(k),\\ \mathbf{z}(k)=\mathbf{x}(k),\end{cases} \tag{13}$$

where

(14) $$\mathbf{A} = \mathbf{I}_n,\ \mathbf{B} = c\mathbf{SG} \in R^{n\times m},\ \mathbf{F} = c\mathbf{I}_n,\ \mathbf{C} = \mathbf{I}_n,$$

where $\mathbf{I}_n$ is $n\times n$ identity matrix, $\overline{\mathbf{x}}_0$ is estimation of initial vector.

2. Analysis of the UTN model

The UTN model (13), (14) describes the traffic flows in saturated mode, i.e. when queues are long enough for planned control reaction. Formally it means that an inequality $\mathbf{x}(k+1)-\mathbf{x}(k) \ge \mathbf{Bu}(k)$ should be always held. In the same time the model does not consider the congestion situation. If $\tilde{\mathbf{x}} = \left[\tilde{x}_1, \tilde{x}_2, ..., \tilde{x}_n\right]^{\mathrm{T}}$ denotes capacities of queues, the admissible states of the system require the satisfaction of upper bound constraints, then $0 \le \mathbf{x}(k) \le \tilde{x}$. The control influences should be also constrained and the relative values $\hat{\mathbf{g}}^e(k)$ must belong to $0 \le \hat{\mathbf{g}}^e(k) \le 1$ and for the control influences it takes values $-\hat{\mathbf{g}}^N \le \mathbf{u}(k) \le 1-\hat{\mathbf{g}}^N$.

The random disturbance processes $\mathbf{w}(k) \in R^n$, $\mathbf{v}(k) \in R^n$ are assumed to be uncorrelated to each other and they are Gaussian centered stationary white noises with known covariance matrices

(15) $$E\left\{\mathbf{w}(k)\mathbf{w}^{\mathrm{T}}(k)\right\} = \mathbf{W} > 0,\quad E\left\{\mathbf{v}(k)\mathbf{v}^{\mathrm{T}}(k)\right\} = \mathbf{V} > 0.$$

They explicitly influence the vehicles queue length, according to (9). Thus the covariance matrix **W** in (15) has positive diagonal elements. Matrix **W** also may have cross correlated components. For instance if there is a direct path from i-th to j-th queue, then processes w_i and w_j are correlated because the component $\psi^s_{i,j}$ is included in both values w_i, w_j with different sign. Hence the (i, j) and (j,i) elements of covariance matrix **W** tend to be negative. Also correlations may exist between processes w_i and w_j when l-th and j-th links are the alternative links from another i-th link, if equality (8) hold. These cases hold if the traffic flow is divided not proportionally over a link and the overflow to one direction means underflow to another one direction. The positive definition of the covariance matrix **W** insists this matrix to have relatively significant positive diagonal elements. For the case when **W** is not positive defined it will be proved that the estimation problem has no solution.

The controllability and observability of the system (13) are assessed according to the values of matrices **A, B, C**. The pair $(\mathbf{A}, \mathbf{B})$ is controllable if the matrix $\mathbf{V} = \left[B, AB, ..., A^{n-1}B\right]$ has full rank, $\operatorname{rank}(\boldsymbol{V}) = n$. The pair $(\mathbf{A}, \mathbf{C})$ is observable if the matrix $\mathbf{U} = \left[C;\ CA;\ ...;\ CA^{n-1}\right]$ has also full rank, $\operatorname{rank}(U) = n$. The right part of matrices *V*, *U* mean matrices concatenation by columns and rows respectively.

The analysis of the observability of the pair $(\mathbf{A},\mathbf{C})$ and the controllability of pair $(\mathbf{A},\mathbf{B})$ is performed by analyzing the additional pair $(\mathbf{A},\bar{\mathbf{N}})$, where $\bar{\mathbf{N}}$ is a result of the factorization $\bar{\mathbf{W}}=\mathbf{FWF}^{\mathrm{T}}=c^2\mathbf{W}=\bar{\mathbf{N}}\bar{\mathbf{N}}^{\mathrm{T}}$. Since $\mathbf{A}$ is an identity matrix, the rank of observability and controllability matrices equal to rank of appropriate matrices $\mathbf{C}$, $\bar{\mathbf{N}}$, $\mathbf{B}$. Hence, the pair $(\mathbf{A},\bar{\mathbf{N}})$ could be stabilized if matrix $\mathbf{W}>0$. The observability of pair $(\mathbf{A},\mathbf{C})$ depends on rank of matrix **C**. Generally **C** may be $r\times n$ real matrix. Matrix **C** has full rank, $\mathrm{rank}(\mathbf{C})=n$, when $r\geq n$, and matrix **C** has n linearly independent rows. Technically this relation requires that the number of sensors to measure the queue lengths should be bigger than the number of queues and each queue has to be measured at least with one sensor. For the case if $\mathbf{C}=\mathbf{I}_n$ the pair $(\mathbf{A},\mathbf{C})$ is always observable.

