



# Applied Mathematics and Nonlinear Sciences

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## New Travelling Wave Solutions for KdV6 Equation Using Sub Equation Method

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### Abstract

This paper proposes obtaining the new wave solutions of time fractional sixth order nonlinear Equation (KdV6) using sub-equation method where the fractional derivatives are considered in conformable sense. Conformable derivative is an understandable and applicable type of fractional derivative that satisfies almost all the basic properties of Newtonian classical derivative such as Leibniz rule, chain rule and etc. Also conformable derivative has some superiority over other popular fractional derivatives such as Caputo and Riemann-Liouville. In this paper all the computations are carried out by computer software called Mathematica.

**Keywords:** Conformable fractional derivative; Sub-Equation method; KdV6 equation, Wave Solution

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## 1 Introduction

In the last years, the number of studies on fractional partial differential equations have increased since they can be used in many fields such as physics, engineering, biology and chemistry [1–3]. Most of these studies focused on obtaining the exact solutions of fractional partial differential equations [13, 14]. But, some of the fractional derivative definitions such as Riemann-Liouville and Caputo do not have capabilities to achieve the exact solutions. Because they do not satisfy some main principles of classical integer order derivative. It is not possible to solve some fractional derivatives by using these definitions. For example

- Riemann-Liouville derivative definition does not satisfy  $D^\alpha c = 0$  where  $c$  is real constant and  $\alpha$  is not a natural number. (This property satisfies for Caputo derivative definition)

- Riemann-Liouville and Caputo derivatives do not satisfy the derivative of the product of two functions.

$$D^\alpha(fg) = gD^\alpha(f) + fD^\alpha(g).$$

- Caputo and Riemann-Liouville derivatives do not satisfy the derivative of the quotient of two functions.

$$D^\alpha\left(\frac{f}{g}\right) = \frac{gD^\alpha(f) - fD^\alpha(g)}{g^2}.$$

- Caputo and Riemann-Liouville derivatives do not satisfy the well-known chain rule.

$$D^\alpha(fog) = f^{(\alpha)}(g(t))g^{(\alpha)}(t).$$

- Caputo and Riemann-Liouville derivatives do not satisfy  $D^\alpha D^\beta(f) = D^{\alpha+\beta}(f)$  usually.
- Caputo derivative definition accepts that the function  $f$  is differentiable.

Although the concept of fractional derivative appeared in the middle of the 17th century, in recent years, interest in this subject has increased. The reason is that physical systems can be expressed clearer by fractional derivative. When the literature is examined, we see that a lot of studies have been carried out on fractional derivatives. Different definitions of fractional derivative have been made since 1730s. Because of the limitations of the popular fractional derivative definitions, the scientists worked to find a new definition for fractional derivative which can satisfy all the main principles. Recently, by Khalil et al. introduced the conformable fractional derivative which is a simple, understandable and efficient fractional derivative definition [4].

**Definition 1.** For all  $t > 0$  and  $\alpha \in (0, 1)$ , an  $\alpha$ -th order “conformable fractional derivative” of a function is defined by [4] as

$$T_\alpha(f)(t) = \lim_{\varepsilon \rightarrow \infty} \frac{f(t + \varepsilon t^{1-\alpha}) - f(t)}{\varepsilon}$$

for  $f : [0, \infty) \rightarrow R$ .

**Definition 2.** If  $f$  is  $\alpha$  differentiable in some  $(0, a)$ ,  $a > 0$  and  $\lim_{t \rightarrow 0^+} f^{(\alpha)}(t)$  exist then define  $f^{(\alpha)}(0) = \lim_{t \rightarrow 0^+} f^\alpha(t)$ . For a function  $f$  starting from  $a \geq 0$ , the conformable fractional integral is defined such as following:

$$I_a^\alpha(f)(t) = \int_a^t f(x) d_\alpha x = \int_a^t \frac{f(x)}{x^{1-\alpha}} dx$$

where  $\alpha \in (0, 1]$  and the integral is the Riemann improper integral.

In the following theorem [1.1], the properties of this new definition are given.

**Theorem 1.1.** Let  $\alpha \in (0, 1]$ ,  $t > 0$  and  $f, g$  be  $\alpha$ -differentiable functions. Then

1.  $T_\alpha(cf + dg) = cT_\alpha(f) + dT_\alpha(g)$ , for all  $a, b \in$ .
2.  $T_\alpha(t^p) = pt^{p-\alpha}$  for all  $p$ .
3.  $T_\alpha(\lambda) = 0$  for all constant functions  $f(t) = \lambda$ .
4.  $T_\alpha(fg) = fT_\alpha(g) + gT_\alpha(f)$ .
5.  $T_\alpha\left(\frac{f}{g}\right) = \frac{gT_\alpha(f) - fT_\alpha(g)}{g^2}$ .
6. If  $f$  is differentiable function, then  $T_\alpha(f)(t) = t^{1-\alpha} \frac{df}{dt}$ .

