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Solution of the Rational Difference Equation $x_{n+1} = \frac{x_{n-13}}{1+x_{n-1}x_{n-3}x_{n-5}x_{n-7}x_{n-9}x_{n-11}}$

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Abstract

In this paper, solution of the following difference equation is examined

$$x_{n+1} = \frac{x_{n-13}}{1+x_{n-1}x_{n-3}x_{n-5}x_{n-7}x_{n-9}x_{n-11}},$$

where the initial conditions are positive real numbers.

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1 Introduction

Difference equations appear naturally as discrete analogs and as numerical solutions of differential and delay differential equations, having applications in biology, ecology, physics.

Recently, a high attention to studying the periodic nature of nonlinear difference equations has been attracted. For some recent results concerning the periodic nature of scalar nonlinear difference equations, among other problems, see, for example, [1–33].

Amleh [1], studied the global stability, the boundedness character, and the periodic nature of the positive solutions of the difference equation:

$$x_{n+1} = \alpha + \frac{x_{n-1}}{x_n}$$

where $\alpha \in [0,\infty)$ and where the initial conditions x_{-1} and x_0 are arbitrary positive real numbers.

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De Vault [8], studied the following problems

$$x_{n+1} = \frac{A}{x_n} + \frac{1}{x_{n-2}}, \quad n = 0, 1, \ldots,$$

and showed every positive solution of the equation where $A \in (0, \infty)$.

Elsayed [15], studied the global result, boundedness, and periodicity of solutions of the difference equation

$$x_{n+1} = a + \frac{bx_{n-1} + cx_{n-k}}{dx_{n-1} + ex_{n-k}}, \quad n = 0, 1, \ldots,$$

where the parameters a, b, c, d and e are positive real numbers and the initial conditions $x_{-t}, x_{-t+1}, \ldots, x_0$ are positive real numbers where $t = max\{l, k\}, l \neq k$.

Ibrahim [18], studied the solutions of non-linear difference equation

$$x_{n+1} = \frac{x_n x_{n-2}}{x_{n-1}(a + bx_n x_{n-2})}, \quad n = 0, 1, \ldots,$$

where the initial values x_0, x_{-1} and x_{-2} non-negative real numbers with $bx_0x_{-2} \neq -a$ and $x_{-1} \neq 0$. He investigated some properties for this difference equation such as the local stability and the boundedness for the solutions.

Simsek et. al. [25–27] and [30], studied the following problems with positive initial values

$$x_{n+1} = \frac{x_{n-3}}{1 + x_{n-1}}, \quad n = 0, 1, \ldots,$$

$$x_{n+1} = \frac{x_{n-5}}{1 + x_{n-2}}, \quad n = 0, 1, \ldots,$$

$$x_{n+1} = \frac{x_{n-5}}{1 + x_{n-1}x_{n-3}}, \quad n = 0, 1, \ldots,$$

$$x_{n+1} = \frac{x_{n-3}}{1 + x_n x_{n-1} x_{n-2}}, \quad n = 0, 1, \ldots,$$

respectively.

In this work, the following non-linear difference equation is studied

$$x_{n+1} = \frac{x_{n-13}}{1 + x_{n-1}x_{n-3}x_{n-5}x_{n-7}x_{n-9}x_{n-11}}, \quad n = 0, 1, \ldots, \tag{1}$$

where $x_{-13}, x_{-12}, \ldots, x_{-1}, x_0 \in (0, \infty)$ is investigated.

2 Main Result

Let $\bar{x}$ be the unique positive equilibrium of the equation 1, then clearly,

$$\bar{x} = \frac{\bar{x}}{1 + \bar{x}.\overline{xxxxx}} \Rightarrow \bar{x} + \bar{x}^7 = \bar{x} \Rightarrow \bar{x}^7 = 0 \Rightarrow \bar{x}0,$$

so, $\bar{x} = 0$ can be obtained. For any $k \geq 0$ and $m > k$, notation $i = \overline{k, m}$ means $i = k, k+1, \ldots, m$.