In the case of controllability of matrix pair $(\mathbf{A},\mathbf{B})$, it depends from the rank(**B**). Since the number of queues n is bigger than the number of independent green times *m*, the pair $(\mathbf{A},\mathbf{B})$ is uncontrollable. This conclusion corresponds to the real observation of a network, which is permanently fed with inflows of vehicles. This means that there is no such control policy, which can reduce the vehicles amount in the network till zero valued queues lengths.

Let the number of controllable dimension of the system (13) is $\tilde{m}=\mathrm{rank}(\mathbf{B})\leq m<n$. If matrix $\mathbf{B}$ has rank $\tilde{m}<m$ it is evident that there is no control (a set of effective green times) that can control the traffic. This situation is not admissible and such behavior of the system should be excluded. For the developments below it has been assumed that the matrix $\mathbf{B}$ has full rank and $m=\tilde{m}$.

3. Linear-quadratic-Gaussian design of UTN

3.1. Overview of LQG design

For a given discrete system (13) with the noises $\mathbf{w}(k)$, $\mathbf{v}(k)$ covariance matrices (15) the LQG optimization design finds control $\mathbf{u}(k)$ which leads to minimum of quadratic cost function

(16)
$$J(\mathbf{u},\bar{\mathbf{x}}_0)=E\left\{\sum_{k=0}^{\infty}\mathbf{x}^{\mathrm{T}}(k)\mathbf{Px}(k)+\mathbf{u}^{\mathrm{T}}(k)\mathbf{Ru}(k)\right\}\rightarrow\min .$$

If the system (13) is controllable the performance criteria (16) does not depend from the initial condition $\bar{\mathbf{x}}_0$.

The control influence $\mathbf{u}(k)$ is found in a feedback form

(17)
$$\mathbf{u}(k)=\mathbf{K}_x\hat{\mathbf{x}}(k),$$

where the feedback gain matrix equal to $\mathbf{K}_x=\left(\mathbf{B}^{\mathrm{T}}\mathbf{XB}+\mathbf{R}\right)^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{XA}$ and $\mathbf{X}$ is a stabilizing solution of the Discrete-time Algebraic Riccati Equation (DARE)

(18) $$\mathbf{X} = \mathbf{A}^{\mathrm{T}}\mathbf{X}\mathbf{A} + \mathbf{P} - \mathbf{A}^{\mathrm{T}}\mathbf{X}\mathbf{B}(\mathbf{B}^{\mathrm{T}}\mathbf{X}\mathbf{B} + \mathbf{R})^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{X}\mathbf{A}\,.$$

Vector $\hat{\mathbf{x}}(k)$ in (17) is an estimation of the real state $\mathbf{x}(k)$ and $e(k) = \hat{\mathbf{x}}(k) - \mathbf{x}(k)$ means an estimation error vector. The estimated vector $\hat{\mathbf{x}}(k)$ may be derived as solution of the Kalman estimator state equation

(19) $$\hat{\mathbf{x}}(k+1) = \mathbf{A}\hat{\mathbf{x}}(k) + \mathbf{B}\mathbf{u}(k) + \mathbf{K}_y\left(\mathbf{y}(k) - \mathbf{C}\hat{\mathbf{x}}(k)\right), \quad \hat{\mathbf{x}}(0) = \hat{\mathbf{x}}_0\,,$$

where an estimator gain matrix $\mathbf{K}_y$ is given by $\mathbf{K}_y = \mathbf{AYC}^{\mathrm{T}}(\mathbf{CYC}^{\mathrm{T}} + \mathbf{V})^{-1}$ and $\mathbf{Y}$ is derived as a solution of DARE in the form

(20) $$\mathbf{Y} = \mathbf{AYA}^{\mathrm{T}} + \bar{\mathbf{W}} - \mathbf{AYC}^{\mathrm{T}}(\mathbf{CYC}^{\mathrm{T}} + \mathbf{V})^{-1}\mathbf{CYA}^{\mathrm{T}},$$

where $\bar{\mathbf{W}} = \mathbf{FWF}^{\mathrm{T}}$. Equation (19)is an optimal steady-state Kalman estimator [16] which constructs a state estimate $\hat{\mathbf{x}}(k)$ that minimizes the mean square error $E\left\{e(k)^{\mathrm{T}}e(k)\right\}$ in the time limit $k \to \infty$.

For the given problem the following conditions should be satisfied:

The pair ($\mathbf{A}$, $\mathbf{B}$) must be stabilizable.

The pair ($\mathbf{A}$, $\mathbf{C}$) must be detectable.

The pair $(\mathbf{A}, \bar{\mathbf{N}})$ must be stabilizable, where $\bar{\mathbf{N}}$ corresponds to factorization $\bar{\mathbf{W}} = \bar{\mathbf{N}}\bar{\mathbf{N}}^{\mathrm{T}}$.

3.2. Numerical simulation of UTN LQG design

The numerical simulation considers an area which is a real urban transportation network in Sofia, Bulgaria and is situated along “Yosif Gurko” Street, and crossed by “Vasil Levski” and “Evlogi and Hristo Georgiev” boulevards, Fig. 3.