The conformable fractional derivative has been used to provide new solutions for existing differential equations by many scientists. For example, Ilie et al. [5] studied general solutions of Riccati and Bernoulli fractional differential equations with conformable fractional derivative. Taşbozan and Kurt [12] obtained new travelling wave solutions of time-space fractional Liouville and Sine-Gordon equations using conformable fractional derivative definition. Many references can be seen in the literature [15–17] that used conformable fractional derivative to build the mathematical model of a natural event.

In this study authors aimed to find the new exact solutions of conformable time fractional (1+1) and (2+1) dimensional KdV6 equations [10, 11] with the aid of sub equation method.

## 2 The Sub-Equation Method

In this section, we mention a brief description of fractional sub-equation method [8]. We assume the nonlinear fractional partial differential equation

$$P(u, D_t^\alpha u, D_x u, D_t^{2\alpha} u, D_x^2 u, \dots) = 0 \tag{2.1}$$

where all the fractional derivatives are in conformable form.  $u(x, t)$  is unknown function and  $D_t^{n\alpha}$  means  $n$  times conformable fractional derivative  $u(x, t)$ . The sub-equation method will be explained step by step as follows:

**Step 1:** Using the wave transformation [7], we get the following equalities.

$$u(x, t) = U(\xi), \quad \xi = kx + w \frac{t^\alpha}{\alpha} \tag{2.2}$$

where  $k, w$  are constants to be examined later and. Equation (2.1) can be rewritten in the form of the following ODE by using chain rule [6]:

$$G(U, U', U'', \dots) = 0 \tag{2.3}$$

where prime indicates the known derivative with respect to  $\xi$ .

**Step 2.** Assume that Equation (2.3) has one solution in the following form

$$U(\xi) = \sum_{i=0}^N a_i \varphi^i(\xi), \quad a_N \neq 0, \tag{2.4}$$

where  $a_i (0 \leq i \leq N)$  are constants to be determined.  $N$  represents a positive integer which is going to found using balancing procedure [9] in Eq. (2.3) and  $\varphi(\xi)$  satisfies the ordinary differential equation below

$$\varphi'(\xi) = \sigma + (\varphi(\xi))^2 \tag{2.5}$$

where  $\sigma$  is a constant. For the Eq. (2.5), some special solutions are given in the following formulas.

$$\varphi(\xi) = \begin{cases} -\sqrt{-\sigma} \tanh(\sqrt{-\sigma}\xi), & \sigma < 0 \\ -\sqrt{-\sigma} \coth(\sqrt{-\sigma}\xi), & \sigma < 0 \\ \sqrt{\sigma} \tan(\sqrt{\sigma}\xi), & \sigma > 0 \\ -\sqrt{\sigma} \cot(\sqrt{\sigma}\xi), & \sigma > 0 \\ -\frac{1}{\xi + \varpi}, \varpi \text{ is a cons.}, & \sigma = 0 \end{cases} \tag{2.6}$$

**Step 3.** Eqs. (2.4) and (2.5) are substituted into Eq. (2.3) and the coefficients of  $\varphi^i(\xi)$  are set to zero. This procedure gives a nonlinear algebraic system in  $a_i (i = 0, 1, \dots, N)$ .

**Step 4.** Finally, solving the obtained non-linear algebraic equations system gives us the values of unknown constants. Substituting obtained constants from the nonlinear algebraic system and by help of the formulas (2.6) the solutions of Eq. (2.5) into Eq. (2.4). This provide the exact solutions for Eq. (2.1).

### 3 Implementation of the Sub-Equation Method

#### 3.1 Solution of (1+1) Dimensional Time Fractional KdV6 Equation

Firstly, we consider (1 + 1) KdV6 equation

$$D_x^6 u + 20D_x u D_x^4 u + 40D_x^2 u D_x^3 u + 120D_x u^2 D_x^2 u + D_x^3 D_t^\alpha u + 8D_x u D_x D_t^\alpha u + 4D_t^\alpha u D_x^2 u = 0. \tag{3.1}$$