Theorem 1. Consider the difference equation 1. Then the following statements are true:

a) The sequences $(x_{14n-13}), (x_{14n-12}), \ldots (x_{14n-1}), (x_{14n})$ are decreased and $a_1, \ldots, a_{14} \geq 0$ is existed in such that:

$$\lim_{n\to\infty} x_{14n-13+k} = a_{1+k}, \quad k = \overline{0, 13}.$$

b) $(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}, a_{11}, a_{12}, a_{13}, a_{14}, \ldots)$ is a solution of 1 having period fourteen.

c)

$$\prod_{k=0}^{6} \lim_{n\to\infty} x_{14n-j+2k} = 0, \quad j = \overline{0,1};$$

or

$$\prod_{k=0}^{6} a_{2k+i} = 0, \quad i = \overline{0,1}.$$

d) If there exist $n_0 \in \mathbb{N}$ such that $x_{n+1} \leq x_{n-11}$ for all $n \geq n_0$, then,

$$\lim_{n\to\infty} x_n = 0.$$

e) The following formulas below:

$$x_{14n+k+1} = x_{-13+k}\Bigg(1 - \frac{x_{-1+k}x_{-3+k}x_{-5+k}x_{-7+k}x_{-9+k}x_{-11+k}}{1+x_{-1+k}x_{-3+k}x_{-5+k}x_{-7+k}x_{-9+k}x_{-11+k}} \times \sum_{j=0}^{n}\prod_{i=1}^{7j} \frac{1}{1+x_{2i-11+k}x_{2i-9+k}x_{2i-7+k}x_{2i-5+k}x_{2i-3+k}x_{2i-1+k}}\Bigg),$$

$$x_{14n+k+3} = x_{-11+k}\Bigg(1 - \frac{x_{-1+k}x_{-3+k}x_{-5+k}x_{-7+k}x_{-9+k}x_{-13+k}}{1+x_{-1+k}x_{-3+k}x_{-5+k}x_{-7+k}x_{-9+k}x_{-11+k}} \times \sum_{j=0}^{n}\prod_{i=1}^{7j+1} \frac{1}{1+x_{2i-11+k}x_{2i-9+k}x_{2i-7+k}x_{2i-5+k}x_{2i-3+k}x_{2i-1+k}}\Bigg),$$

$$x_{14n+k+5} = x_{-9+k}\Bigg(1 - \frac{x_{-1+k}x_{-3+k}x_{-5+k}x_{-7+k}x_{-11+k}x_{-13+k}}{1+x_{-1+k}x_{-3+k}x_{-5+k}x_{-7+k}x_{-9+k}x_{-11+k}} \times \sum_{j=0}^{n}\prod_{i=1}^{7j+2} \frac{1}{1+x_{2i-11+k}x_{2i-9+k}x_{2i-7+k}x_{2i-5+k}x_{2i-3+k}x_{2i-1+k}}\Bigg),$$

$$x_{14n+k+7} = x_{-7+k}\Bigg(1 - \frac{x_{-1+k}x_{-3+k}x_{-5+k}x_{-9+k}x_{-11+k}x_{-13+k}}{1+x_{-1+k}x_{-3+k}x_{-5+k}x_{-7+k}x_{-9+k}x_{-11+k}} \times \sum_{j=0}^{n}\prod_{i=1}^{7j+3} \frac{1}{1+x_{2i-11+k}x_{2i-9+k}x_{2i-7+k}x_{2i-5+k}x_{2i-3+k}x_{2i-1+k}}\Bigg),$$

$$x_{14n+k+9} = x_{-5+k}\Bigg(1 - \frac{x_{-1+k}x_{-3+k}x_{-7+k}x_{-9+k}x_{-11+k}x_{-13+k}}{1+x_{-1+k}x_{-3+k}x_{-5+k}x_{-7+k}x_{-9+k}x_{-11+k}} \times \sum_{j=0}^{n}\prod_{i=1}^{7j+4} \frac{1}{1+x_{2i-11+k}x_{2i-9+k}x_{2i-7+k}x_{2i-5+k}x_{2i-3+k}x_{2i-1+k}}\Bigg),$$

$$x_{14n+k+11} = x_{-3+k}\Bigg(1 - \frac{x_{-1+k}x_{-5+k}x_{-7+k}x_{-9+k}x_{-11+k}x_{-13+k}}{1+x_{-1+k}x_{-3+k}x_{-5+k}x_{-7+k}x_{-9+k}x_{-11+k}} \times \sum_{j=0}^{n}\prod_{i=1}^{7j+5} \frac{1}{1+x_{2i-11+k}x_{2i-9+k}x_{2i-7+k}x_{2i-5+k}x_{2i-3+k}x_{2i-1+k}}\Bigg),$$