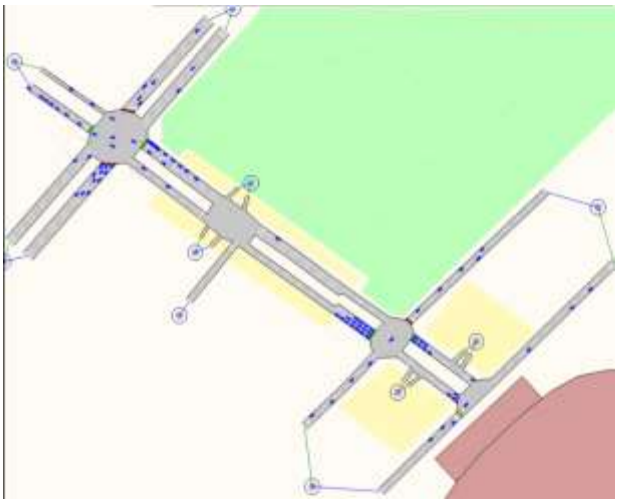

Fig. 3. The Sofia UTN area

The trafic flows on this UTN area are given on Fig. 4.

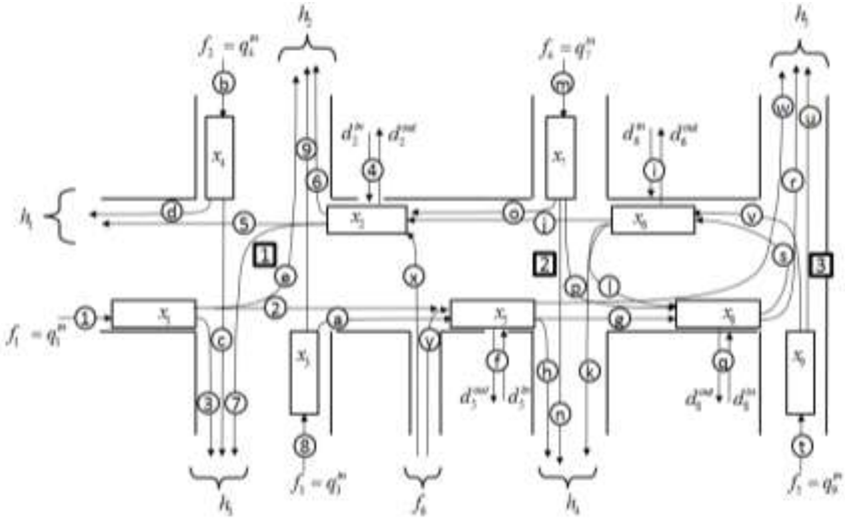

Fig. 4. Traffic flows diagram

This UTN consists of six external sources $n^{\mathrm{src}} = 6$; five vehicles sinks $n^{\mathrm{snk}} = 5$; there are three traffic light controlled junctions $p = 3$, and nine links and corresponding vehicle queues, $n = 9$. The junctions are numbered from one to three with a digit in a rectangle. Every junction controls two traffic lights phases and in total there are six effective green times $g^e_{r,j}$ control influences (index *r* corresponds to the number of phase, *j* correspond to the number of junction, $r = 1, 2$; $j = 1, 2, 3$). The network has 34 paths for vehicle motion from one direction to another. They are marked by symbols $(1-9,\ a-y)$, written in rounds. The cycle time for all junctions is assumed a constant value $c = 60$ s. Table 1 represents the relations in sets $\Omega_{r,j}$ and defines which queues have right of way on *j*-th junction at *r*-th phase during an appropriate effective green time $g^e_{r,j}$.

Table 1. Right of way sets $\Omega_{r,j}$

j / *r*	Junction 1	Junction 2	Junction 3
Phase 1	{1, 2}	{5, 6}	{9}
Phase 2	{3, 4}	{7}	{8}

According to Table 1 the green time vector has components

$$\mathbf{g} = \left[g_1,\ g_2, \ldots,\ g_9\right]^{\mathrm{T}} = \left[g^e_{1,1},\ g^e_{1,1},\ g^e_{2,1},\ g^e_{2,1},\ g^e_{1,2},\ g^e_{1,2},\ g^e_{2,2},\ g^e_{1,3},\ g^e_{2,3}\right]^{\mathrm{T}}. \tag{21}$$

Let $g^e_{1,1}$, $g^e_{1,2}$, $g^e_{2,3}$ are chosen as independent variables and, according to (2) the dependent variables are as follows:

$$g^e_{2,1} = \frac{1}{\left(1+\mu_{2,1}\right)} c - \frac{\left(1+\mu_{1,1}\right)}{\left(1+\mu_{2,1}\right)} g^e_{1,1},$$

$$g_{2,2}^{e}=\frac{1}{\left(1+\mu_{2,2}\right)}c-\frac{\left(1+\mu_{1,2}\right)}{\left(1+\mu_{2,2}\right)}g_{1,2}^{e},\quad g_{1,3}^{e}=\frac{1}{\left(1+\mu_{1,3}\right)}c-\frac{\left(1+\mu_{2,3}\right)}{\left(1+\mu_{1,3}\right)}g_{2,3}^{e}.$$

