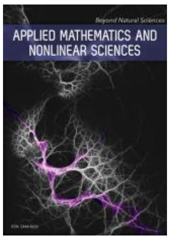

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New Applications on the Flows of Space-like Curves Specified by Normal Acceleration in Minkowski Space $\mathbb{R}^{2,1}$

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Abstract

This paper investigates the kinematic motions of space-like and time-like curves specified by acceleration fields in Minkowski space $\mathbb{R}^{2,1}$. Through the motion, the relationship between the acceleration fields and velocity fields is determined. In this study, we focus on studying the flows of inextensible space-like curves with a space-like principal normal vector specified by a normal acceleration that equals the curvature of the curve. Through the motion of the inextensible space-like curve with normal acceleration, we prove that the position vector of the curve satisfies a one-dimensional wave equation. We present some novel applications and visualize the flows of curves and their curvatures.

Keywords: Flows of curves; evolution of curves; motion of space-like curves; motion of time-like curves; motion by acceleration field; time evolution equations.
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1 Introduction

The topic of the motion of curves or the flows of curves has many applications in physics, such as the study of the evolution of a front propagating along its normal vector field by a speed function of curvature, the motion of fronts in viscous fingering in a Hele-Shaw cell, and the vortex filament motion in incompressible and inviscid fluids [1–3].

Researchers studied the flow of curves in various spaces with different frames, such as Euclidean space, Lorentz space, and Galilean space. The motion of curves in Euclidean space $\mathbb{R}^3$ and Minkowski space $\mathbb{R}^{2,1}$ was studied by many authors: Nakayama et al. [4], studied the motion of curves given by the acceleration in $\mathbb{R}^3$. Nassar et al. [5], studied the motion of curves in $\mathbb{R}^n$ that were described by acceleration. Bas et al. [6], studied inextensible flows of spacelike curves (IFSPC) on oriented space-like surfaces in M_1^3 and obtained the necessary and sufficient conditions (NSCs) for (IFSPC) on oriented spacelike surfaces. Ergut et al. [7], studied the (IFSPC) with Sabban frame on S_1^2 and obtained partial differential equations (PDEs) in terms of these flows of the space-like curves (SPC) that are related to the Sabban frame on S_1^2. Korpinar et al. [8], constructed a novel approach to the inextensible flow of curves in $\mathbb{R}^3$ by employing the Frenet frame of the given curve, and provided some characterizations for the curvatures of the curve. Bektas et al. [9], obtained the (NSCs) for the space-like curves (SPC) in the three-dimensional light-like cone of E_1^4 to be inextensible through their study of the (IFSPC).

Arroyo et al. [10], investigated the binormal flow of curves specified by curvatures depending on velocity and sweeping out immersed surfaces. By employing the Gauss-Codazzi equations, the filaments that evolving by constant torsion were constructed. Yuzbasi, et al. [11], investigated the inextensible flows (IFC) on a lightlike surface in $\mathbb{R}^{2,1}$, and obtained (NSCs) for (IFC) as a PDE involving the curvatures of the curve on a lightlike surface. In addition, the lightlike ruled surfaces in $\mathbb{R}^{2,1}$ were classified, and the inextensible evolution of a lightlike curve on a lightlike tangent developable surface was characterized.

The novel coupled non-linear partial differential equations (CNLPDE) were developed by Yuzbasi, et al. [12], to obtain the temporal evolution of the curvatures of the developing curve in Galilean space. The exact solutions were obtained for these novel CNLPDEs. These novel CNLPDE were also subjected to Lie symmetry analysis. The algebra of their Lie point symmetries were identified. Kaymanli et al. [13], investigated the ruled surfaces obtained by normal and binormal vectors along a time-like space curve by using the q-frame in $\mathbb{R}^{2,1}$. The directrices of quasi-normal and quasi-binormal ruled surfaces are used to study their directional evolution. Some geometric features, including inextensibility, developability, and minimality, of these ruled surfaces, were investigated.

Abdo [14], obtained a relationship between the curve evolution and the soliton equations in $\mathbb{R}^{2,1}$ for the (SPC) with a space-like principal normal vector. Yoon et al. [15], obtained the time evolution equations (TEEs) of the curvature and torsion for evolving space-like curves in Minkowski space. Additionally, the inextensible evolutions of time-like ruled surfaces formed by the time-like normal and space-like binormal vector fields of space-like curves were given. The (NSCs) for the evolution of inelastic surface were also provided. Moreover, the coefficients of the first and second fundamental forms, the Gauss, and the mean curvatures for time-like special ruled surfaces were determined.

The quaternionic curves in $\mathbb{R}^3$ and $\mathbb{R}^4$ were described by Eren [16], and the motions of inextensible quaternionic curves were characterized by the modified Korteweg-de Vries (mKdV) equations. In addition, the evolution of inextensible quaternionic curves with the Frenet frame was obtained. Hussien and Gaber [17], studied the (IFC) in $\mathbb{R}^3$ and constructed the generated surfaces from the motion of inextensible curves. The geometric properties of the generated surfaces were investigated and visualized. Gaber [18], studied the (IFC) in spherical space S^3 and derived the (TEEs) of the orthonormal frame and curvatures. Moreover, some novel explicit examples of motions of inextensible curves in S^3 were presented. Gaber [19] studied the binormal motions of time-like curves and space-like curves with a time-like normal vector in De Sitter space $\mathbb{S}^{2,1}$, and constructed Hashimoto surfaces.

Recently, Gaber [20], obtained the (TEEs) for the type-1 Bishop frame and Bishop curvatures of curves in $\mathbb{R}^3$. Gaber and Sorour [21], studied the (IFC) specially time-like curves with a quasi-frame in $\mathbb{R}^{2,1}$. Gaber and Al Elaiw [22], investigated the flows of a null Cartan curve described through the velocity and acceleration fields and proved that the binormal velocity influences the tangential and normal velocities. In addition, the (TEEs) were derived for the torsion of the null curves and also for the Cartan frame. Furthermore, a family of inextensible null Cartan curves were constructed.

In the present work, we consider the kinematic motion of (SPC) and (TIC) in $\mathbb{R}^{2,1}$. We derive the second (TEEs) of the Frenet frame and obtain a connection between acceleration and velocity fields. We study the motion of an inextensible (SPC) with normal acceleration that equals its curvature and prove that its position vector satisfies the (PDE) called a one-dimensional wave equation. We solve this equation for specific initial conditions, providing novel applications for flows of inextensible (SPC). These applications include explicit parametrization of the curve and its flows, computation of Frenet frame and curvature flows, as well as visualization of these flows.

This paper is organized as follows: In Section [2], we give some geometric properties of curves in $\mathbb{R}^{2,1}$. In Section [3], we provide the main results for the motions of (SPC) and (TIC) via the acceleration fields in $\mathbb{R}^{2,1}$. In Section [4], we study the (IFSPC) with a space-like principal normal vector (SPNV) via the normal acceleration that equals the curvature of the curve. We present some novel applications and investigate the one-dimensional wave equation arising from the motion of inextensible (SPC) according to the normal acceleration. Finally, we give our conclusions and a discussion.

2 Preliminaries

In this part, we outline some characteristics of vectors and curves in Minkowski space $\mathbb{R}^{2,1}$.

Definition 1. [23] The Minkowski space $\mathbb{R}^{2,1}$ is a three-dimensional $\mathbb{R}$-vector space with the vectors $\{X = (x_0, x_1, x_2) \mid x_0, x_1, x_2 \in \mathbb{R}\}$. It has the following properties:

- Metric: $-dx_0^2 + dx_1^2 + dx_2^2$.
- Inner product: $\langle a, b\rangle = -a_0b_0 + a_1b_1 + a_2b_2$, where, $a = (a_0, a_1, a_2), b = (b_0, b_1, b_2) \in \mathbb{R}^{2,1}$.
- Vector product: $a \times b = (a_2b_1 - a_1b_2, a_2b_0 - a_0b_2, a_0b_1 - a_1b_0)$, $a = (a_0, a_1, a_2), b = (b_0, b_1, b_2) \in \mathbb{R}^{2,1}$.
- The vector $v \in \mathbb{R}^{2,1}$ is space-like if $\langle v, v\rangle > 0$, time-like if $\langle v, v\rangle < 0$, and null (light-like) if $\langle v, v\rangle = 0$.
- The signature of the vector v is 1 if v is space-like, -1 if U is time-like, and 0 if v is light-like.