$$x_{14n+k+13} = x_{-1+k}\Big(1 - \frac{x_{-3+k}x_{-5+k}x_{-7+k}x_{-9+k}x_{-11+k}x_{-13+k}}{1+x_{-1+k}x_{-3+k}x_{-5+k}x_{-7+k}x_{-9+k}x_{-11+k}} \times \sum_{j=0}^{n}\prod_{i=1}^{7j+6}\frac{1}{1+x_{2i-11+k}x_{2i-9+k}x_{2i-7+k}x_{2i-5+k}x_{2i-3+k}x_{2i-1+k}}\Big),$$

$k=\overline{0,1}$, holds.

f) If $x_{14n+1+k}\to a_{k+1}\neq 0$, $x_{14n+3+k}\to a_{k+3}\neq 0$, $x_{14n+5+k}\to a_{k+5}\neq 0$, $x_{14n+7+k}\to a_{k+7}\neq 0$, $x_{14n+9+k}\to a_{k+9}\neq 0$, $x_{14n+11+k}\to a_{k+11}\neq 0$ then $x_{14n+13+k}\to 0$ as $n\to\infty, k=\overline{0,1}$.

Proof.

a) Firstly, from 1,

$$x_{n+1}(1+x_{n-1}x_{n-3}x_{n-5}x_{n-7}x_{n-9}x_{n-11}) = x_{n-13},$$

is obtained. If $x_{n-1}x_{n-3}x_{n-5}x_{n-7}x_{n-9}x_{n-11}\in(0,+\infty)$, then $1+x_{n-1}x_{n-3}x_{n-5}x_{n-7}x_{n-9}x_{n-11}\in(1,+\infty)$. Since

$$x_{n+1} < x_{n-13},$$

$n\in\mathbb{N}$,

$$\lim_{n\to\infty} x_{14n-13+k} = a_{1+k}, \quad for\ \ k=\overline{0,13}$$

existed formulas are obtained.

b) $(a_1,a_2,a_3,a_4,a_5,a_6,a_7,a_8,a_9,a_{10},a_{11},a_{12},a_{13},a_{14},\ldots)$ is a solution of 1 having period fourteen.

c) In view of the 1,

$$n=14n \Rightarrow x_{14n+1} = \frac{x_{14n-13}}{1+\prod_{k=0}^{5}\lim_{n\to\infty} x_{14n-11+2k}},$$

is obtained. If the limits are put on both sides of the above equality,

$$\prod_{k=0}^{6}\lim_{n\to\infty} x_{14n-13+2k} = 0 \quad or \quad \prod_{k=0}^{6} a_{2k+1} = 0,$$

is obtained. Similarly for $n=14n+1$,

$$n=14n+1 \Rightarrow x_{14n+2} = \frac{x_{14n-12}}{1+\prod_{k=0}^{5}\lim_{n\to\infty} x_{14n-10+2k}},$$

is obtained. If the limits are put on both sides of the above equality,

$$\prod_{k=0}^{6}\lim_{n\to\infty} x_{14n-12+2k} = 0 \quad or \quad \prod_{k=0}^{6} a_{2k+2} = 0,$$

is obtained.

d) If there exist $n_0\in\mathbb{N}$ such that $x_{n+1}\leq x_{n-11}$ for all $n\geq n_0$, then, $a_1\leq a_3\leq a_5\leq a_7\leq a_9\leq a_{11}\leq a_{13}\leq a_1$, $a_2\leq a_4\leq a_6\leq a_8\leq a_{10}\leq a_{12}\leq a_{14}\leq a_2$. Using (c) we get

$$\prod_{k=0}^{6} a_{2k+i} = 0, \quad i=\overline{1,2}.$$

Then we see that,

$$\lim_{n\to\infty} x_n = 0.$$

Hence the proof of (d) completed.

