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ROBUST EXTREMUM SEEKING FOR A SECOND ORDER UNCERTAIN PLANT USING A SLIDING MODE CONTROLLER

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This paper suggests a novel continuous-time robust extremum seeking algorithm for an unknown convex function constrained by a dynamical plant with uncertainties. The main idea of the proposed method is to develop a robust closed-loop controller based on sliding modes where the sliding surface takes the trajectory around a zone of the optimal point. We assume that the output of the plant is given by the states and a measure of the function. We show the stability and zone-convergence of the proposed algorithm. In order to validate the proposed method, we present a numerical example.

Keywords: convex optimization, extremum seeking, continuous-time gradient algorithm, dynamical constrained optimization, unknown functions.

1. Introduction

Run-time optimization requires estimation of the gradient. For solving this problem, we consider the synchronous detection method (SDM) proposed by Armstrong (1914). Raju and Rao (2009), Ulusoy *et al*. (2011), as well as Montesinos-García and Martínez-Guerra (2017) employed this concept in communication systems, modulation/demodulation signals and security. Jignesh *et al*. (2013) use the SDM for satellite communications.

In the control and optimization arena, see the work of Nana *et al*. (2012) for robust and optimal controllers, and Liu *et al*. (2015), who presented a real-time control and optimization framework for embedded systems. Related work on building run-time robust optimization framework algorithms and computational procedures has been reported in the literature. In the robust optimization systems area, Apkarian and Tuan (2000) proposed several applications to local and global robust optimization based on an LMI characterization considering additional algebraic constrains. Dimitrova and Krastanov (2009) considered a nonlinear model of a biological wastewater treatment process, based on two microbial populations and two substrates. Ghadimi and Lan (2012) suggested an optimal algorithm for stochastic strongly convex optimization considering stochastic and probabilistic uncertainties. An application to photovoltaic systems which maximizes the power of the system is given by Bazzi and Krein (2011). Zhang and Ordóñez (2012) set forth a numerical optimization based approach for extremum seeking control. Sahneh *et al*. (2012) introduced a time-varying extremum point optimization algorithm. Cassandras and Lin (2013) presented an optimization approach for multi-agent persistent monitoring systems with performance constraints.

For second order systems, qun Mei (2013) presented an optimal control based on Fourier transform. Wang *et al*. (2014) described an extremum seeking approach with an accelerating convergence parameter. For unknown maps, Mills and Krstic (2015) presented an algorithm to obtain a derivative estimation employing an extremum

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seeking technique. Also, Sarkar *et al*. (2017) introduced a robust and optimal controller based on sliding modes for event triggered systems. Solis *et al*. (2018b) proposed a continuous-time extremum seeking algorithm with function measurements disturbed by a stochastic noise. Solis *et al*. (2018a) suggested some advances for constrained stochastic static maps optimization, and presented a continuous-time optimization method for an unknown convex function restricted to a dynamic plant with an available output including a stochastic noise (Solis *et al*., 2019).

To realize an effective on-line optimization, controlling measurable output of an uncertain dynamic plant, the *sliding mode control* (SMC) technique is suggested to be applied in this paper. The advantages of SMC approach being compared with other methods are as follows:

- SMC technique does not require any exact description of the dynamic model of the controlled plant: neither the structure nor exact values of the parameters are required, only some prior upper estimate of the acceleration ($\ddot{x}$) is used during the controller designing process;
- the right-hand side of the dynamic equation may also include an external bounded-noise perturbations whose effect can be effectively dismissed by the corresponding sliding-mode controller.

For the description of SMC theory, see the works of Bartoszewicz and Leśniewski (2014), Davila and Poznyak (2010), Eichfelder *et al*. (2017), Liu *et al*. (2018) and Poznyak (2018). Perruquetti and Barbot (2002) introduced basic theory and applications on SMC for systems in engineering. Shtessel *et al*. (2014) suggested a theoretical design of controllers and observers based on sliding modes with some practical applications. Shi *et al*. (2006) presented an interesting application of the SMC to stochastic jump systems with repercussions in engineering. Alnejaili *et al*. (2015) and Belkaid *et al*. (2016) showed important applications of SMC to renewable power systems and a solar panel with battery storage.

The *main contributions* of the paper are as follows:

(i) a continuous-time robust extremum seeking algorithm for a strongly convex functions are suggested;

(ii) a wide class of unknown function which are twice differentiable with a bounded Hessian is considered;

(iii) uncertain dynamic systems, which may include bounded uncertainties are admitted to be considered as an optimization instrument;

(iv) stability analysis is realized and zone-convergence of the proposed algorithm is shown.

In order to validate the contribution of this paper, we developed two numerical examples. The first example is a generic dynamic system, and the second example is a simplified optimal mechanical control system for a solar panel.

The paper is organized as follows. The next section presents the problem formulation. Section 3 suggests the robust extremum seeking algorithm. We proved the stability and the convergence of the proposed method in Section 4. Section 5 presents the numerical examples (generic plant and simplified mechanical control system for a solar panel). Section 6 concludes the paper.