Junctions 2, 3 are crossed by pedestrian routs and for coefficients $\mu_{r,j}$ it has been assumed values, which corresponds to the pedestrian intensity to appropriate routs $\mu_{1,2}=\mu_{2,2}=\mu_{1,3}=\mu_{2,3}=0.15$, $\mu_{1,1}=\mu_{2,1}=0.05$. Then the relation (21) can be rewritten in form (3) where

$$\mathbf{G}=\begin{bmatrix}1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1\end{bmatrix}^{\mathrm{T}},\ \mathbf{g}^{e}=\left[g_{1,1}^{e},\ g_{1,2}^{e},\ g_{2,3}^{e}\right]^{\mathrm{T}},$$

$$\mathbf{L}=\begin{bmatrix}0 & 0 & \ell_1 & \ell_1 & 0 & 0 & \ell_2 & \ell_2 & 0\end{bmatrix}^{\mathrm{T}},\ \ell_1=0.95,\ \ell_2=0.86.$$

The UTN on morning and evening peaks is oversaturated and this traffic area needs optimization. The traffic conditions were collected partially by natural observing and by using AIMSUN microscopic traffic flow simulator. The UTN observing was evaluated on evening-peak at 6:00 P.M., 8:00 P.M. For this period long vehicle queues persist. The AIMSUN simulations were done under fixed-time signal control settings. The simulation results confirmed the observations and showed that under fixed-time control, long vehicle queues are created at the links. In Table 2 the green times for the different paths are presented.

The estimated traffic data were used for the definition of the UTN model as a discrete-time linear state-space model (13), (14). The system variables and parameters are defined by

(22) $$\mathbf{x}=\left[x_1,\ x_2,\dots,\ x_9\right]^{\mathrm{T}},\ \ \mathbf{u}=\left[u_1,\ u_2,\ u_3\right]^{\mathrm{T}},\ \ \mathbf{w}=\left[w_1,\ w_2,\dots,\ w_9\right]^{\mathrm{T}},$$

(23) $$\mathbf{B}=\begin{bmatrix}-18.84 & -16.17 & 45.43 & 52.7 & 9.27 & 0 & 0 & 0 & 0\\ 0 & 6.49 & 0 & 0 & -20.12 & -9.13 & 15.12 & 1.22 & 0\\ 0 & 0 & 0 & 0 & 0 & 6.39 & 0 & 3.06 & -33.28\end{bmatrix}^{\mathrm{T}}$$

(24) $$\mathbf{y}=\left[y_1,\ y_2,\dots,\ y_9\right]^{\mathrm{T}},\ \ \mathbf{v}=\left[v_1,\ v_2,\dots,\ v_9\right]^{\mathrm{T}},\ \mathbf{A}=\mathbf{C}=\mathbf{I}_9,\ \mathbf{F}=60\mathbf{I}_9,$$

The network flow demand vector is estimated to

$$\mathbf{N}=\begin{bmatrix}0.109 & 0.037 & -0.26 & -0.307 & 0.113 & 0.001 & -0.105 & -0.033 & 0.25\end{bmatrix}^{\mathrm{T}}$$

veh/s. The nominal green times vector $\mathbf{g}^{N}$ is found as a solution of problem (12) and $\mathbf{g}^{N}=\begin{bmatrix}0.35 & 0.5 & 0.5\end{bmatrix}^{\mathrm{T}}$ s. The fiction flows vector is assumed

(25) $$\boldsymbol{\Delta}=\begin{bmatrix}0 & 0 & 0 & 0 & 0 & -0.02 & 0 & 0 & -0.02\end{bmatrix}^{\mathrm{T}}.$$

The numerical solution of LQG traffic optimization problem requires an effective algorithm for solving the problem DARE in (18) and (20). Each DARE problem is defined by quadruple of matrices and for (18) these matrices are $\mathbf{A}, \mathbf{B}, \mathbf{P}, \mathbf{R}$. If existing, the stabilizing solution $\mathbf{X}$ of (18) can be denoted as

$\mathbf{X} = \mathrm{Ric}(\mathbf{A}, \mathbf{B}, \mathbf{P}, \mathbf{R})$. Respectively for DARE (20) the solution is noted as $\mathbf{Y} = \mathrm{Ric}(\mathbf{A}^{\mathrm{T}}, \mathbf{C}^{\mathrm{T}}, \bar{\mathbf{W}}, \mathbf{V})$.