e) Subracting x_{n-13} from the left and right-hand sides in 1,

$$x_{n+1}-x_{n-13}=\frac{1}{1+x_{n-1}x_{n-3}x_{n-5}x_{n-7}x_{n-9}x_{n-11}}(x_{n-1}-x_{n-15}),$$

is obtained and the following formula is produced below, for $n\geq 2$

$$\begin{aligned} x_{2n-3}-x_{2n-17}&=(x_1-x_{-13})\prod_{i=1}^{n-2}\frac{1}{1+x_{2i-1}x_{2i-3}x_{2i-5}x_{2i-7}x_{2i-9}x_{2i-11}},\\ x_{2n-2}-x_{2n-16}&=(x_1-x_{-12})\prod_{i=1}^{n-2}\frac{1}{1+x_{2i}x_{2i-2}x_{2i-4}x_{2i-6}x_{2i-8}x_{2i-10}}, \end{aligned} \quad (2)$$

$7j$ inserted in 2 by replacing n, $j=0$ to $j=n$ is obtained by summing, for $k=\overline{0,1}$,

$$x_{14n+1+k}-x_{-13+k}=(x_{1+k}-x_{-13+k})\sum_{j=0}^{n}\prod_{i=1}^{7j}\frac{1}{1+x_{2i-1+k}x_{2i-3+k}x_{2i-5+k}x_{2i-7+k}x_{2i-9+k}x_{2i-11+k}}.$$

Also, $7j+1$ inserted in 2 by replacing n, $j=0$ to $j=n$ is obtained by summing, for $k=\overline{0,1}$,

$$x_{14n+3+k}-x_{-11+k}=(x_{3+k}-x_{-11+k})\sum_{j=0}^{n}\prod_{i=1}^{7j+1}\frac{1}{1+x_{2i-1+k}x_{2i-3+k}x_{2i-5+k}x_{2i-7+k}x_{2i-9+k}x_{2i-11+k}}.$$

Also, $7j+2$ inserted in 2 by replacing n, $j=0$ to $j=n$ is obtained by summing, for $k=\overline{0,1}$,

$$x_{14n+5+k}-x_{-9+k}=(x_{5+k}-x_{-9+k})\sum_{j=0}^{n}\prod_{i=1}^{7j+2}\frac{1}{1+x_{2i-1+k}x_{2i-3+k}x_{2i-5+k}x_{2i-7+k}x_{2i-9+k}x_{2i-11+k}}.$$

Also, $7j+3$ inserted in 2 by replacing n, $j=0$ to $j=n$ is obtained by summing, for $k=\overline{0,1}$,

$$x_{14n+7+k}-x_{-7+k}=(x_{7+k}-x_{-7+k})\sum_{j=0}^{n}\prod_{i=1}^{7j+3}\frac{1}{1+x_{2i-1+k}x_{2i-3+k}x_{2i-5+k}x_{2i-7+k}x_{2i-9+k}x_{2i-11+k}}.$$

Also, $7j+4$ inserted in 2 by replacing n, $j=0$ to $j=n$ is obtained by summing, for $k=\overline{0,1}$,

$$x_{14n+9+k}-x_{-5+k}=(x_{9+k}-x_{-5+k})\sum_{j=0}^{n}\prod_{i=1}^{7j+4}\frac{1}{1+x_{2i-1+k}x_{2i-3+k}x_{2i-5+k}x_{2i-7+k}x_{2i-9+k}x_{2i-11+k}}.$$

Also, $7j+5$ inserted in 2 by replacing n, $j=0$ to $j=n$ is obtained by summing, for $k=\overline{0,1}$,

$$x_{14n+11+k}-x_{-3+k}=(x_{11+k}-x_{-3+k})\sum_{j=0}^{n}\prod_{i=1}^{7j+5}\frac{1}{1+x_{2i-1+k}x_{2i-3+k}x_{2i-5+k}x_{2i-7+k}x_{2i-9+k}x_{2i-11+k}}.$$

Also, $7j+6$ inserted in 2 by replacing n, $j=0$ to $j=n$ is obtained by summing, for $k=\overline{0,1}$,

$$x_{14n+13+k}-x_{-1+k}=(x_{13+k}-x_{-1+k})\sum_{j=0}^{n}\prod_{i=1}^{7j+6}\frac{1}{1+x_{2i-1+k}x_{2i-3+k}x_{2i-5+k}x_{2i-7+k}x_{2i-9+k}x_{2i-11+k}}.$$