Definition 2. [23] Let $\mathbb{R}^{2,1}$ be $3-$dimensional Minkowski space. Assume that $\beta = \beta(u) : I \to \mathbb{R}^{2,1}$, be a regular parameterized curve, where $u \in I$ is the curve's parameter and let $\dot{\beta}(u)$ be the tangent vector to the curve, where $(\dot{}) = \frac{d}{du}$. The curve $\beta(u)$ is: space-like if $\langle \dot{\beta}, \dot{\beta}\rangle > 0$, time-like if $\langle \dot{\beta}, \dot{\beta}\rangle < 0$, and null (light-like) if $\langle \dot{\beta}, \dot{\beta}\rangle = 0$.

Definition 3. [24] Consider β be a regular (SPC) or time-like curve (TIC) and let $s(u) = \int_0^u \|\dot{\beta}(\sigma)\| d\sigma$ be the arc-length of the curve $\beta(u)$, we define the metric $g > 0$ by $\frac{ds}{du} = \|\dot{\beta}\| = \sqrt{g}$. In case of $\|\dot{\beta}\| = 1$, $\forall u \in I$, then $\beta = \beta(s)$ is called an arc-length parameterized curve.

Definition 4. [24] Let $\beta = \beta(s)$ be (SPC) or (TIC), parameterized by arc-length, and assume that the curvature of the curve $k \neq 0$. Assume that the curve β moves according to the orthonormal Frenet frame $\{T, N, B\}$, where T, N and B are the unit tangent, unit principal normal and unit binormal vector fields to the curve $\beta(s)$, respectively. The characteristics of the Frenet vectors in $\mathbb{R}^{2,1}$ are given as follows:

- $sign(T) = \varepsilon_1$, $sign(T') = \varepsilon_2$, where $\beta' = T$, and $\varepsilon_1 = \pm 1, \varepsilon_2 = \pm 1$, $(') = \frac{d}{ds}$.
- $\langle N, N\rangle = \varepsilon_2$, $\langle B, B\rangle = -\varepsilon_1\varepsilon_2$.

- $\langle T,N\rangle = \langle T,B\rangle = \langle N,B\rangle = 0$.
- $N\times B = -\varepsilon_2 T$, $B\times T = -\varepsilon_1 N$, and $T\times N = B$.

Definition 5. [24] Assume that k and τ depicts the curvature and torsion of the (SPC) or (TIC) curve respectively, and they are defined as follows:

$$k = \varepsilon_2\langle T',N\rangle,\ \tau = -\varepsilon_1\varepsilon_2\langle N',B\rangle.$$

Lemma 1. *[24] Let $\beta = \beta(s)$ be (SPC) or (TIC), the Frenet Frame $\{T,N,B\}$ for the (SPC) or (TIC) in $\mathbb{R}^{2,1}$ satisfies the following Ordinary Differential Equations (ODEs):*

$$\beta_s = T, \quad T_s = kN, \quad N_s = -\varepsilon_1\varepsilon_2 kT + \tau B, \quad B_s = \varepsilon_1\tau N. \tag{1}$$

2.1 Motions of space-like and time-like curves in $\mathbb{R}^{2,1}$ specified by the velocity fields

In this section, we summarize some important results proved by [11, 15] for the motions of (SPC) or (TIC) in $\mathbb{R}^{2,1}$ specified by the velocity fields.

Definition 6. Let $C_0 = \beta(s,0) : I = [0,l] \longrightarrow \mathbb{R}^{2,1}$ be the initial (SPC) or (TIC) in $\mathbb{R}^{2,1}$. Let t depicts the time parameter for the curve flow, and assume that the (SPC) or (TIC) curves with their flows are denoted by C_t and defined by $C_t = \beta(s,t) : I = [0,l]\times[0,\infty) \longrightarrow \mathbb{R}^{2,1}$. The kinematics motion is described in terms of the velocity fields W,U,V of the points on the (SPC) or (TIC) as the following (PDE):

$$\frac{\partial\beta}{\partial t} = WT + UN + VB. \tag{2}$$

Lemma 2. *The first (TEEs) of the Frenet frame of (SPC) or (TIC) in $\mathbb{R}^{2,1}$ are given by:*

$$\begin{aligned} T_t &= f_1 N + f_2 B\ , \\ N_t &= -\varepsilon_1\varepsilon_2 f_1 T + \xi\ B\ , \\ B_t &= \varepsilon_2 f_2 T + \varepsilon_1 \xi N\ . \end{aligned} \tag{3}$$

Definition 7. The (SPC) or (TIC) in $\mathbb{R}^{2,1}$ is said to be inextensible if it preserves its arclength through the motion. So $g_t = 0$.

Lemma 3. *Consider the motion of (SPC) or (TIC) in $\mathbb{R}^{2,1}$. The (NSCs) for the (SPC) or (TIC) to be inextensible ($g_t = 0$) leads to the following relationship between the velocity functions:*

$$W_s = \varepsilon_1\varepsilon_2 kU, \tag{4}$$

and the (TEEs) for the curvature and torsion take the following form:

$$\begin{aligned} k_t &= \varepsilon_1 f_2\tau + f_{1,s}\ , \\ \tau_t &= \varepsilon_1\varepsilon_2 f_2 k + \xi_s\ , \end{aligned} \tag{5}$$

where:

$$\begin{aligned} f_1 &= kW + U_s + \varepsilon_1\tau V, \\ f_2 &= V_s + \tau\ U\ , \\ \xi &= \frac{1}{k}\left(\frac{\partial f_2}{\partial s} + f_1\tau\right). \end{aligned} \tag{6}$$

3 Main results

The main purpose of this section is study of the motion of the inextensible (SPC) and (TIC) in $\mathbb{R}^{2,1}$ specified by acceleration fields. Consider that the inextensible (SPC) or (TIC) evolves by the acceleration functions E,F, and G in the direction of the tangent vector, principal normal vector, and principal binormal vector. The acceleration functions E,F, and G are functions of the curvature, the torsion of the curve, and their derivatives. The evolving equation is described by the following (PDE):

$$\frac{\partial^2\beta}{\partial t^2} = E\,T + F\,N + G\,B. \tag{7}$$

Lemma 4. *The connection between the acceleration fields E,F and G that describe the evolving of the inextensible (SPC) and (TIC) by (7) and the velocity fields W,U,V that describe the evolving of the inextensible (SPC) and (TIC) by (2) is given by:*

$$\begin{aligned} E &= W_t - \varepsilon_1\varepsilon_2 f_1 U + \varepsilon_2 f_2 V,\\ F &= U_t + f_1 W + \varepsilon_1 \xi V,\\ G &= V_t + f_2 W + \xi U. \end{aligned} \tag{8}$$

Proof. Take the derivative of (2) with respect to t, and using (3), hence

$$\beta_{tt} = (W_t - \varepsilon_1\varepsilon_2 f_1 U + \varepsilon_2 f_2 V)T + (U_t + f_1 W + \varepsilon_1\xi V)N + (V_t + f_2 W + \xi U)B. \tag{9}$$

Equating (7) and (9), hence the lemma holds.

Theorem 5. *The second (TEEs) for the Frenet frame vectors T,N,B are given by:*

$$R_{tt} = \Omega\cdot R, \tag{10}$$

where: $R = \begin{pmatrix} T\\ N\\ B\end{pmatrix}$ *and* $\Omega = \begin{pmatrix} \varepsilon_2(f_2^2-\varepsilon_1 f_1^2) & f_{1,t}+\varepsilon_1\xi f_2 & f_{2,t}+\xi f_1\\ \varepsilon_2(-\varepsilon_1 f_{1,t}+\xi f_2) & \varepsilon_1(-\varepsilon_2 f_1^2+\xi^2) & \xi_t-\varepsilon_1\varepsilon_2 f_1 f_2\\ \varepsilon_2(f_{2,t}-\xi f_1) & \varepsilon_1\xi_t+\varepsilon_2 f_1 f_2\xi & \varepsilon_2 f_2^2+\varepsilon_1\xi^2 \end{pmatrix}$.