Table 2. Path parameters

Path	1	2	3	4	5	6	7	8	9
Type	$f_{1,1}$	$s_{5,1}$	$h_{3,1}$	d_2^{dif}	$h_{1,2}$	$h_{2,2}$	$h_{3,2}$	$f_{3,3}$	$h_{2,3}$
Average split	0.26	0.63	0.11	0	0.24	0.09	0.01	0.38	0.53
Average flow	0.42	0.42	0.42	0.8	0.8	0.8	0.8	1.21	1.21
Path	a	b	c	d	e	f	g	h	i
Type	$s_{5,3}$	$f_{4,2}$	$h_{3,4}$	$h_{1,4}$	$h_{2,1}$	d_5^{dif}	$s_{8,5}$	$h_{4,5}$	d_6^{dif}
Average split	0.09	1	0.89	0.02	0.01	0	0.06	0.07	0
Average flow	1.21	0.53	0.97	0.97	0.42	0.59	0.59	0.59	0.16
Path	j	k	l	m	n	o	p	q	r
Type	$s_{2,6}$	$h_{4,6}$	$s_{8,6}$	$f_{7,4}$	$h_{4,7}$	$s_{2,7}$	$s_{8,7}$	d_8^{dif}	$h_{5,8}$
Average split	0.89	0.04	0	1	0.8	0.15	0.05	0	0.97
Average flow	0.16	0.16	0.16	0.09	0.25	0.25	0.25	0.05	0.05
Path	s	t	u	v	w	x	y	–	–
Type	$s_{6,8}$	$f_{9,5}$	$h_{5,9}$	$s_{6,9}$	$h_{5,5}$	$f_{2,6}$	$f_{5,6}$	–	–
Average split	0.03	0.43	0.81	0.19	0.44	0.38	0.62	–	–
Average flow	0.05	0.59	0.55	0.55	0.59	0.01	0.01	–	–

For a real case the UTN may be significantly large and the dimensions of the optimization problem will increase. Such large problem requires an effective and fast algorithm for solving the LQG optimization problem. For real time control process, the LQG optimization problem must be solved repetitively by means to implement intelligent urban transportation control policy, which corresponds to real traffic conditions.

For the case of real time LQG problem solution it is proposed here a representation for fast solution of the DARE matrix equations. The representation is derived on the special data of the traffic control problem which allows reduce of the computational workload for solving DARE. The particular case of the quadruple $\mathbf{A} = \mathbf{I}_n, \mathbf{B}, \mathbf{P}, \mathbf{R}$ results in DARE solution

(26) $$\mathbf{X} = \mathbf{P}\left(\frac{1}{2}\mathbf{I}_n + \sum_{k=0}^{n-1} \eta_{n-1-k} \bar{\mathbf{A}}^k\right).$$

where $\bar{\mathbf{A}} = \mathbf{I}_n - \left(\mathbf{I}_n + \frac{1}{4}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{P}\right)^{-1}$; $\eta_m = \gamma_0\delta_m + \gamma_1\delta_{m-1} + \ldots + \gamma_m\delta_0$, $\delta_0 = 1$, $m = 0, 1, \ldots, n-1$; $\delta_1, \delta_2, \ldots, \delta_n$ are coefficients of polynomial $\delta(x) = \det(x\mathbf{I}_n - \bar{\mathbf{A}}) = \sum_{i=0}^{n} \delta_i x^{n-i}$ and $\gamma_r = \frac{1}{\pi}\int_0^1 \{h_r(z) - g_r(z)\}\, d\tau$, $z = -\tau^2$, $r = 0, 1, \ldots, n-1$. The subintegral functions $g_r(x) = \frac{x^r}{\delta(x)}$, $h_r(\xi) = \frac{\xi^{n-r-1}}{\mu(\xi)}$ can be computed with effective recurrent formulas as $g_r(x) = x g_{r-1}(x)$, $r = 1, 2, \ldots, n-1$,

$g_0(x) = \dfrac{1}{\delta(x)}$; $\mu(\xi)$ is a polynomial with flipped to polynomial $\delta(x)$ coefficients, such that $\mu(\xi) = \dfrac{1}{x^n}\delta(x)$, $\xi = \dfrac{1}{x}$.

If the system matrix $\mathbf{A}$ is an identity one, the required conditions for stability and detectability of the system, given in the end of section 0 become sufficient conditions, proved in section 0.

Since the UTN system (13), (14) is not controllable there is no such control law that can make the system asymptotically stable. For that case it is recommended [17] to decompose the original system (13), (14) to m dimensions controllable subsystem, and $(n-m)$ dimensions uncontrollable subsystem.

According to [17], such decomposition may be implemented by linear transformation of state vector $\mathbf{x} = \mathbf{Q}\begin{bmatrix}\mathbf{x}^c \\ \mathbf{x}^u\end{bmatrix}$ with orthogonal matrix $\mathbf{Q}$, $\left(\mathbf{Q}^{-1} = \mathbf{Q}^{\mathrm{T}}\right)$. Such transformation provides new representation of matrix $\mathbf{B} = \mathbf{Q}\begin{bmatrix}\mathbf{B}^c \\ \mathbf{0}\end{bmatrix}$, $\mathbf{B}^c \in R^{m\times m}$. The system is decomposed to two subsystems with formal descriptions