Employing the chain rule [6] and wave transform [7] in Eq. (3.1) and integrating once yield following differential equation

$$k^6 u^{(v)} + k^3 w u''' + 6k^2 w (u')^2 + 20k^5 u''' u' + 10k^5 (u'')^2 + 40k^4 (u')^3 = 0 \tag{3.2}$$

where prime denotes integer order derivative of function  $u(\xi)$  with respect to variable  $\xi$ . Assuming the solution of Equation (3.2) is denoted by the following series

$$u(\xi) = \sum_{i=0}^N a_i \varphi^i(\xi), a_N \neq 0 \tag{3.3}$$

where  $\varphi(\xi)$  is the exact solutions of Riccati differential equation (2.5). With aid balancing procedure [9], we get  $N = 1$ . Substituting all the obtained values in Eq. (3.2) led to an equation with respect to  $\varphi(\xi)$ . Equating all the coefficients of  $\varphi^i(\xi)$  to zero we obtain an equation system. After solving the equation system by using Mathematica, we get,

$$a_1 = -k, w = 4k^3 \sigma.$$

Due to the this solution set the new wave solutions of Eq. (3.1) can be obtained as

$$\begin{aligned} u_1(x,t) &= a_0 + k\sqrt{-\sigma} \tanh\left(\sqrt{-\sigma}\left(kx + \frac{4k^3 t^\alpha \sigma}{\alpha}\right)\right), \\ u_2(x,t) &= a_0 + k\sqrt{-\sigma} \coth\left(\sqrt{-\sigma}\left(kx + \frac{4k^3 t^\alpha \sigma}{\alpha}\right)\right), \\ u_3(x,t) &= a_0 - k\sqrt{\sigma} \tan\left(\sqrt{\sigma}\left(kx + \frac{4k^3 t^\alpha \sigma}{\alpha}\right)\right), \\ u_4(x,t) &= a_0 + k\sqrt{\sigma} \cot\left(\sqrt{\sigma}\left(kx + \frac{4k^3 t^\alpha \sigma}{\alpha}\right)\right). \end{aligned}$$

#### 3.2 Solution of (2+1) Dimensional KdV6 Equation

Consider the time fractional (2+1) KdV6 equation as follows:

$$D_x(D_x^6 u + 20D_x u D_x^4 u + 40D_x^2 u D_x^3 u + 120D_x u^2 D_x^2 u + D_x^3 D_t^\alpha u + 8D_x u D_x D_t^\alpha u + 4D_t^\alpha u D_x^2 u) + D_y^3 u = 0$$

then applying chain rule [6] and with the aid of the conformable wave transform [7]  $\xi = kx + w \frac{t^\alpha}{\alpha} + ly, u(x, y, t) = u(\xi)$ , and integrating once we have the following ODE

$$k^6 u^{(v)} + k^3 w u''' + 6k^2 w (u')^2 + 20k^5 u''' u' + 10k^5 (u'')^2 + 40k^4 (u''')^3 + l^3 u' = 0 \tag{3.4}$$

Using balancing principle [9] we have  $N = 1$ . Subrogating all the obtained results in Eq. (3.2) we have an equation system. Solving the system yields

$$w = -20k^3 \sigma, a_1 = -k, l = -22^{2/3} 3^{1/3} k^{7/3} \sigma^{2/3}.$$

Using the solution set above, we acquire the exact solutions for time fractional (2+1) dimensional KdV6 Equation

$$u_1(x, y, t) = a_0 + k\sqrt{-\sigma} \tanh \left( \sqrt{-\sigma} \left( kx - 22^{2/3} 3^{1/3} k^{7/3} y \sigma^{2/3} - \frac{20k^3 t^\alpha \sigma}{\alpha} \right) \right),$$

$$u_2(x, y, t) = a_0 + k\sqrt{-\sigma} \coth \left( \sqrt{-\sigma} \left( kx - 22^{2/3} 3^{1/3} k^{7/3} y \sigma^{2/3} - \frac{20k^3 t^\alpha \sigma}{\alpha} \right) \right),$$

$$u_3(x, y, t) = a_0 - k\sqrt{\sigma} \tan \left( \sqrt{\sigma} \left( kx - 22^{2/3} 3^{1/3} k^{7/3} y \sigma^{2/3} - \frac{20k^3 t^\alpha \sigma}{\alpha} \right) \right),$$

$$u_4(x, y, t) = a_0 + k\sqrt{\sigma} \cot \left( \sqrt{\sigma} \left( kx - 22^{2/3} 3^{1/3} k^{7/3} y \sigma^{2/3} - \frac{20k^3 t^\alpha \sigma}{\alpha} \right) \right).$$

#### 4 Conclusions

In study the sub equation method is implemented to get the new traveling wave solutions of time fractional (1+1) and (2+1) dimensional KdV6 equation successfully. The obtained results indicate that the sub equation method is an efficient, reliable and applicable technique for obtaining the exact solutions of fractional derivatives in conformable sense.

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