Now we obtained of the above formulas:

$$x_{14n+k+1}=x_{-13+k}\Big(1-\frac{x_{-1+k}x_{-3+k}x_{-5+k}x_{-7+k}x_{-9+k}x_{-11+k}}{1+x_{-1+k}x_{-3+k}x_{-5+k}x_{-7+k}x_{-9+k}x_{-11+k}} \times\sum_{j=0}^{n}\prod_{i=1}^{7j}\frac{1}{1+x_{2i-11+k}x_{2i-9+k}x_{2i-7+k}x_{2i-5+k}x_{2i-3+k}x_{2i-1+k}}\Big),$$

$$x_{14n+k+3} = x_{-11+k}\Big(1 - \frac{x_{-1+k}x_{-3+k}x_{-5+k}x_{-7+k}x_{-9+k}x_{-13+k}}{1+x_{-1+k}x_{-3+k}x_{-5+k}x_{-7+k}x_{-9+k}x_{-11+k}} \times \sum_{j=0}^{n}\prod_{i=1}^{7j+1}\frac{1}{1+x_{2i-11+k}x_{2i-9+k}x_{2i-7+k}x_{2i-5+k}x_{2i-3+k}x_{2i-1+k}}\Big),$$

$$x_{14n+k+5} = x_{-9+k}\Big(1 - \frac{x_{-1+k}x_{-3+k}x_{-5+k}x_{-7+k}x_{-11+k}x_{-13+k}}{1+x_{-1+k}x_{-3+k}x_{-5+k}x_{-7+k}x_{-9+k}x_{-11+k}} \times \sum_{j=0}^{n}\prod_{i=1}^{7j+2}\frac{1}{1+x_{2i-11+k}x_{2i-9+k}x_{2i-7+k}x_{2i-5+k}x_{2i-3+k}x_{2i-1+k}}\Big),$$

$$x_{14n+k+7} = x_{-7+k}\Big(1 - \frac{x_{-1+k}x_{-3+k}x_{-5+k}x_{-9+k}x_{-11+k}x_{-13+k}}{1+x_{-1+k}x_{-3+k}x_{-5+k}x_{-7+k}x_{-9+k}x_{-11+k}} \times \sum_{j=0}^{n}\prod_{i=1}^{7j+3}\frac{1}{1+x_{2i-11+k}x_{2i-9+k}x_{2i-7+k}x_{2i-5+k}x_{2i-3+k}x_{2i-1+k}}\Big),$$

$$x_{14n+k+9} = x_{-5+k}\Big(1 - \frac{x_{-1+k}x_{-3+k}x_{-7+k}x_{-9+k}x_{-11+k}x_{-13+k}}{1+x_{-1+k}x_{-3+k}x_{-5+k}x_{-7+k}x_{-9+k}x_{-11+k}} \times \sum_{j=0}^{n}\prod_{i=1}^{7j+4}\frac{1}{1+x_{2i-11+k}x_{2i-9+k}x_{2i-7+k}x_{2i-5+k}x_{2i-3+k}x_{2i-1+k}}\Big),$$

$$x_{14n+k+11} = x_{-3+k}\Big(1 - \frac{x_{-1+k}x_{-5+k}x_{-7+k}x_{-9+k}x_{-11+k}x_{-13+k}}{1+x_{-1+k}x_{-3+k}x_{-5+k}x_{-7+k}x_{-9+k}x_{-11+k}} \times \sum_{j=0}^{n}\prod_{i=1}^{7j+5}\frac{1}{1+x_{2i-11+k}x_{2i-9+k}x_{2i-7+k}x_{2i-5+k}x_{2i-3+k}x_{2i-1+k}}\Big),$$

$$x_{14n+k+13} = x_{-1+k}\Big(1 - \frac{x_{-3+k}x_{-5+k}x_{-7+k}x_{-9+k}x_{-11+k}x_{-13+k}}{1+x_{-1+k}x_{-3+k}x_{-5+k}x_{-7+k}x_{-9+k}x_{-11+k}} \times \sum_{j=0}^{n}\prod_{i=1}^{7j+6}\frac{1}{1+x_{2i-11+k}x_{2i-9+k}x_{2i-7+k}x_{2i-5+k}x_{2i-3+k}x_{2i-1+k}}\Big).$$