(27) $$\mathbf{x}^c(k+1) = \mathbf{x}^c(k) + \mathbf{B}^c\mathbf{u}(k) + \mathbf{w}^c(k),$$

$$\mathbf{x}^u(k+1) = \mathbf{x}^u(k) + \mathbf{w}^u(k), \quad \begin{bmatrix}\mathbf{w}^c \\ \mathbf{w}^u\end{bmatrix} = \mathbf{Q}^{\mathrm{T}}\mathbf{F}\mathbf{w}.$$

Now, for the controllable subsystem (27) it stabilizing control law is in the feedback form

(28) $$\mathbf{u}(k) = \mathbf{K}_x^c\mathbf{x}^c(k).$$

This feedback control can be numerically found and implemented for the original system (13), (14). From practical reasons it worths to rearrange the sequence of the system states in (13), (14) by means the control policy to serve with priority the most important vehicle queues. Particularly for this research it has been chosen to serve first the links that are close to overflow. The level of the overflow is evaluated by the relative value $\dfrac{\tilde{x}_i - x_i(0)}{\tilde{x}_i}$. Then, as next level of the rearrangement, it is reasonable to serve states that correspond to big values of the fiction positive flows Δ_i (these states are tend to become overflowed).

Formally, the reordering of the vector elements is the result of vector multiplication with some permutation matrix, $\mathbf{x} = \mathbf{M}\mathbf{x}^n$. The appropriate permutation should be also performed for matrix $\mathbf{B}$. For the formal descriptions to keep simplicity for the notations matrix $\mathbf{B}$ will be used, but one have to consider that the real content is $\mathbf{M}^{\mathrm{T}}\mathbf{B}$, or $\mathbf{B} \leftarrow \mathbf{M}^{\mathrm{T}}\mathbf{B}$.

Let the initial state of traffic UTN model (13), (22)-(24) is estimated to $\overline{\mathbf{x}}_0 = \begin{bmatrix}5 & 16 & 35 & 35 & 14 & 6 & 25 & 4 & 30\end{bmatrix}^{\mathrm{T}}$. Applying the rearrangement strategy the permutation matrix $\mathbf{M}$ is equal to 9×9 identity matrix with reordered columns $\{4, 3, 9, 7, 2, 5, 6, 1, 8\}$.

Then, by using a standard **QR** decomposition procedure the new form of matrix $\mathbf{B}=\mathbf{Q}\begin{bmatrix}\mathbf{B}^c\\ \mathbf{0}\end{bmatrix}$ is found, $\mathbf{B}^c \in R^{m\times m}$. The corresponding subsystem (27) contains

$$\mathbf{B}^c = \begin{bmatrix} -74.46 & 3.91 & 0 \\ 0 & -27.30 & 2.002 \\ 0 & 0 & 33.97 \end{bmatrix}.$$

The solution of an optimal control problem, defined with (27) and performance index

$$J(\mathbf{u}) = E\left\{\sum_{k=0}^{\infty} \mathbf{x}^{c\mathrm{T}}(k)\mathbf{P}^c\mathbf{x}^c(k) + \mathbf{u}^{\mathrm{T}}(k)\mathbf{R}\mathbf{u}(k)\right\} \to \min,$$

$\mathbf{P}^c = \mathbf{I}_3$, $\mathbf{R} = 10^3 \mathrm{diag}([10 \quad 0.9 \quad 1])$ gives feedback matrix

$$\mathbf{K}_x^c = \begin{bmatrix} -0.0069 & -0.0007 & 0 \\ 0.001 & -0.021 & 0.0009 \\ 0 & 0.0003 & 0.018 \end{bmatrix}. \quad (29)$$

Assuming system response as the total values of the vehicles in the UTM, $y(k) = \sum_{i=1}^{n} x_i(k)$, the evaluated feedback control (28), (29) implemented to the original system (13), (22)-(24) gives system response which is presented in Fig. 5 (in blue dashed line). The black solid line describes the system response with absence of fiction flow (25). The number of samples means the number of traffic light cycle, applied on the network.

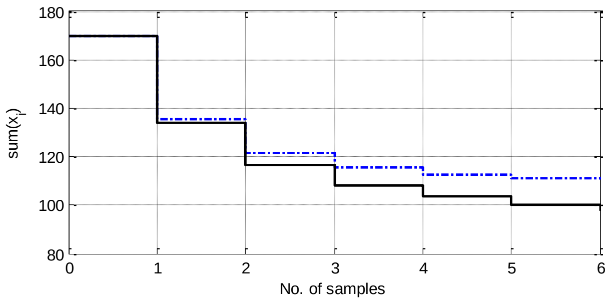

Fig. 5. Vehicles in the queues

For these both cases the reduction of total vehicles amount in the queues equals to 35% and 42% respectively in comparison with the lack of control and fixed time phase durations (currently established). The illustrations of control influences are given in Fig. 6. As it is seen, control values satisfy upper valued constraints.