2. Problem formulation

Consider the problem of an approximate extremum seeking of an unknown (but measurable) function by the state variation of a second order dynamic plant, which may contain both bounded uncertainties and disturbances in the description of its model. Formally, it is formulated as follows: Design a control $\mathbf{u} \in \mathbb{R}^n$ based on the system dynamics

$$\begin{cases} \dot{\mathbf{x}}_1 = \mathbf{x}_2, \\ \dot{\mathbf{x}}_2 = g(\mathbf{x}_1, \mathbf{x}_2, t) + \mathbf{u}, \\ y(\cdot) = f(\cdot), \end{cases} \tag{1}$$

which asymptotically tends to a small $\varkappa$-neighborhood ($\varkappa$ should be estimated) of the extremal point $\mathbf{x}_1^* = \arg_{\mathbf{x}_1 \in \mathbb{R}^n} \min f(\cdot)$ of the optimized function $f : \mathbb{R}^n \to \mathbb{R}$, namely,

$$\mathbf{x}_1(t) \underset{t\to\infty}{\longrightarrow} X_\varkappa := \{\mathbf{x}_1 \in \mathbb{R}^n \mid \|\mathbf{x}_1 - \mathbf{x}_1^*\| \le \varkappa\} \tag{2}$$

with the following assumptions:

1. $\mathbf{x}_1,\ \mathbf{x}_2 \in \mathbb{R}^n$ the known states of the plant and $\mathbf{u} \in \mathbb{R}^n$ the input control;
2. $g : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ defined by $g(\mathbf{x}_1, \mathbf{x}_2, t)$ is a smooth function that represents the description of the the dynamical plant with nonmodeled uncertainties, which satisfies
$$\|g(\mathbf{x}_1, \mathbf{x}_2, t)\| \le c_0 + c_1\|\mathbf{x}_1\| + c_2\|\mathbf{x}_2\|$$
for some known constants $c_0,\ c_1,\ c_2 > 0$;
3. $f : \mathbb{R}^n \to \mathbb{R}$ is an unknown twice differentiable strongly convex function such that $h_-\mathbb{I} < \nabla^2 f(\mathbf{x}_1) < h_+\mathbb{I}$ for some known constants $h_+,\ h_- > 0$,
4. $y(\cdot) \in \mathbb{R}^n$ is a measurable output signal about the aforementioned function.

The problem considers a continuous-time strongly convex unknown function $f(\cdot)$ with bounded dynamical constraints including uncertainties. We shall propose a robust closed-loop controller that stabilizes the system around an optimal convergence zone.

In the next section the method of on-line gradient estimation, will be considered based only on function measurements. The functionality of the suggested robust extrema seeking method will be also discussed.

3. Robust extremum seeking

Let x be a $\mathbb{R}^n$ vector. We define $\text{sign}(x) := [\text{sign}(x_m)]_m$ with $m \in [\![1, n]\!]$.

The suggested algorithm to solve the problem (1) is as follows:

$$\begin{cases} \hat{\mathcal{D}} = \frac{1}{\alpha}\zeta_t y\left(\mathbf{x}_1 + \alpha\zeta_t\right), \\ \dot{\mathbf{J}}_o = \beta\left(\hat{\mathcal{D}} - \mathbf{J}_o\right) \\ \dot{\mathbf{J}} = \beta\left(\mathbf{J}_o - \mathbf{J}\right), \\ \\ \mathbf{u} = -\mu\dot{\mathbf{J}} - U_{\max}\left[c_0 + c_1\left\|\mathbf{x}_1\right\|\right. \\ \left.+c_2\left\|\mathbf{x}_2\right\|\right]\text{sign}\left(\mu\mathbf{J} + \mathbf{x}_2\right) \end{cases} \tag{3}$$

with the following assumptions:

(i) constants $\beta,\ \mu,\ \alpha > 0$ and $U_{\max} \geq 1$,

(ii) the auxiliary states $\hat{\mathcal{D}},\ \mathbf{J},\ \mathbf{J}_0 \in \mathbb{R}^n$,

(iii) the dither signal ζ_t is defined as follows:

$$\zeta_t := \left[\sin\left(\omega_0 t\right), \sin\left(2\omega_0 t\right), \ldots, \sin\left(n\omega_0 t\right)\right]^{\mathsf{T}}, \tag{4}$$

where $\omega_0 > 0$ is sufficiently large. The dither signal ζ_t is defined as in (4) to satisfy the orthogonality properties of the system and to obtain a correct representation for the time-dependent functions. Figure 1 shows the block structure of the proposed algorithm given in (3).

3.1. Gradient estimation. For simplicity, in this section, we consider the time-dependent vector $x_t := \mathbf{x}_1$ and we introduce the following definition and notation.