Proof. Take the derivative of (3) with respect to t, and using (3), then we obtain:

$$T_{tt} = \varepsilon_2(f_2^2-\varepsilon_1 f_1^2)T + (f_{1,t}+\varepsilon_1\xi f_2)N + (f_{2,t}+\xi f_1)B. \tag{11}$$

Taking the derivative of the second equation of (3) with respect to the parameter t, and using (3), then we have:

$$N_{tt} = \varepsilon_2(-\varepsilon_1 f_{1,t}+\xi f_2)T + \varepsilon_1(-\varepsilon_2 f_1^2+\xi^2)N + (\xi_t-\varepsilon_1\varepsilon_2 f_1 f_2)B. \tag{12}$$

Taking the $t-$derivative of the third equation of (3), with respect to the parameter t, and using (3), then we have:

$$B_{tt} = \varepsilon_2(f_{2,t}-\xi f_1)T + (\varepsilon_1\xi_t+\varepsilon_2 f_1 f_2\xi)N + (\varepsilon_2 f_2^2+\varepsilon_1\xi^2)B. \tag{13}$$

Hence the theorem holds.

Theorem 6. *The functions f_1, f_2 and ξ that defined by (6) can be given in terms of acceleration fields E,F,G as the following system of (PDEs):*

$$\begin{aligned} 2(\sqrt{g})_t f_1 + \sqrt{g}(f_{1,t}+\varepsilon_1\xi f_2) &= F_u + \sqrt{g}(kE+\varepsilon_1\tau G)\,,\\ 2(\sqrt{g})_t f_2 + \sqrt{g}(f_{2,t}+\xi f_1) &= G_u + \sqrt{g}\tau F\,,\\ (\sqrt{g})_{tt} + \sqrt{g}\varepsilon_2(f_2^2-\varepsilon_1 f_1^2) &= E_u - \varepsilon_1\varepsilon_2\sqrt{g}k\,F\,. \end{aligned} \tag{14}$$

Proof. Taking the derivative of (7), with respect to the parameter u, then we have

$$\beta_{ttu} = \Big(E_u - \varepsilon_1\varepsilon_2\sqrt{g}k\,F\Big)T + \Big(F_u + \sqrt{g}(k\,E + \varepsilon_1\tau\,G)\Big)N + \Big(G_u + \sqrt{g}\tau\,F\Big)B. \tag{15}$$

Since $\beta_u = \sqrt{g}\,\beta_s = \sqrt{g}\,T$, by taking the second and third derivative of this equation with respect to t, and by using (3), then we get:

$$\begin{aligned}\beta_{utt} = &\Big((\sqrt{g})_{tt} + \sqrt{g}\varepsilon_2(f_2^2 - \varepsilon_1 f_1^2)\Big)T + \Big(2(\sqrt{g})_t f_1 + \sqrt{g}(f_{1,t} + \varepsilon_1\xi f_2)\Big)N \\ &+ \Big(2(\sqrt{g})_t f_2 + \sqrt{g}(f_{2,t} + \xi f_1)\Big)B.\end{aligned} \tag{16}$$

Using the compatibility condition $\beta_{ttu} = \beta_{utt}$, hence the theorem holds.

Remark 1. Consider the arclength parameterized (SPC) or (TIC) via the acceleration fields E,F, and G. Since $E_u = \sqrt{g}\,E_s$, $F_u = \sqrt{g}\,F_s$, and $G_u = \sqrt{g}\,G_s$, and assume that the curve is inextensible ($g_t = 0$), hence we obtain the following relations for (IFSPC):

$$\begin{aligned} f_{1,t} + \varepsilon_1\xi f_2 &= F_s + kE + \varepsilon_1\tau G\,, \\ f_{2,t} + \xi f_1 &= G_s + \tau F\,, \\ \varepsilon_2(f_2^2 - \varepsilon_1 f_1^2) &= E_s - \varepsilon_1\varepsilon_2 kF\,. \end{aligned} \tag{17}$$

3.1 Motion of (IFSPC) with space-like principal normal vector

Consider the (IFSPC) with space-like principal normal vector (SPNV) specified by the acceleration fields E,F, and G, so $\varepsilon_1 = 1, \varepsilon_2 = 1$. Hence from **Theorem 5**, we obtain the next Lemma:

Lemma 7. *The second (TEEs) for the Frenet frame for (IFSPC) with (SPNV) is given by:*

$$R_{tt} = \Omega\cdot R, \tag{18}$$

where $R = \begin{pmatrix} T \\ N \\ B \end{pmatrix}$ *and* $\Omega = \begin{pmatrix} f_2^2 - f_1^2 & f_{1,t} + \xi f_2 & f_{2,t} + \xi f_1 \\ -f_{1,t} + \xi f_2 & -f_1^2 + \xi^2 & \xi_t - f_1 f_2 \\ f_{2,t} - \xi f_1 & \xi_t + f_1 f_2\xi & f_2^2 + \xi^2 \end{pmatrix}.$

Lemma 8. *The (TEEs) for the curvatures of the (IFSPC) with (SPNV) are given by*

$$\begin{aligned} k_t &= \tau f_2 + f_{1,s}\,, \\ \tau_t &= k f_2 + \xi_{,s}\,, \end{aligned} \tag{19}$$

where, the functions f_1, f_2 and ξ are given in terms of accelerations fields E,F and G as the following system of PDEs:

$$\begin{aligned} f_{1,t} + \xi f_2 &= F_s + k\,E + \tau\,G, \\ f_{2,t} + \xi f_1 &= G_s + \tau\,F, \\ f_1^2 - f_2^2 &= -E_s + k\,F. \end{aligned} \tag{20}$$

4 The one-dimensional wave equation arising from (IFSPC) with normal acceleration

Consider (IFSPC) with (SPNV) specified by the normal acceleration $F = k$ and assume that $E = 0$, $G = 0$. So, the evolution equation (7) that describes (IFSPC) takes the following form:

$$\frac{\partial^2\beta}{\partial t^2} = kN. \tag{21}$$

We study a special case for $(f_2 = 0)$, then by substituting in (20), we get:

$$\begin{aligned} f_{1,t} &= k_s\,, \\ \xi f_1 &= \tau k\,, \\ f_1^2 &= k^2\,. \end{aligned} \tag{22}$$

Hence, we get $f_1 = k, \xi = \tau$.

Lemma 9. *Consider (IFSPC) with (SPNV) specified by the normal acceleration $F = k$, and for $f_1 = k, f_2 = 0$, and $\xi = \tau$, then the second (TEEs) of the Frenet frame that given by (18), takes the form:*

$$\begin{aligned} T_{tt} &= -k^2 T + k_t N + \tau k B\,, \\ N_{tt} &= -k_t T + (-k^2 + \tau^2) N + \tau_t B\,, \\ B_{tt} &= -\tau k T + \tau_t N + \tau^2 B\,. \end{aligned} \tag{23}$$

Lemma 10. *Let $C_t(s,t) = \beta(s,t) = (\beta_1(s,t), \beta_2(s,t), \beta_3(s,t))$ be inextensible (SPC) with (SPNV) and its flows specified by the normal acceleration $F = k$. Then the position vector $\beta(s,t)$ and the Frenet frame vectors T, N, B satisfy the one-dimensional wave equation as follows:*

$$\begin{aligned} \beta_{ss} &= \beta_{tt}\,, \\ T_{ss} &= T_{tt}\,, \\ N_{ss} &= N_{tt}\,, \\ B_{ss} &= B_{tt}. \end{aligned} \tag{24}$$

Proof. By taking the derivative of (1), with respect to the arclength s, and comparing the results with (21) and (23), hence the lemma holds.

Lemma 11. *Consider (IFSPC) with (SPNV) specified by a normal acceleration $F = k$, and for $f_1 = k, f_2 = 0$, and $\xi = \tau$, then the (TEEs) for curvatures that are given by (19), take the following new form:*

$$\begin{aligned} k_t &= k_s, \\ \tau_t &= \tau_s, \end{aligned} \tag{25}$$

this system of (PDE) represents the heat equations. The general solution takes the following form:

$$\begin{aligned} k(s,t) &= C_1(s+t), \\ \tau(s,t) &= C_2(s+t), \end{aligned} \tag{26}$$

where $C_1(s+t)$ and $C_2(s+t)$ are arbitrary functions.