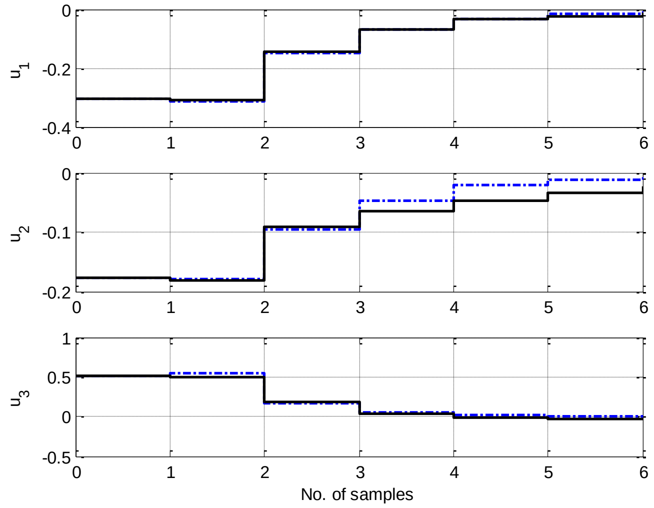

Fig. 6. Control law

Further the control policy has been complicated by introducing noise in the system estimation and measurements. For that case an additional optimal filtering problem is solved. Taking into account the particular topology of the UTM it has been assumed that the queue lengths on sixth and the eighth network links are measured without noise (due to the short lengths of these links). Thus the random components v_6, v_8 have zero standard deviation. For the estimation and measurement processes $\mathbf{w}(k)$, $\mathbf{v}(k)$ it was defined the following covariance matrices

$$\mathbf{W} = \frac{1}{c^2}\begin{bmatrix} 9 & 0 & 0 & 0 & -2.4 & 0 & 0 & 0 & 0 \\ 0 & 16 & 0 & 0 & 0 & -1.2 & -3.2 & 0 & 0 \\ 0 & 0 & 36 & 0 & -4.8 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 36 & 0 & 0 & 0 & 0 & 0 \\ -2.4 & 0 & -4.8 & 0 & 16 & 0 & 0 & -1.2 & 0 \\ 0 & -1.2 & 0 & 0 & 0 & 2.25 & 0 & -0.45 & -1.2 \\ 0 & -3.2 & 0 & 0 & 0 & 0 & 16 & -1.2 & 0 \\ 0 & 0 & 0 & 0 & -1.2 & -0.45 & -1.2 & 2.25 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1.2 & 0 & 0 & 16 \end{bmatrix},$$

$$\mathbf{V} = \mathrm{diag}\left(\begin{bmatrix} 4 & 9 & 16 & 16 & 9 & 0 & 4 & 0 & 4 \end{bmatrix}\right).$$

Matrix $\mathbf{V}$ is not positive defined and respectively the estimation problem is singular. To make the problem nonsingular it has been excluded the sixth and eighth state vector components from the UTN model. Applying Kalman filtering reduction technique [16], the new reduced-order system has a corresponding state vector $\mathbf{x} = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_7 & x_9 \end{bmatrix}^{\mathrm{T}}$ which has to be estimated. Using the particular form of matrices $\mathbf{A} = \mathbf{C} = \mathbf{I}_7$ the modifies covariance matrices are obtained

$$\bar{\mathbf{W}} = \begin{bmatrix} 9 & 0 & 0 & 0 & -2.4 & 0 & 0 \\ 0 & 16 & 0 & 0 & 0 & -3.2 & 0 \\ 0 & 0 & 36 & 0 & -4.8 & 0 & 0 \\ 0 & 0 & 0 & 36 & 0 & 0 & 0 \\ -2.4 & 0 & -4.8 & 0 & 16 & 0 & 0 \\ 0 & -3.2 & 0 & 0 & 0 & 16 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 16 \end{bmatrix},$$

$$\mathbf{V} = \mathrm{diag}\left(\begin{bmatrix} 4 & 9 & 16 & 16 & 9 & 4 & 4 \end{bmatrix}\right).$$

The solution of the Kalman estimator problem gives gain matrix

$$\mathbf{K}_y = \begin{bmatrix} 0.74 & 0 & -0.0018 & 0 & -0.021 & 0 & 0 \\ 0 & 0.70 & 0 & 0 & 0 & -0.042 & 0 \\ -0.0073 & 0 & 0.74 & 0 & -0.042 & 0 & 0 \\ 0 & 0 & 0 & 0.75 & 0 & 0 & 0 \\ -0.047 & 0 & -0.023 & 0 & 0.7 & 0 & 0 \\ 0 & -0.018 & 0 & 0 & 0 & 0.82 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.82 \end{bmatrix}.$$

On Fig. 7 are presented the numerical results for the estimated queue lengths measurements under fixed time control strategy. The notation $e_i^v(k)$ represents the measurement error and $e_i(k)$ is the estimated error of the filtered data for the vehicle queues.