Definition 1. (*Fourier transform*) For a time-dependent integrable real value function $h(t)$, the Fourier transform, denoted by $\overline{h(t)}(\omega)$ in the frequency domain is given by

$$\overline{h(t)}(\omega) := \int_{-\infty}^{\infty} h(t)\exp(-i\omega t)\ \mathrm{d}t. \tag{5}$$

Lemma 1. (Stade, 2005) *For an integrable real-valued function $h(t)$ and for some constant $\omega_0 > 0$ we have that*

$$\overline{\sin(n\omega_0 t)\, h(t)}(\omega) = \frac{1}{2i}\left(\overline{h(t)}(\omega + n\omega_0) - \overline{h(t)}(\omega - n\omega_0)\right), \tag{6}$$

$$\begin{aligned} &\overline{\sin(n\omega_0 t)\sin(m\omega_0 t)\, h(t)}(\omega) \\ &\quad = \frac{1}{4}\Big(\overline{h(t)}(\omega - (m-n)\omega_0) \\ &\qquad + \overline{h(t)}(\omega - (n-m)\omega_0) \\ &\qquad - \overline{h(t)}(\omega - (n+m)\omega_0) \\ &\qquad - \overline{h(t)}(\omega + (m+n)\omega_0)\Big). \end{aligned} \tag{7}$$

In particular, we have that

$$\begin{aligned} &\overline{\sin^2(n\omega_0 t)\, h(t)}(\omega) \\ &\quad = \frac{1}{2}\overline{h(t)}(\omega) \\ &\qquad - \frac{1}{4}\left(\overline{h(t)}(\omega - 2n\omega_0) + \overline{h(t)}(\omega + 2n\omega_0)\right). \end{aligned} \tag{8}$$

Notice that if Eqn. (6) for $\omega_0 > 0$ the spectrum of $h(t)$ is displaced $\pm\omega_0$ around central frequency and in (8) there is a spectral component around the origin. This property will be used to separate the spectrum of the gradient estimation signal from other components.

Definition 2. (*Heaviside function*) Let us define the complex value Heaviside function as follows:

$$\text{Heaviside}(z) := \begin{cases} 1, & \Re(z) > 0, \\ 0, & \Re(z) \leq 0, \end{cases}$$

where $\Re(z)$ is the real part of $z \in \mathbb{C}$. This definition can be extended to a vector form as follows: let z be a complex vector in $\mathbb{C}^n$; then $\text{Heaviside}(z) := [\text{Heaviside}(z_m)]_m$ for every entry $m \in [\![1, n]\!]$.

Proposition 1. (Frequency analysis for the estimator) *Let x_t be a time-dependent vector in $\mathbb{R}^n$. Consider a function $f(\cdot)$ given in (1) which admits a Fourier transform given by $\overline{f(\cdot)}(\omega)$. On the above assumptions, let ω_d be the bandwidth of the gradient $\nabla f(\cdot)$. Then for some small $\alpha > 0$ and a sufficiently large dither frequency ω_0 of the dither signal ζ_t, we have that*

$$\overline{\frac{1}{\alpha}\zeta_t f(x_t + \alpha\zeta_t)}(\omega) \cdot H(\omega) = \frac{1}{2}\overline{\nabla f(\epsilon_t)}(\omega), \tag{9}$$

where $\epsilon_t = a_t x_t + (1 - a_t)(x_t + \alpha\zeta_t)$ for some $a_t \in [0, 1]$ and

$$H(\omega) := \text{Heaviside}(\omega + \omega_d) - \text{Heaviside}(\omega - \omega_d) \tag{10}$$

with $0 < \omega_d \ll \omega_0$.

Proof. Recall the mean value theorem:

$$f(x_t + \alpha\zeta_t) - f(x_t) = (\alpha\zeta_t)^{\mathsf{T}}\nabla f(\epsilon_t), \tag{11}$$

where ϵ_t is given as in the formulation of the proposition. Rewriting (11) and pre-multiplying it by $\frac{1}{\alpha}\zeta_t$, we obtain

$$\frac{1}{\alpha}\zeta_t f(x_t + \alpha\zeta_t) = \frac{1}{\alpha}\zeta_t f(x_t) + \zeta_t\zeta_t^{\mathsf{T}}\nabla f(\epsilon_t).$$

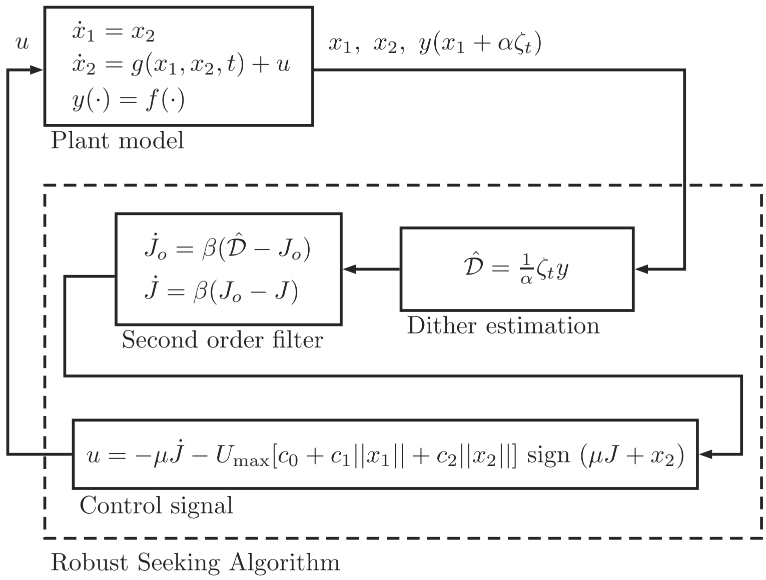

Fig. 1. Block diagram structure for the algorithm.