4.1 Applications on one-dimensional wave equation arising from the (IFSPC)

Consider (IFSPC) with (SPNV) specified by a normal acceleration $F = k$. Since the position vector $\beta(s,t) = (\beta_1(s,t), \beta_2(s,t), \beta_3(s,t))$ of the (SPC) and its flows satisfies one-dimensional wave equation $\beta_{ss} = \beta_{tt}$, so:

$$\begin{aligned} \beta_{1,ss} &= \beta_{1,tt}, \\ \beta_{2,ss} &= \beta_{2,tt}, \\ \beta_{3,ss} &= \beta_{3,tt}. \end{aligned} \tag{27}$$

To solve the 1−dimensional wave equation (27), we choose some initial conditions by considering the initial conditions: $\beta(s,0)=(\beta_1(s,0),\beta_2(s,0),\beta_3(s,0))$, and $\beta_t(s,0)=(\beta_{1,t}(s,0),\beta_{2,t}(s,0),\beta_{3,t}(s,0))$, where:

$$\begin{aligned} \beta_1(s,0)&=\phi_1(s), & \beta_{1,t}(s,0)&=\psi_1(s),\\ \beta_2(s,0)&=\phi_2(s), & \beta_{2,t}(s,0)&=\psi_2(s),\\ \beta_3(s,0)&=\phi_3(s), & \beta_{3,t}(s,0)&=\psi_3(s), \end{aligned} \tag{28}$$

with arclength parameter $s\geq 0$ and time $t\geq 0$. Then, the 1−dimensional wave equation (27) has the following general solution:

$$\begin{aligned} \beta_1(s,t)&=\frac{1}{2}\big(\phi_1(s+t)+\phi_1(s-t)+\int_{s-t}^{s+t}\psi_1(x)dx\big),\\ \beta_2(s,t)&=\frac{1}{2}\big(\phi_2(s+t)+\phi_2(s-t)+\int_{s-t}^{s+t}\psi_2(x)dx\big),\\ \beta_3(s,t)&=\frac{1}{2}\big(\phi_3(s+t)+\phi_3(s-t)+\int_{s-t}^{s+t}\psi_3(x)dx\big), \end{aligned} \tag{29}$$

where the solution in the formula (29) is called the d'Alamber's formula for the initial value problem for the one-dimensional wave equation. According to the properties of the Frenet vectors that are given by **Definition 4** and **Definition 5**, then the functions $\beta_1(s,t),\beta_2(s,t)$ and $\beta_3(s,t)$ satisfy the following system of (PDEs):

$$\begin{aligned} -\beta_{1,s}^2+\beta_{2,s}^2+\beta_{3,s}^2&=1,\\ -\beta_{1,ss}^2+\beta_{2,ss}^2+\beta_{3,ss}^2&=C_1^2(s+t),\\ \eta_1(s,t)\beta_{1,sss}-\eta_2(s,t)\beta_{2,sss}-\eta_3(s,t)\beta_{3,sss}&=C_1^2(s+t)C_2(s+t), \end{aligned} \tag{30}$$

where,

$$\begin{aligned} \eta_1(s,t)&=\beta_{2,ss}\beta_{3,s}-\beta_{2,s}\beta_{3,ss},\\ \eta_2(s,t)&=\beta_{1,ss}\beta_{3,s}-\beta_{1,s}\beta_{3,ss},\\ \eta_3(s,t)&=\beta_{1,s}\beta_{2,ss}-\beta_{1,ss}\beta_{2,s}. \end{aligned} \tag{31}$$

Application 1. *Consider the (SPC) $\beta(s,t)=(\beta_1(s,t),\beta_2(s,t),\beta_3(s,t))$ with (SPNV) which moves with normal acceleration $F=k(s,t)$ and satisfies the one-dimensional wave equation (27) with the following initial conditions:*

$$\begin{aligned} \beta(s,0)&=(\beta_1(s,0),\beta_2(s,0),\beta_3(s,0)),\\ \beta_t(s,0)&=(\beta_{1,t}(s,0),\beta_{2,t}(s,0),\beta_{3,t}(s,0)). \end{aligned} \tag{32}$$

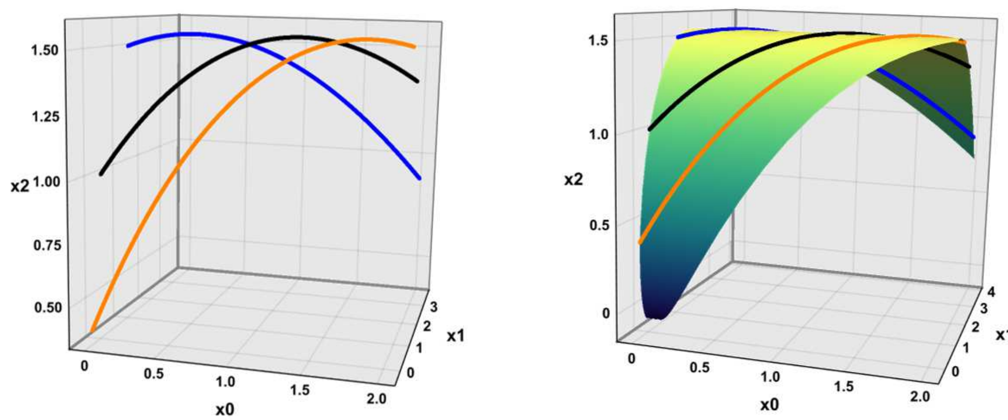
where

$$\begin{aligned} \beta_1(s,0)&=\phi_1(s)=s, & \beta_{1,t}(s,0)&=\psi_1(s)=0,\\ \beta_2(s,0)&=\phi_2(s)=s\sin(\log(s)), & \beta_{2,t}(s,0)&=\psi_2(s)=\sin(\log(s))+\cos(\log(s)),\\ \beta_3(s,0)&=\phi_3(s)=s\cos(\log(s)), & \beta_{3,t}(s,0)&=\psi_3(s)=-\sin(\log(s))+\cos(\log(s)). \end{aligned} \tag{33}$$

Substitute from (33) into (29), then we get the general solution:

$$\beta(s,t)=\big(s,(s+t)\sin(\log(s+t)),(s+t)\cos(\log(s+t))\big). \tag{34}$$

The flows of the (SPC) (34) are illustrated by Figure 1.



(a) (b)

Fig. 1: The flows of the (SPC) $\beta(s,t)$ for $s \in [0.01,2]$ and $t \in [0,2]$. The flows for $t = 0.5, 1, 1.8$ are depicted by orange, black, and blue curves, respectively.

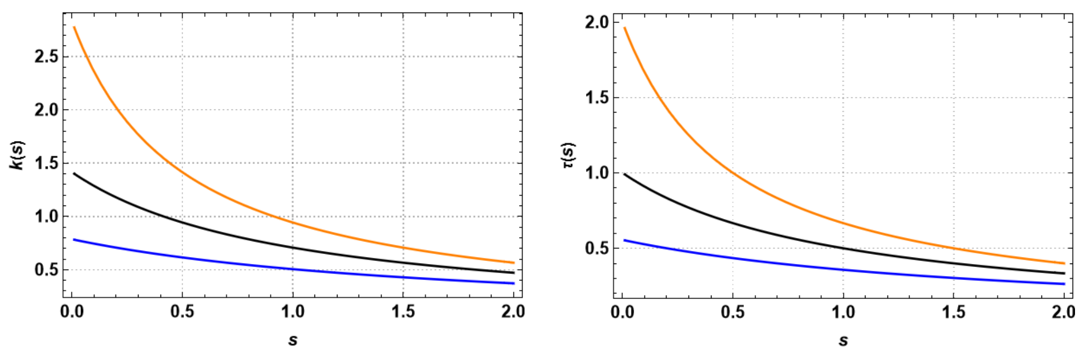
The flows of the Frenet frame are:

$$
\begin{aligned}
T(s,t) &= (1, \cos(\log(s+t)) + \sin(\log(s+t)), \cos(\log(s+t)) - \sin(\log(s+t))),\\
N(s,t) &= \frac{1}{\sqrt{2}}(0, \cos(\log(s+t)) - \sin(\log(s+t)), -\cos(\log(s+t)) - \sin(\log(s+t))),\\
B(s,t) &= \frac{1}{\sqrt{2}}(2, \cos(\log(s+t)) + \sin(\log(s+t)), \cos(\log(s+t)) - \sin(\log(s+t))).
\end{aligned}
\tag{35}
$$