The results on Fig. 7 illustrate that the noise level has been considerably reduced by applying the filtering strategy. The standard deviations of error vector $\mathbf{e}^v = \begin{bmatrix} e_1^v & e_2^v & e_3^v & e_4^v & e_5^v & e_7^v & e_9^v \end{bmatrix}^{\mathrm{T}}$ are {1.99, 2.99, 3.99, 4.00, 2.99, 1.99, 1.99} while the standard deviations of estimation error vector $\mathbf{e} = \begin{bmatrix} e_1 & e_2 & e_3 & e_4 & e_5 & e_7 & e_9 \end{bmatrix}^{\mathrm{T}}$ are {1.54, 2.22, 3.08, 3.1, 2.21, 1.67, 1.67}. These values are quit low, which proves the efficiency of the filtering optimization. The improvement of estimation process in comparison with the lack of optimization and filtering traffic control is $\{23, \ 26, \ 23, \ 23, \ 26, \ 16, \ 16\}$ percentages respectively.

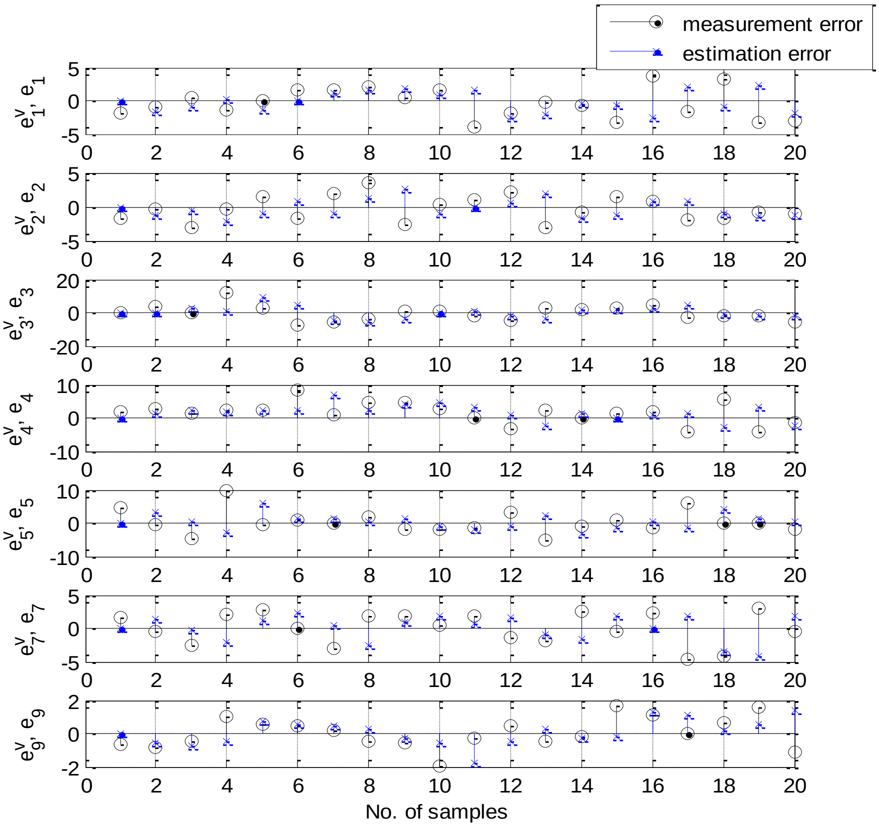

Fig. 7. Values of measured and estimated data

Finally, according to separation technique [16], the solution of LQG problem is a combination of solutions of optimal observer design and optimal linear state feedback control design. The implementation of the feedback control law (28), (29), $\mathbf{x}^c = \mathbf{Q}^{\mathrm{T}}\hat{\mathbf{x}}$, derived by separation, compared with the original system (13), (22)-(24) gives approximately the same result, illustrated on Fig. 5.

4. Conclusion

This research formalizes a traffic-responsive urban control in the form of linear-quadratic regulator. It has been defined a LQG optimization problem of UTN with explicit description of random processes in the system behavior. Some results are achieved, which increase the application domain of the classical control to store-and-forward model but with explicit inclusion of random processes for state estimation and measurements.

Particular solutions have been tested, when the initial model is not stable and additional one in form (12) is introduced. This allows obtaining stable discrete-time linear state-space problem solution. An extension for the case of design of state estimator is presented for the case of not controllable unstable dimension of the system(13).

The LQG control problem has been modified using the special structure of UTN. This modification results in numerical implementation of fast algorithms for solving the initial LQG optimization problem using relation (26) .

The formal UTM model and control policies were numerically tested on real network from Sofia city. The obtained results give benefit to the implementation of the LQG optimization in comparison to the fixed time plan of traffic control, which is currently established.

For completeness of this research some perspectives and problems are mentioned here, which need additional considerations. Currently the separation of the original system (13), (14) to controllable and uncontrollable subsystems theoretically may not give the minimum of system performance index (16). But nevertheless such separation, the solution of a linear-quadratic control problem always gives optimal solutions via terms of controllable subsystem performance index.

A perspective direction for future extension of this approach is also the assumption for dependences between the external disturbance $w_i(k)$ and the control values of green time vector $\hat{\mathbf{g}}(k)$. Such assumption can influence the precise estimation of the expected junction flows $\tilde{\mathbf{z}}(k)$, which are not currently optimized by the LQG problem. These considerations need additional developments because there are potentials for improving the presented traffic response control policy with the application of LQG control.

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