Apply the Fourier transform:

$$\begin{aligned}&\overline{\frac{1}{\alpha}\zeta_t f\left(x_t + \alpha\zeta_t\right)}\left(\omega\right)\\ &\qquad = \frac{1}{\alpha}\overline{\zeta_t f\left(x_t\right)}\left(\omega\right) + \overline{\zeta_t\zeta_t^{\intercal}\nabla f\left(\epsilon_t\right)}\left(\omega\right).\end{aligned}$$

This is an $\mathbb{R}^n$ vector for any entry $m \in [\![1, n]\!]$. Then applying (6) to entry m for the first sum we obtain

$$\begin{aligned}&\left[\overline{\zeta_t f\left(x_t\right)}\left(\omega\right)\right]_m\\ &\quad = \frac{1}{2i}\left(\overline{f\left(x_t\right)}\left(\omega + m\omega_0\right) - \overline{f\left(x_t\right)}\left(\omega - m\omega_0\right)\right).\end{aligned}$$

For the second sum we apply (7) and (8) to the entry m. From the properties of the matrix multiplication we obtain

$$\begin{aligned}&\left[\overline{\zeta_t\zeta_t^{\intercal}\nabla f\left(\epsilon_t\right)}\left(\omega\right)\right]_m\\ &\quad = \frac{1}{4}\sum_{j=1}^{n}\left(\left[\overline{\nabla f\left(\epsilon_t\right)}\left(\omega - \left(m - j\right)\omega_0\right)\right]_j\right.\\ &\qquad + \left[\overline{\nabla f\left(\epsilon_t\right)}\left(\omega - \left(j - m\right)\omega_0\right)\right]_j\\ &\qquad - \left[\overline{\nabla f\left(\epsilon_t\right)}\left(\omega - \left(j + m\right)\omega_0\right)\right]_j\\ &\qquad \left. - \left[\overline{\nabla f\left(\epsilon_t\right)}\left(\omega + \left(m + j\right)\omega_0\right)\right]_j\right).\end{aligned}$$

Consider an ideal low pass filter with a transfer function given by the Heaviside function Eqn. (10) with $\omega_0 \gg \omega_d$. Then the product of this Heaviside function with the Fourier transform yields a cutting frequency resultant and we obtain (9). ■

Remark 1. Observe that (9) represents the frequency spectrum of the gradient $\nabla f\left(\cdot\right)$ with some error introduced by ϵ_t. Notice that $H\left(\omega\right)$ is a transfer function for an ideal filter implying that it is not realizable. Then, for practical considerations it is important to design a low-pass filter with a flat response and a minimum phase change in the output.

In order to illustrate the previous remark, calculate the Taylor expansion of $f\left(x_t + \alpha\zeta_t\right)$. Taking into account (9), after some algebra we have that

$$\nabla f\left(\epsilon_t\right) = \nabla f\left(x_t\right) + o\left(\alpha^2\right).$$

Now, consider the realizable second order filter given by

$$\begin{cases}\dot{\mathbf{J}}_o = \beta\left(\hat{\mathcal{D}} - \mathbf{J}_o\right),\\ \dot{\mathbf{J}} = \beta\left(\mathbf{J}_o - \mathbf{J}\right).\end{cases} \tag{12}$$

It is possible to analyze every entry as a transfer function as follows:

$$\frac{\left[\overline{\mathbf{J}}\right]_m}{\left[\overline{\hat{\mathcal{D}}}\right]_m} = \frac{1}{\left(\frac{1}{\beta}i\omega + 1\right)^2}.$$

Then we have that

$$\left[\overline{\mathbf{J}}\right]_m = \frac{\left[\overline{\frac{1}{\alpha}\zeta f\left(x_t + \alpha\zeta_t\right)}\right]_m\left(\omega\right)}{\left(\frac{1}{\beta}i\omega + 1\right)^2}.$$

Clearly, it is not possible to obtain an analytic inverse of the Fourier transform. However, we can consider the limit points to approximate the expression. Then, remembering

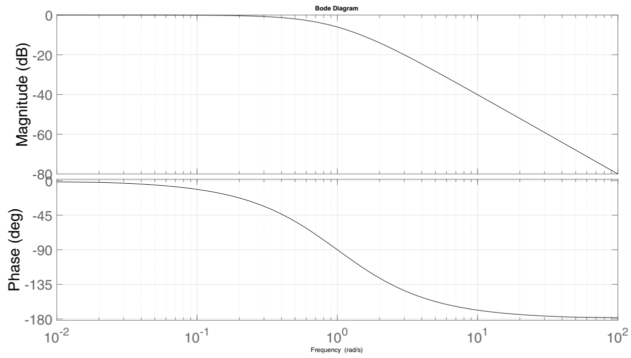

Fig. 2. Bode diagram design.

that $\nabla f(\cdot)$ is a limited bandwidth ω_d, it is possible to choose $\beta > 0$ such that for $\omega \geq 0$,

$$[\overline{\mathbf{J}}]_m \approx \begin{cases} \overline{\frac{1}{\alpha}\zeta_t f(x_t + \alpha\zeta_t)}(\omega) \\ \cdot H(\omega), & \omega \leq \omega_d \ll \beta \ll \omega_0, \\ \\ 0, & \omega \gg \beta, \end{cases}$$