The Frenet frame vectors (35) *coincide with the properties in* ***Definition 4*** *and* ***Definition 5****. By substitute from* (34) *into* (30) *and* (31)*, then we obtain the curvature and torsion:*

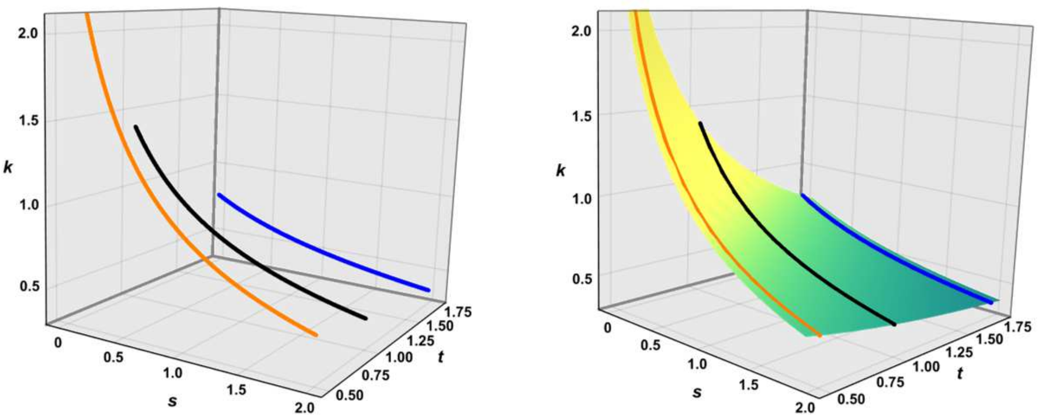
$$
\begin{aligned}
k(s,t) &= C_1(s+t) = \frac{\sqrt{2}}{s+t},\\
\tau(s,t) &= C_2(s+t) = \frac{1}{s+t}.
\end{aligned}
\tag{36}
$$

The curvature and torsion of the (SPC) and their flows are illustrated by (Figure 2), (Figure.3), and (Figure.4).



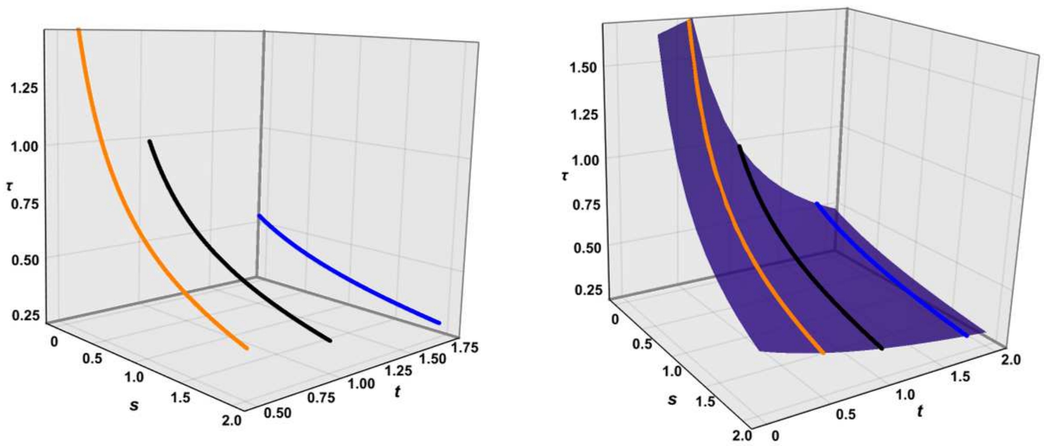
(a) (b)

Fig. 2: The flows of curvature and torsion of the (SPC) for $s \in [0.01,2]$ at $t = 0.5, 1, 1.8$, are depicted by orange, black, and blue curves, respectively.



(**a**) (**b**)

Fig. 3: The flows of the curvature for the (SPC) $k(s,t) = \frac{\sqrt{2}}{s+t}$ **for** $s \in [0.01, 2]$ **and** $t \in [0, 2]$**. The flows for** $t = 0.5, 1, 1.8$ **are depicted by the orange, black, and blue curves, respectively.**



(**a**) (**b**)

Fig. 4: The flows of the torsion for the (SPC) $\tau(s,t) = \frac{1}{s+t}$ **for** $s \in [0.01, 2]$ **and** $t \in [0, 2]$**. The flows at** $t = 0.5, 1, 1.8$ **are depicted by the orange, black, and blue curves, respectively.**

Graphic interpretations on application 1

- *Figure 1(b) shows the three-dimensional graph (3-D) graph of the flows of the (SPC) (34) for* $s \in [0.01, 2]$ *and* $t = [0, 2]$. *The orange, black, and blue curves in Figure 1(a), and Figure 1(b), represents the flows at* $t = 0.5, 1, 1.8$, *respectively.*

- *Figure 2(a) and Figure 2(b) depict the (2-D) graphs of the flows of curvature* $k(s)$ *and torsion* $\tau(s)$ *for* $s \in [0.01, 2]$ *at* $t = 0.5, 1, 1.8$. *The color curves (orange, black, and blue curves) represent the flows at*

$t = 0.5, 1, 1.8$, respectively.

- *In Figure 3(a) and Figure 3(b), the color curves (orange, black, and blue curves) depict the flows of the curvature $k(s,t) = \frac{\sqrt{2}}{s+t}$ at $t = 0.5, 1, 1.8$, respectively. Figure 3(b) depicts the (3-D) graph of the flows of the curvature for $s \in [0.01, 2]$ and $t = [0, 2]$.*
- *In Figure 4(a) and Figure 4(b), the color curves (orange, black, and blue curves) depict the flows of the torsion $\tau(s,t) = \frac{1}{(s+t)}$ at $t = 0.5, 1, 1.8$, respectively. Figure 4(b) depicts the (3-D) graph of the flows of the torsion for $s \in [0.01, 2]$ and $t \in [0, 2]$.*

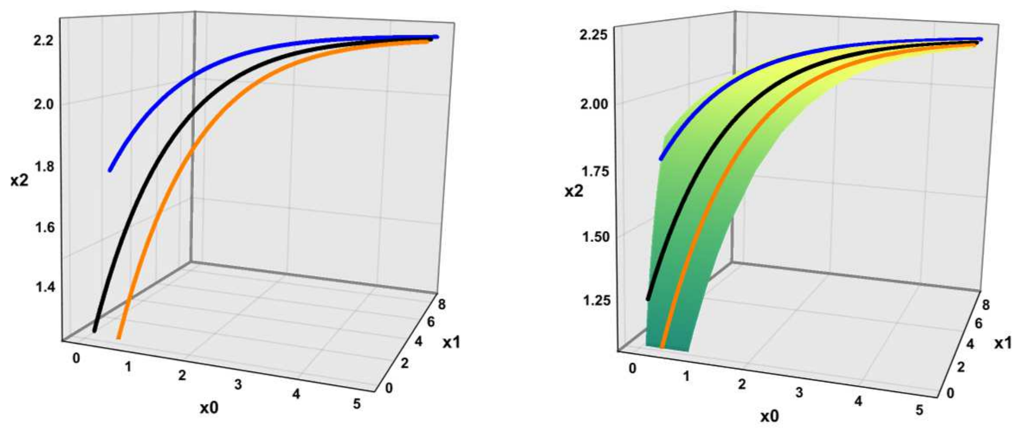
Application 2. *Consider the (SPC) $\beta(s,t) = (\beta_1(s,t), \beta_2(s,t), \beta_3(s,t))$ with (SPNV) which moves with normal acceleration $F = k(s,t)$ and satisfies the one-dimensional wave equation (27) with the following initial conditions:*

$$\begin{aligned}
&\beta(s,0) = (\beta_1(s,0), \beta_2(s,0), \beta_3(s,0)),\\
&\beta_t(s,0) = (\beta_{1,t}(s,0), \beta_{2,t}(s,0), \beta_{3,t}(s,0)),\\
&\beta_1(s,0) = \phi_1(s) = s \quad , \quad \beta_{1,t}(s,0) = \psi_1(s) = 0,\\
&\beta_2(s,0) = \phi_2(s) = \sqrt{2}\log\cosh s \quad , \quad \beta_{2,t}(s,0) = \psi_2(s) = \sqrt{2}\tanh s,\\
&\beta_3(s,0) = \phi_3(s) = 2\sqrt{2}\arctan(\tanh(\frac{s}{2})) \quad , \quad \beta_{3,t}(s,0) = \psi_3(s) = \sqrt{2}\mathrm{sech}\, s.
\end{aligned} \tag{37}$$

Substitute from (37) into (29), then we obtain the general solution:

$$\beta(s,t) = \big(s, \sqrt{2}\log\cosh(s+t), 2\sqrt{2}\arctan(\tanh(\frac{s+t}{2}))\big), \tag{38}$$

The flows of the (SPC) (38) are illustrated by Figure 5.