f) Suppose that $a_1 = a_3 = a_5 = a_7 = a_9 = a_{11} = a_{13} = 0$. By (e), we have

$$\lim n \to \infty x_{14n+1} = \lim_{n\to\infty} x_{-13}\left(1 - \frac{x_{-1}x_{-3}x_{-5}x_{-7}x_{-9}x_{-11}}{1+x_{-1}x_{-3}x_{-5}x_{-7}x_{-9}x_{-11}}\sum_{j=0}^{n}\prod_{i=1}^{7j}\frac{1}{1+x_{2i-1}x_{2i-3}x_{2i-5}x_{2i-7}x_{2i-9}x_{2i-11}}\right),$$

$$a_1 = x_{-13}\left(1 - \frac{x_{-1}x_{-3}x_{-5}x_{-7}x_{-9}x_{-11}}{1+x_{-1}x_{-3}x_{-5}x_{-7}x_{-9}x_{-11}}\sum_{j=0}^{n}\prod_{i=1}^{7j}\frac{1}{1+x_{2i-1}x_{2i-3}x_{2i-5}x_{2i-7}x_{2i-9}x_{2i-11}}\right)$$

$$a_1 = 0 \Rightarrow \frac{1+x_{-1}x_{-3}x_{-5}x_{-7}x_{-9}x_{-11}}{x_{-1}x_{-3}x_{-5}x_{-7}x_{-9}x_{-11}} = \sum_{j=0}^{n}\prod_{i=1}^{7j}\frac{1}{1+x_{2i-1}x_{2i-3}x_{2i-5}x_{2i-7}x_{2i-9}x_{2i-11}}. \quad (3)$$

Similarly,

$$a_3 = 0 \Rightarrow \frac{1+x_{-1}x_{-3}x_{-5}x_{-7}x_{-9}x_{-11}}{x_{-1}x_{-3}x_{-5}x_{-7}x_{-9}x_{-13}} = \sum_{j=0}^{n}\prod_{i=1}^{7j+1}\frac{1}{1+x_{2i-1}x_{2i-3}x_{2i-5}x_{2i-7}x_{2i-9}x_{2i-11}}. \tag{4}$$

Similarly,

$$a_5 = 0 \Rightarrow \frac{1+x_{-1}x_{-3}x_{-5}x_{-7}x_{-9}x_{-11}}{x_{-1}x_{-3}x_{-5}x_{-7}x_{-11}x_{-13}} = \sum_{j=0}^{n}\prod_{i=1}^{7j+2}\frac{1}{1+x_{2i-1}x_{2i-3}x_{2i-5}x_{2i-7}x_{2i-9}x_{2i-11}}. \tag{5}$$

Similarly,

$$a_7 = 0 \Rightarrow \frac{1+x_{-1}x_{-3}x_{-5}x_{-7}x_{-9}x_{-11}}{x_{-1}x_{-3}x_{-5}x_{-9}x_{-11}x_{-13}} = \sum_{j=0}^{n}\prod_{i=1}^{7j+3}\frac{1}{1+x_{2i-1}x_{2i-3}x_{2i-5}x_{2i-7}x_{2i-9}x_{2i-11}}. \tag{6}$$

Similarly,

$$a_9 = 0 \Rightarrow \frac{1+x_{-1}x_{-3}x_{-5}x_{-7}x_{-9}x_{-11}}{x_{-1}x_{-3}x_{-7}x_{-9}x_{-11}x_{-13}} = \sum_{j=0}^{n}\prod_{i=1}^{7j+4}\frac{1}{1+x_{2i-1}x_{2i-3}x_{2i-5}x_{2i-7}x_{2i-9}x_{2i-11}}. \tag{7}$$

Similarly,

$$a_{11} = 0 \Rightarrow \frac{1+x_{-1}x_{-3}x_{-5}x_{-7}x_{-9}x_{-11}}{x_{-1}x_{-5}x_{-7}x_{-9}x_{-11}x_{-13}} = \sum_{j=0}^{n}\prod_{i=1}^{7j+5}\frac{1}{1+x_{2i-1}x_{2i-3}x_{2i-5}x_{2i-7}x_{2i-9}x_{2i-11}}. \tag{8}$$