Figure 2 illustrates a method to tune the filter, where β is the cutting angular frequency. For practical purposes it is common to use $\beta = 10\omega_d = 0.1\omega_0$. It is possible to design high order filters different from the proposed filter, but the delay group time increases implying a general delay in the global system. Then the output of the filter is as follows:

$$\mathbf{J} = \nabla f(x_t) + o(\alpha^2). \tag{13}$$

3.2. Proof of stability and of convergence.

Lemma 2. (Surface stability) *With the assumptions and notation given in (1) and (3) the dynamics $\dot{\mathbf{x}}_1 = -\mu\mathbf{J}$ is stable in a $\varkappa$-zone around the optimal point $\mathbf{x}_1^*$, that is,*

$$\limsup_{t\to\infty} \|\mathbf{x}_1^* - \mathbf{x}_1(t)\| \leq \varkappa := \frac{h_+}{h_-^2}\left|o(\alpha^2)\right|. \tag{14}$$

Proof. For notational simplicity we employ the following symbols:

$$\mathcal{D} := \nabla f(\mathbf{x}_1),$$
$$\mathcal{H} := \nabla^2 f(\mathbf{x}_1).$$

Consider the Lyapunov function

$$V := \mathcal{D}^\intercal\mathcal{D}.$$

Taking the time-derivative, we get

$$\dot{V} = 2\mathcal{D}^\intercal\mathcal{H}\dot{\mathbf{x}}_1 = -2\mu\mathcal{D}^\intercal\mathcal{H}\mathbf{J}$$

and in view of $\mathbf{J} = \mathcal{D} + o(\alpha^2)$ and by Eqn. (13) we have

$$\begin{aligned}\dot{V} &= -2\mu\mathcal{D}^\intercal\mathcal{H}\mathcal{D} - 2\mu\mathcal{D}^\intercal\mathcal{H}o(\alpha^2) \\ &\leq -2\mu h_- \|\mathcal{D}\|^2 + 2\mu h_+ \|\mathcal{D}\| \left|o(\alpha^2)\right|.\end{aligned}$$

For some $\epsilon > 0$ (by Λ-inequality)

$$\begin{aligned}\dot{V} &\leq -2\mu h_- \|\mathcal{D}\|^2 + \mu h_+ \left(\epsilon\|\mathcal{D}\|^2 + \epsilon^{-1}\left|o(\alpha^4)\right|\right) \\ &= -\mu\|\mathcal{D}\|^2(2h_- - h_+\epsilon) + \mu h_+\epsilon^{-1}\left|o(\alpha^4)\right|.\end{aligned}$$

Then, choosing $0 < \epsilon < 2h_-/h_+$ and renaming $\rho := \mu(2h_- - h_+\epsilon)$ and $\epsilon_0 := \mu h_+\epsilon^{-1}\left|o(\alpha^4)\right|$, we have that

$$\dot{V} \leq -\rho V + \epsilon_0, \tag{15}$$

implying

$$\limsup_{t\to\infty} V \leq \frac{\epsilon_0}{\rho} = \frac{h_+\left|o(\alpha^4)\right|}{(2h_- - h_+\epsilon)\epsilon}.$$

Notice that if we select $\epsilon = h_-/h_+$, we obtain the following minimum bound:

$$\limsup_{t\to\infty} V \leq \frac{h_+^2}{h_-^2}\left|o(\alpha^4)\right|.$$

Returning to the original notation, this implies

$$\limsup_{t\to\infty} \|\nabla f(\mathbf{x}_1)\| \leq \frac{h_+}{h_-}\left|o(\alpha^2)\right|.$$

Applying Cauchy's mean value theorem to the gradient and taking account of the fact that $\nabla f(\mathbf{x}_1^*) = 0$, we have

$$\nabla f(\mathbf{x}_1) = \nabla^2 f((c-1)\mathbf{x}_1 + c\mathbf{x}_1^*)(\mathbf{x}_1^* - \mathbf{x}_1),$$

where $c \in [0, 1]$. Majoring by taking the norm and the assumptions for $f(\cdot)$ yields (14). This completes the proof. ■

Remark 2. The following facts are easy to check:

(i) Substituting the value ϵ in ρ, we obtain $\rho = \mu h_-$ ($\mu > 0$) which controls the speed of convergence. Now, for (15) it is possible to solve the inequality

$$V(t) \leq \frac{\epsilon_0}{\mu h_-} + V(0)\exp(-\mu h_- t).$$

(ii) If we choose a very small parameter $\alpha > 0$, it is possible to guarantee the convergence to a small zone. It is important to note that the term $\hat{\mathcal{D}}$ can present an undesirable numerical singularity.

Theorem 1. (Global stability and convergence) *Under the above assumptions the optimal seeking algorithm given in (3) applied to the problem (1) satisfies the following:*

(i) the dynamics $\sigma := \mu\mathbf{J} + \mathbf{x}_2 = 0$ is stable,

(ii) the global system reaches the sliding surface $\sigma = 0$ in finite time,

(iii) the global system is stable and converges to a zone around the optimal point.