(a) **(b)**

Fig. 5: The flows of the (SPC) $\beta(s,t)$ for $s \in [0,5]$ and $t \in [0,2]$. The flows for $t = 0.5, 1, 1.8$, are depicted by the orange, black, and blue curves, respectively. respectively.

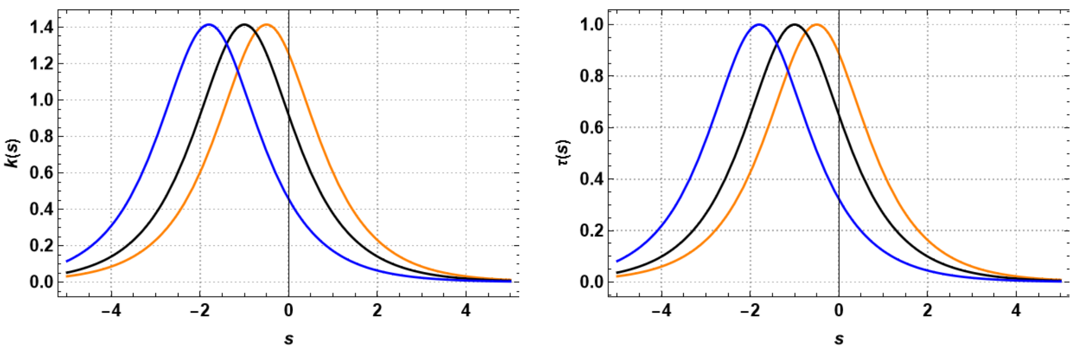
The flows of the Frenet frame are:

$$\begin{aligned}
T(s,t) &= (1, \sqrt{2}\tanh(s+t), \sqrt{2}\mathrm{sech}\,(s+t)),\\
N(s,t) &= (0, \mathrm{sech}\,(s+t), -\tanh(s+t)),\\
B(s,t) &= (\sqrt{2}, \tanh(s+t), \mathrm{sech}\,(s+t)).
\end{aligned} \tag{39}$$

We can verify that the Frenet frame vectors (39) *satisfy the properties in* ***Definition 4*** *and* ***Definition 5****. Substitute from* (38) *into* (30) *and* (31)*, then we get the curvature and torsion:*

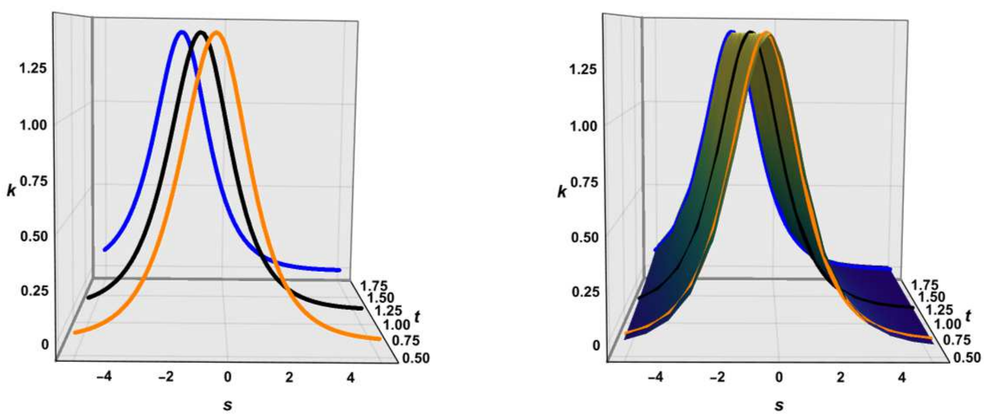
$$
\begin{aligned}
k(s,t) &= C_1(s+t) = \sqrt{2}\operatorname{sech}(s+t), \\
\tau(s,t) &= C_2(s+t) = \operatorname{sech}(s+t).
\end{aligned}
\tag{40}
$$

We plot the curvature and torsion of the (SPC) and their flows as illustrated by (Figure 6), (Figure 7), and (Figure 8).



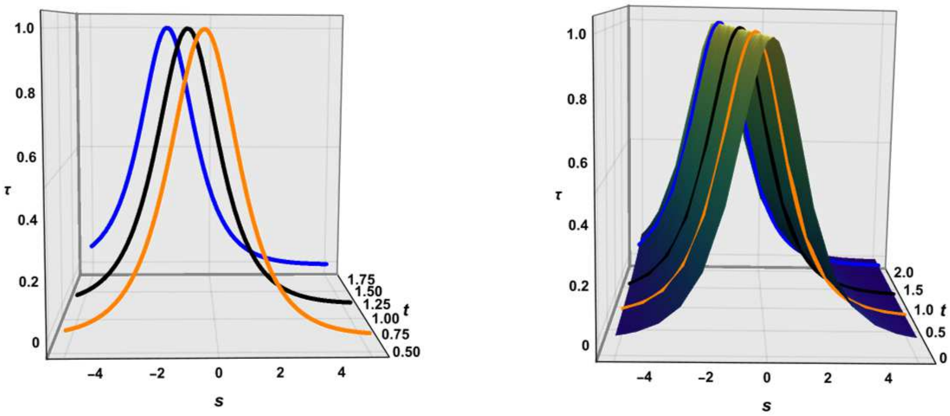
(a) **(b)**

Fig. 6: The flows of curvature and torsion of the for $s \in$ **[-5**$,5]$ **at** $t = 0.5, 1, 1.8$**, are depicted by the orange, black, and blue curves, respectively.**



(a) **(b)**

Fig. 7: The flows of curvature for the (SPC) $k(s,t) = \sqrt{2}\operatorname{sech}(s+t)$ **for** $s \in$ **[-5**$,5]$ **and** $t \in [0,2]$**. The flows at** $t = 0.5, 1, 1.8$**, are depicted by the orange, black, and blue curves, respectively.**



(a) (b)

Fig. 8: The flows of the torsion for the (SPC) $\tau(s,t) = \operatorname{sech}(s+t)$ for $s \in$ [-5, 5] and $t \in [0,2]$. The flows at $t = 0.5, 1, 1.8$, are depicted by the orange, black, and blue curves, respectively.

Graphic interpretations on application 2

- *Figure 5(b) represents the (3-D) graph of the flows of the (SPC) (34) for $s \in [0,5]$ and $t \in [0,2]$. The orange, black, and blue curves in Figure 5(a), and Figure 5(b), represents the flows at $t = 0.5, 1, 1.8$, respectively.*
- *Figure 6(a) and Figure 6(b) illustrate the (2-D) graphs of the flows of curvature $k(s)$ and torsion $\tau(s)$ for $s \in$ [-5, 5] at $t = 0.5, 1, 1.8$. The orange, black, and blue curves represent the flows at $t = 0.5, 1, 1.8$, respectively.*
- *In Figure 7(a) and 7(b), the color curves (orange, black, and blue curves) depict the flows of the curvature $k(s,t) = \frac{\sqrt{2}}{s+t}$ at $t = 0.5, 1, 1.8$, respectively. Figure 7(b) represents the (3-D) graph of the flows of the curvature for $s \in$ [-5, 5] and $t \in [0,2]$.*
- *In Figure 8(a) and 8(b), the color curves (orange, black, and blue curves) depict the (3-D) graph of the flows of the torsion $\tau(s,t) = \operatorname{sech}(s+t)$ at $t = 0.5, 1, 1.8$, respectively. Figure 8(b) illustrates the (3-D) graph of the flows of the torsion for $s \in$ [-5, 5] and $t \in [0,2]$.*
- *It is obvious that in Figure 6, Figure 7, Figure 8, the flows of curvature and torsion have a shift to the left by increasing the time values.*