Similarly,

$$a_{13} = 0 \Rightarrow \frac{1+x_{-1}x_{-3}x_{-5}x_{-7}x_{-9}x_{-11}}{x_{-3}x_{-5}x_{-7}x_{-9}x_{-11}x_{-13}} = \sum_{j=0}^{n}\prod_{i=1}^{7j+6}\frac{1}{1+x_{2i-1}x_{2i-3}x_{2i-5}x_{2i-7}x_{2i-9}x_{2i-11}}. \tag{9}$$

From 3 and 4,

$$\frac{1+x_{-1}x_{-3}x_{-5}x_{-7}x_{-9}x_{-11}}{x_{-1}x_{-3}x_{-5}x_{-7}x_{-9}x_{-11}} = \sum_{j=0}^{n}\prod_{i=1}^{7j}\frac{1}{1+x_{2i-1}x_{2i-3}x_{2i-5}x_{2i-7}x_{2i-9}x_{2i-11}} >$$
$$\frac{1+x_{-1}x_{-3}x_{-5}x_{-7}x_{-9}x_{-11}}{x_{-1}x_{-3}x_{-5}x_{-7}x_{-9}x_{-13}} = \sum_{j=0}^{n}\prod_{i=1}^{7j+1}\frac{1}{1+x_{2i-1}x_{2i-3}x_{2i-5}x_{2i-7}x_{2i-9}x_{2i-11}},$$

thus, $x_{-13} > x_{-11}$. From 4 and 5,

$$\frac{1+x_{-1}x_{-3}x_{-5}x_{-7}x_{-9}x_{-11}}{x_{-1}x_{-3}x_{-5}x_{-7}x_{-9}x_{-13}} = \sum_{j=0}^{n}\prod_{i=1}^{7j+1}\frac{1}{1+x_{2i-1}x_{2i-3}x_{2i-5}x_{2i-7}x_{2i-9}x_{2i-11}} >$$
$$\frac{1+x_{-1}x_{-3}x_{-5}x_{-7}x_{-9}x_{-11}}{x_{-1}x_{-3}x_{-5}x_{-7}x_{-11}x_{-13}} = \sum_{j=0}^{n}\prod_{i=1}^{7j+2}\frac{1}{1+x_{2i-1}x_{2i-3}x_{2i-5}x_{2i-7}x_{2i-9}x_{2i-11}},$$

thus, $x_{-11} > x_{-9}$. From 5 and 6,

$$\frac{1+x_{-1}x_{-3}x_{-5}x_{-7}x_{-9}x_{-11}}{x_{-1}x_{-3}x_{-5}x_{-7}x_{-11}x_{-13}} = \sum_{j=0}^{n}\prod_{i=1}^{7j+2}\frac{1}{1+x_{2i-1}x_{2i-3}x_{2i-5}x_{2i-7}x_{2i-9}x_{2i-11}} >$$
$$\frac{1+x_{-1}x_{-3}x_{-5}x_{-7}x_{-9}x_{-11}}{x_{-1}x_{-3}x_{-5}x_{-9}x_{-11}x_{-13}} = \sum_{j=0}^{n}\prod_{i=1}^{7j+3}\frac{1}{1+x_{2i-1}x_{2i-3}x_{2i-5}x_{2i-7}x_{2i-9}x_{2i-11}},$$

thus, $x_{-9} > x_{-7}$. From 6 and 7,

$$\frac{1+x_{-1}x_{-3}x_{-5}x_{-7}x_{-9}x_{-11}}{x_{-1}x_{-3}x_{-5}x_{-9}x_{-11}x_{-13}} = \sum_{j=0}^{n}\prod_{i=1}^{7j+3}\frac{1}{1+x_{2i-1}x_{2i-3}x_{2i-5}x_{2i-7}x_{2i-9}x_{2i-11}} > \frac{1+x_{-1}x_{-3}x_{-5}x_{-7}x_{-9}x_{-11}}{x_{-1}x_{-3}x_{-7}x_{-9}x_{-11}x_{-13}} = \sum_{j=0}^{n}\prod_{i=1}^{7j+4}\frac{1}{1+x_{2i-1}x_{2i-3}x_{2i-5}x_{2i-7}x_{2i-9}x_{2i-11}}$$