Proof. Let us consider the sliding surface $\sigma \in \mathbb{R}^n$ such that

$$\sigma := \mu \mathbf{J} + \mathbf{x}_2 = 0. \tag{16}$$

By (1) and Lemma 2 we have that $\sigma = \mu \mathbf{J} + \dot{\mathbf{x}}_1$, which is a stable surface, proving (i). Then, we can apply sliding mode control theory to the problem. Considering a Lyapunov-like function of the form

$$\mathcal{V}(t) = \frac{1}{2}\sigma^{\mathsf{T}}\sigma$$

and taking its time-derivative we have that

$$\begin{aligned} \dot{\mathcal{V}}(t) &= \sigma^{\mathsf{T}}\dot{\sigma} \\ &= \sigma^{\mathsf{T}}\left[\mu\dot{\mathbf{J}} + g(\mathbf{x}_1, \mathbf{x}_2, t) + \mathbf{u}\right]. \end{aligned} \tag{17}$$

Select the control as $\mathbf{u} := -\mu\dot{\mathbf{J}} + \mathbf{u}_0$. Suppose that for $[\sigma]_m < 0$ we need $[\mathbf{u}_0]_m > \|[g(\mathbf{x}_1, \mathbf{x}_2, t)]_m\|$ and for $[\sigma]_m > 0$ we need $[\mathbf{u}_0]_m < -\|[g(\mathbf{x}_1, \mathbf{x}_2, t)]_m\|$, but

$$\begin{aligned} \|[g(\mathbf{x}_1, \mathbf{x}_2, t)]_m\| &\le \|g(\mathbf{x}_1, \mathbf{x}_2, t)\| \\ &\le c_0 + c_1\|\mathbf{x}_1\| + c_2\|\mathbf{x}_2\|. \end{aligned}$$

Then, choose the control law $\mathbf{u}_0$ as follows:

$$\mathbf{u}_0 = -U_{\max}\left[c_0 + c_1\|\mathbf{x}_1\| + c_2\|\mathbf{x}_2\|\right]\operatorname{sign}(\sigma)$$

for some $U_{\max} \ge 1$. Hence, the control signal is given by

$$\begin{aligned} \mathbf{u} := &-\mu\dot{\mathbf{J}} - U_{\max}\left[c_0 + c_1\|\mathbf{x}_1\| + c_2\|\mathbf{x}_2\|\right] \\ &\times \operatorname{sign}(\mu\mathbf{J} + \mathbf{x}_2). \end{aligned} \tag{18}$$

Substituting this result in (17) and applying the triangle inequality, we get

$$\dot{\mathcal{V}}(t) \le -U_{\max}|\sigma| = -U_{\max}\sqrt{2\mathcal{V}(t)}.$$

Now, by solving the ordinary differential inequality, the reaching time to the sliding surface is bounded by

$$t_{\text{reach}} \le \frac{1 + \sqrt{U_{\max}(U_{\max} - 1)}}{U_{\max}}\sqrt{2\mathcal{V}(0)}$$

proving (ii).

When the sliding surface is reached, the behavior of the global system is given by the dynamics $\dot{\mathbf{x}}_1 = -\mu\mathbf{J}$ and by Lemma 2 it is possible to conclude (iii). ■

Remark 3. (*Practical selection of the parameters*) In Sections 3 and 4 we present the theoretical background of the real-time optimal algorithm under conditions and constraints presented in Section 2. In order to select the parameters, we suggest the following:

1. Verify that the global plant satisfies approximately the conditions given in Section 2.
2. Try to stabilize the plant choosing $0 < \alpha < 1$, $0 < \beta < 10$, $0 < \mu < 2$, $1 < U_{\max} < 2$, and $0 < c_0$, $0 < c_1$ and $0 < c_2$.
3. If the plant is stable then reduce the values for c_0, c_1 and c_2 to decrease the input control signal, otherwise increase these values.
4. To decrease the time of convergence, increase the values $0 < \mu$ and $1 < U_{\max}$ such that the actuators and the global system plant allows it.
5. To obtain the best convergence zone decrease the values $0 < \alpha$ and $0 < \beta$ taking care of avoiding saturation of the numerical error given by $1/\alpha$ in the algorithm.

The next section demonstrates the effectiveness of the suggested technique being applied to two examples: one is a simple benchmark example and another deals with the control of a solar panel.

4. Numerical examples

4.1. Example 1: A generic plant. Consider the plant given by

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = \eta(t) + x_2 + \dfrac{x_1}{1 + x_1} + u, \\ y(x_1) = \cosh\left(\frac{1}{5}(x_1 - 15)\right), \end{cases} \tag{19}$$

where $\eta(t) := \sin(t)$ is the disturbance in the state. Note that

$$\begin{aligned} |g(x_1, x_2, t)| &= |\eta(t) + x_2 + \frac{x_1}{1 + x_1}| \\ &\le 1 + |x_1| + |x_2|. \end{aligned}$$

Then for our algorithm we select the following parameters: $\alpha = 0.1$, $\beta = 10\pi$, $\omega_0 = 1000\pi$, $\mu = 3$, $c_0, c_1, c_2 = 1$, $U_{\max} = 1.5$.

The optimal point is $x_1^* = 15$. Figure 3 show the convergence of the state to the optimal point. The optimization of the functional is presented in Fig. 4. Finally, Figs. 5 and 6 show the control signal and the reached sliding surface, respectively.