Application 3. *Consider the (SPC) $\beta(s,t) = (\beta_1(s,t), \beta_2(s,t), \beta_3(s,t))$ with (SPNV) which moves with normal acceleration $F = k(s,t)$ and satisfies the one-dimensional wave equation (27) with the following initial conditions:*

$$\begin{aligned}
&\beta(s,0) = (\beta_1(s,0), \beta_2(s,0), \beta_3(s,0)),\\
&\beta_t(s,0) = (\beta_{1,t}(s,0), \beta_{2,t}(s,0), \beta_{3,t}(s,0)),\\
&\beta_1(s,0) = \phi_1(s) = s \quad , \quad \beta_{1,t}(s,0) = \psi_1(s) = 0,\\
&\beta_2(s,0) = \phi_2(s) = \frac{2}{3}s^{\frac{3}{2}} \quad , \quad \beta_{2,t}(s,0) = \psi_2(s) = \sqrt{s},\\
&\beta_3(s,0) = \phi_3(s) = \frac{-2}{3}(2-s)^{\frac{3}{2}} \quad , \quad \beta_{3,t}(s,0) = \psi_3(s) = \sqrt{2-s}.
\end{aligned} \tag{41}$$

Substitute from (41) *into* (29)*, then we obtain the general solution:*

$$\beta(s,t) = \big(s, \frac{2}{3}(s+t)^{\frac{3}{2}}, -\frac{2}{3}(2-s-t)^{\frac{3}{2}}\big). \tag{42}$$

The flows of the (SPC) (42) *are illustrated by Figure 9.*

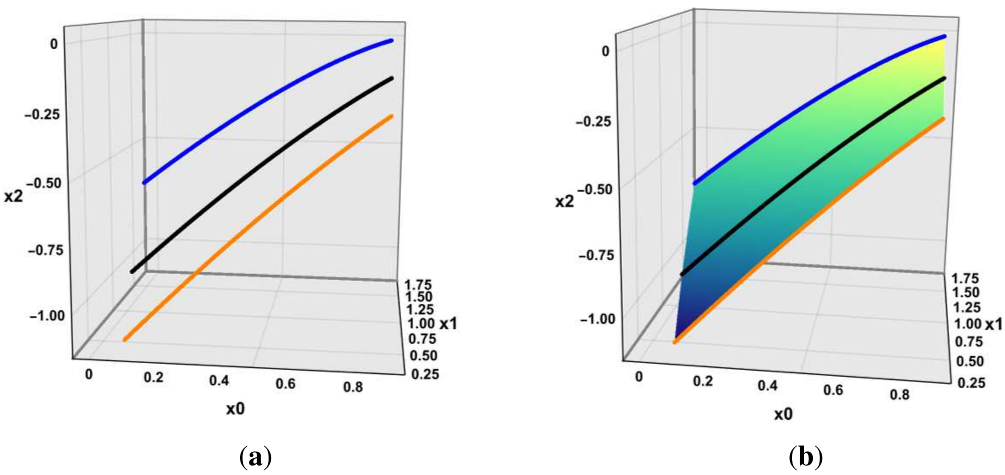


Fig. 9: Flows of the (SPC) $\beta(s,t)$ for $s \in [0.1, 0.9]$ and $t \in [0.5, 1]$. The orange, black, and blue curves depict the flows at $t = 0.5, 0.7, 1$, respectively.

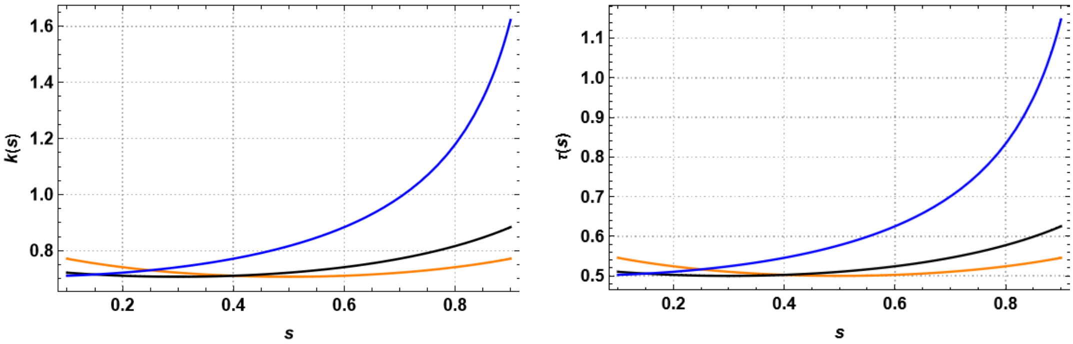
The flows of the Frenet frame are:

$$\begin{aligned} T(s,t) &= (1, \sqrt{s+t}, \sqrt{2-s-t}),\\ N(s,t) &= (0, \frac{\sqrt{2-s-t}}{\sqrt{2}}, -\frac{\sqrt{s+t}}{\sqrt{2}}),\\ B(s,t) &= (\sqrt{2}, \frac{\sqrt{s+t}}{\sqrt{2}}, \frac{\sqrt{2-s-t}}{\sqrt{2}}). \end{aligned} \tag{43}$$

The Frenet frame vectors (43) *satisfy the properties in **Definition 4** and **Definition 5**. Substitute from* (42) *into* (30) *and* (31)*, then we get the curvature and torsion:*

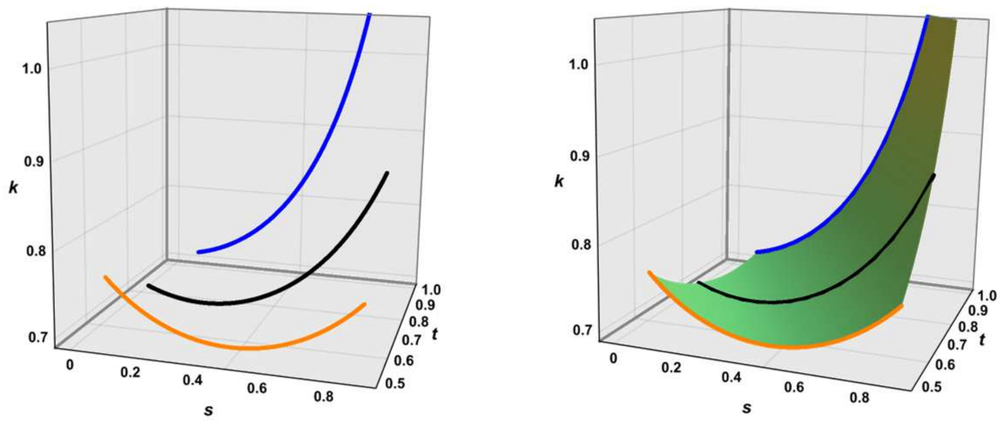
$$\begin{aligned} k(s,t) &= C_1(s+t) = \frac{1}{\sqrt{2(s+t)(2-s-t)}},\\ \tau(s,t) &= C_2(s+t) = \frac{1}{2\sqrt{(s+t)(2-s-t)}}. \end{aligned} \tag{44}$$

We plot the curvature and torsion of the (SPC) and their flows as illustrated by (Figure 10), (Figure 11), and (Figure 12).