thus, $x_{-7} > x_{-5}$. From 7 and 8,

$$\frac{1+x_{-1}x_{-3}x_{-5}x_{-7}x_{-9}x_{-11}}{x_{-1}x_{-3}x_{-7}x_{-9}x_{-11}x_{-13}} = \sum_{j=0}^{n}\prod_{i=1}^{7j+4}\frac{1}{1+x_{2i-1}x_{2i-3}x_{2i-5}x_{2i-7}x_{2i-9}x_{2i-11}} > \frac{1+x_{-1}x_{-3}x_{-5}x_{-7}x_{-9}x_{-11}}{x_{-1}x_{-5}x_{-7}x_{-9}x_{-11}x_{-13}} = \sum_{j=0}^{n}\prod_{i=1}^{7j+5}\frac{1}{1+x_{2i-1}x_{2i-3}x_{2i-5}x_{2i-7}x_{2i-9}x_{2i-11}}$$

thus, $x_{-5} > x_{-3}$. From 8 and 9,

$$\frac{1+x_{-1}x_{-3}x_{-5}x_{-7}x_{-9}x_{-11}}{x_{-1}x_{-5}x_{-7}x_{-9}x_{-11}x_{-13}} = \sum_{j=0}^{n}\prod_{i=1}^{7j+5}\frac{1}{1+x_{2i-1}x_{2i-3}x_{2i-5}x_{2i-7}x_{2i-9}x_{2i-11}} > \frac{1+x_{-1}x_{-3}x_{-5}x_{-7}x_{-9}x_{-11}}{x_{-3}x_{-5}x_{-7}x_{-9}x_{-11}x_{-13}} = \sum_{j=0}^{n}\prod_{i=1}^{7j+6}\frac{1}{1+x_{2i-1}x_{2i-3}x_{2i-5}x_{2i-7}x_{2i-9}x_{2i-11}},$$

thus, $x_{-3} > x_{-1}$. From here we obtain $x_{-13} > x_{-11} > x_{-9} > x_{-7} > x_{-5} > x_{-3} > x_{-1}$. Similarly, we can obtain $x_{-12} > x_{-10} > x_{-8} > x_{-6} > x_{-4} > x_{-2} > x_0$. We arrive at a contradiction which completes the proof of theorem.

Example 2. If the initial conditions are selected as follows:

$$x_{-13} = 0.98,\ x_{-12} = 0.97,\ x_{-11} = 0.96,\ x_{-10} = 0.95,\ x_{-9} = 0.94,\ x_{-8} = 0.93,\ x_{-7} = 0.92,$$
$$x_{-6} = 0.91,\ x_{-5} = 0.9,\ x_{-4} = 0.89,\ x_{-3} = 0.88,\ x_{-2} = 0.87,\ x_{-1} = 0.86,\ x_0 = 0.85.$$

The graph of the solution is given below, $x_n = \{0.62601,\ 0.634341,\ 0.701375,\ 0.701974,\ 0.737178,$ $0.734195,\ 0.753815,\ 0.748866,\ 0.759718,\ 0.753567,\ 0.759006,\ 0.752062,\ 0.753929,\ 0.746432,\ 0.535307,$ $0.545309,\ 0.621059,\ 0.622964,\ 0.664753,\ 0.66286,\ 0.687738, 0.683736,\ 0.698929,\ 0.693618,\ 0.702724,$ $0.696539,\ 0.701557,\ 0.694753,\ 0.487602,\ 0.498104, 0.576757,\ 0.579084,\ 0.623219,\ 0.621705,\ 0.64857,$ $0.644917, 0.661839, 0.656857, 0.667494, \ldots\}$

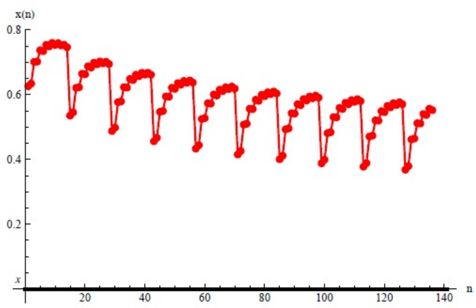

Fig. 1 x_n graph of the solution.

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