4.2. Example 2: A simplified mechanical control system for a solar panel. Consider the problem of tracking the position of the sun in order to expose a simple solar panel to a maximum radiation at a given time. The main goal of a solar tracking system is to obtain the best solar panel orientation at a given time of the day. The solar

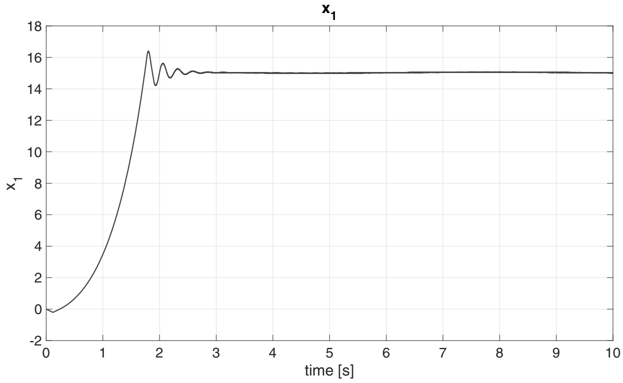

Fig. 3. State $x\,(t)$.

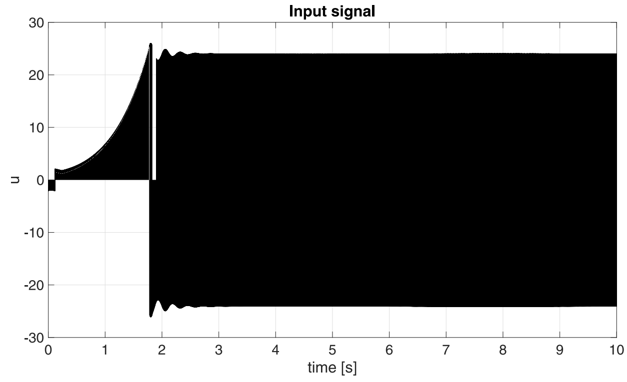

Fig. 5. Control signal.

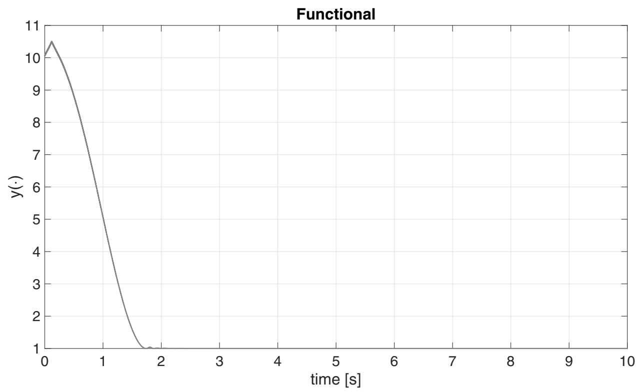

Fig. 4. Optimization of the functional $f\,(\cdot)$.

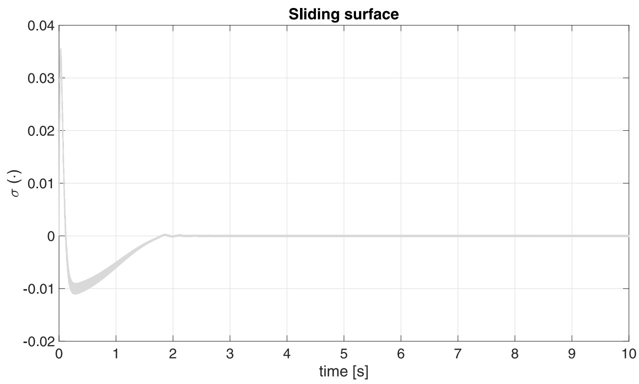

Fig. 6. Sliding surface.

panel is represented in Fig. 7 and its simplified dynamical system given by

$$\begin{cases} I\ddot{\theta} + k\dot{\theta} = \tau + \tau_\eta, \\ y(\cdot) = v_1 \cdot v_2, \end{cases} \tag{20}$$

where I is the rotational inertia of the solar panel, k a viscous friction of the rotor, τ the motor torque, τ_η a disturbance torque over the solar panel, θ the angle from the horizontal plane, v_1 and v_2 are unitary vectors as in the figure. Clearly $y(\cdot)$ is a dot product of two unitary vectors. Then the minimum is -1 when the vectors are oriented in opposite directions, that is, the maximum light intensity. Note that

$$|g(x_1, x_2, t)| = |-\frac{k}{I}x_2 + \frac{1}{I}\tau_\eta| \leq \frac{1}{I}\tau_\eta + \frac{k}{I}|x_2|.$$

For simulation we choose $I = 5$, $k = 0.5$, $\tau_\eta = 1$.

Choosing the parameters $\alpha = 0.1$, $\beta = 1/2$, $\omega = 200$, $\mu = 10$, $c_0 = 1$, $c_1 = 0$, $c_2 = 1$ y $U_{\max} = 1.5$, we can see that the functional tends to the minimum. In Fig. 9 it is possible to see the real-time optimization of the function, that is, the solar panel tends to follow the maximum of the light. In Fig. 8 we emphasize the turning around to the optimal angle of the solar panel. Finally, Fig. 11 shows the reached sliding surface.

5. Conclusions

This paper presented a new continuous-time robust extremum seeking algorithm for strongly unknown (Hessian bounded) convex functionals constrained by dynamic plants with bounded uncertainties. The outputs of the plant are given by the states of the dynamical and a measure of the functional. The mathematical analysis of the stability and convergence of the method was presented. The main advantages of the suggested approach being compared with other methods are the following:

- *it does not require any exact description of the dynamic model* of the controlled plant: neither the structure nor exact values of the parameters are required; only some a priory upper estimate of the acceleration are used during the controller design

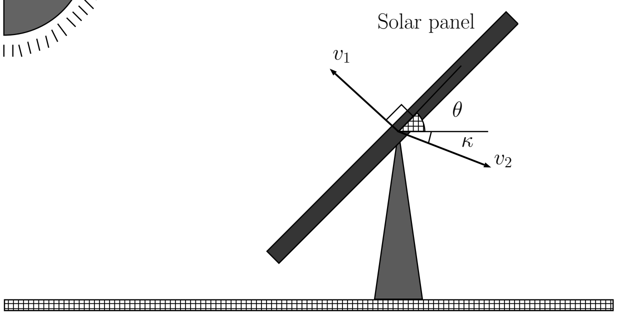

Fig. 7. Solar panel.

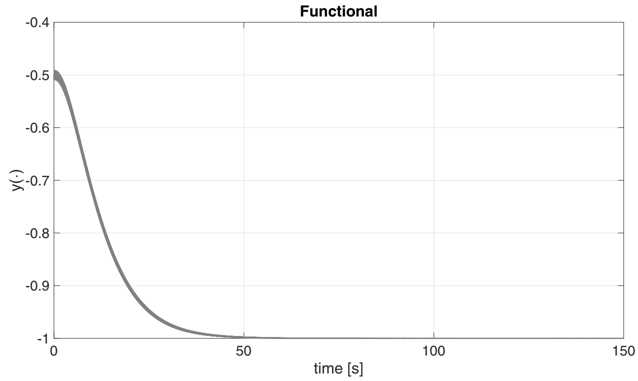

Fig. 9. Functional minimization.

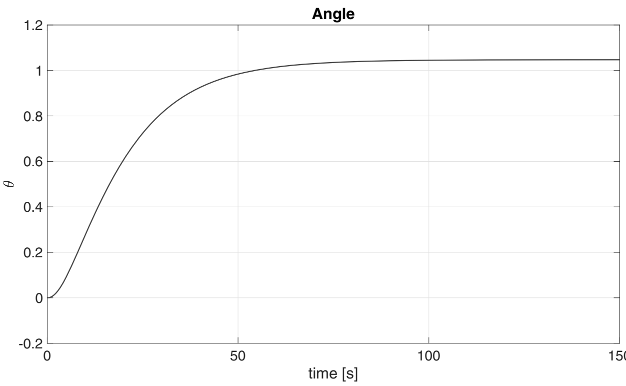

Fig. 8. Angle θ of the solar panel.

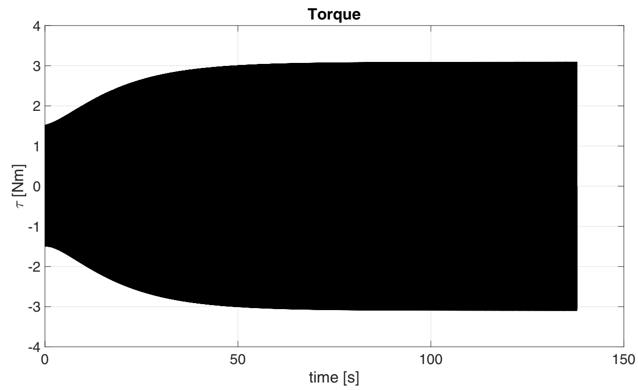

Fig. 10. Motor torque.

process;

- the right-hand side of the dynamic equation *may also include an external bounded noise-perturbations* whose impact can be effectively dismissed by the suggested sliding-mode based controller.

To validate the theoretical background, we presented two numerical examples: (a) the first example consists in a stabilization and optimization of a generic plant considering a strongly convex functional, and (b) the second example applied to a mechanical optimization problem which tracks the position of the sun in order to expose a solar panel to maximum radiation at any given time. Both examples exhibited an optimal real-time behavior. As a future work, we consider improving this algorithm by including algebraic convex constraints and stochastic noise.

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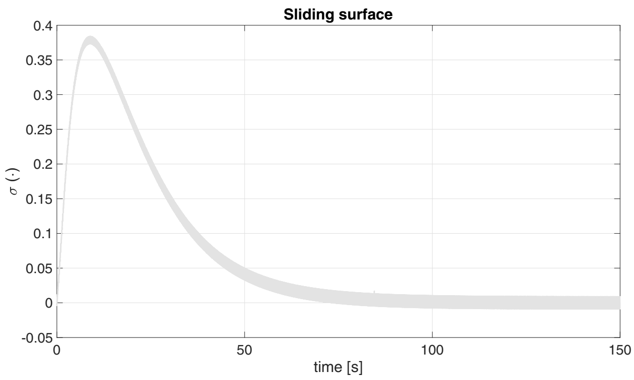

Fig. 11. Sliding surface convergence.

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