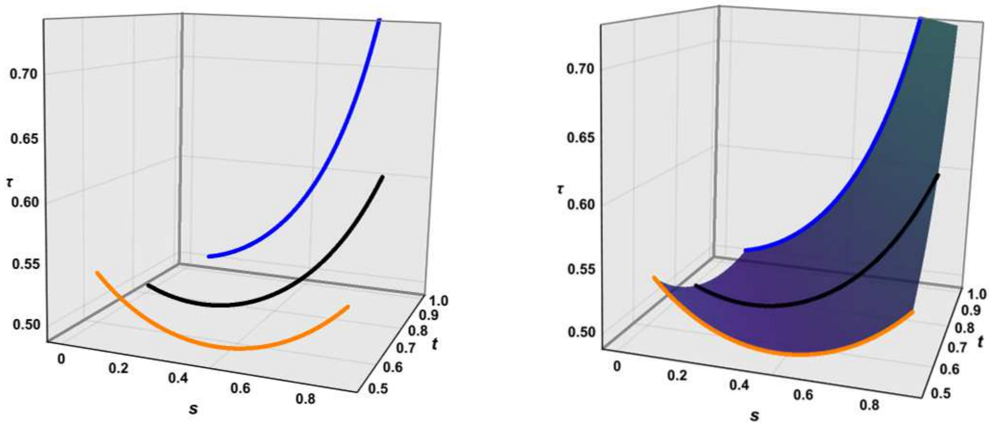
(**a**) (**b**)

Fig. 10: The orange, black, and blue curves depict the flows of the curvature and torsion of the (SPC) for $s \in [0.1, 0.9]$ **at** $t = 0.5, 0.7, 1$**, respectively.**



(**a**) (**b**)

Fig. 11: The flows of curvature for the (SPC) $k(s,t) = \frac{1}{\sqrt{2(s+t)(2-s-t)}}$ **for** $s \in [0.1, 0.9]$ **and** $t \in [0.5, 1]$**. The flows at** $t = 0.5, 0.7, 1$**, are depicted by the orange, black, and blue curves, respectively.**



(a) **(b)**

Fig. 12: The flows of torsion for the (SPC) $\tau(s,t) = \frac{1}{2\sqrt{(s+t)(2-s-t)}}$ for $s \in [0.1, 0.9]$ and $t \in [0.5, 1]$. The orange, black, and blue curves represent the flows at $t = 0.5, 0.7, 1$, respectively.

Graphic interpretations on application 3

- *Figure 9(b) represents the (3-D) graph of the flows of the (SPC) (42) for $s \in [0.1, 0.9]$ and $t \in [0.5, 1]$. The orange, black, and blue curves in Figure 9(a), and Figure 9(b), represents the flows at $t = 0.5, 0.7, 1$, respectively.*

- *Figure 10(a) and Figure 10(b) represent the (2-D) graphs of the flows of curvature $k(s)$ and torsion $\tau(s)$ for $s \in [0.1, 0.9]$ and $t = 0.5, 0.7, 1$. The orange, black, and blue curves represent the flows at $t = 0.5, 0.7, 1.$, respectively.*

- *In Figure 11(a) and Figure 11(b), the color curves (orange, black, and blue curves) represent the flows of the curvature $k(s,t) = \frac{1}{\sqrt{2(s+t)(2-s-t)}}$ at $t = 0.5, 0.7, 1$, respectively. Figure 11(b) represents the (3-D) graph of the flows of the curvature for $s \in$ [-5,5] and $t \in [0, 2]$.*

- *In Figure 12(a) and 12(b), the color curves (orange, black, and blue curves) represent the flows of the torsion $\tau(s,t) = \frac{1}{2\sqrt{(s+t)(2-s-t)}}$ at $t = 0.5, 0.7, 1$, respectively. Figure 12(b) represents the (3-D) graph of the flows of the torsion for $s \in [0.1, 0.9]$ and $t \in [0.5, 1]$.*

Application 4. *Consider the (SPC) $\beta(s,t) = (\beta_1(s,t), \beta_2(s,t), \beta_3(s,t))$ with (SPNV) which moves with normal acceleration $F = k(s,t)$ and satisfies the one-dimensional wave equation (27) with the following initial conditions:*

$$\begin{aligned}
&\beta(s,0) = (\beta_1(s,0), \beta_2(s,0), \beta_3(s,0)),\\
&\beta_t(s,0) = (\beta_{1,t}(s,0), \beta_{2,t}(s,0), \beta_{3,t}(s,0)),\\
&\beta_1(s,0) = \phi_1(s) = \log(\sec s + \tan s) \quad , \quad \beta_{1,t}(s,0) = \psi_1(s) = \sec s,\\
&\beta_2(s,0) = \phi_2(s) = \sqrt{2}s \quad , \quad \beta_{2,t}(s,0) = \psi_2(s) = 0,\\
&\beta_3(s,0) = \phi_3(s) = \log(\sec s) \quad , \quad \beta_{3,t}(s,0) = \psi_3(s) = \tan s.
\end{aligned} \tag{45}$$

Substitute from (45) into (29), then we obtain the general solution:

$$\beta(s,t) = \big(\log(\sec(s+t) + \tan(s+t)), \sqrt{2}s, \log(\sec(s+t)\big). \tag{46}$$

The flows of the (SPC) (46) *are illustrated by Figure 13.*

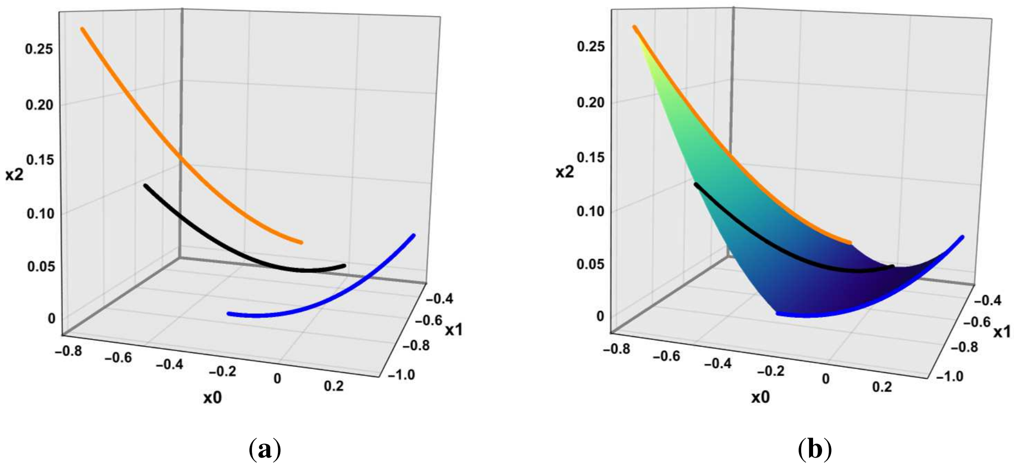


Fig. 13: The flows of the (SPC) $\beta(s,t)$ for $s \in$[- 0.7,-0.21] and $t \in [0,0.5]$. The orange, black, and blue curves represent the flows at $t = 0, 0.2, 0.5$, respectively.

The flows of the Frenet frame are:

$$\begin{aligned} T(s,t) &= (\sec(s+t), \sqrt{2}, \tan(s+t)), \\ N(s,t) &= (\tan(s+t), 0, \sec(s+t)), \\ B(s,t) &= (-\sqrt{2}\sec(s+t), -1, -\sqrt{2}\tan(s+t)). \end{aligned} \tag{47}$$

The Frenet frame vectors (47) *satisfy the properties in **Definition 4** and **Definition 5**. Substitute from* (46) *into* (30) *and* (31)*, then we get the curvature and torsion:*

$$\begin{aligned} k(s,t) &= C_1(s+t) = \sec(s+t), \\ \tau(s,t) &= C_2(s+t) = -\sqrt{2}\sec(s+t). \end{aligned} \tag{48}$$

We plot the flows of the curvature and torsion of the (SPC) as illustrated by (Figure 14), (Figure 15), and (Figure 16).

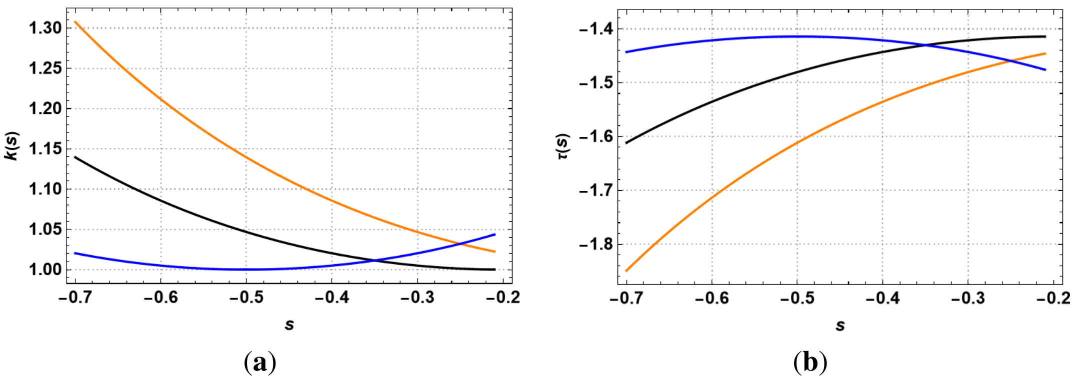
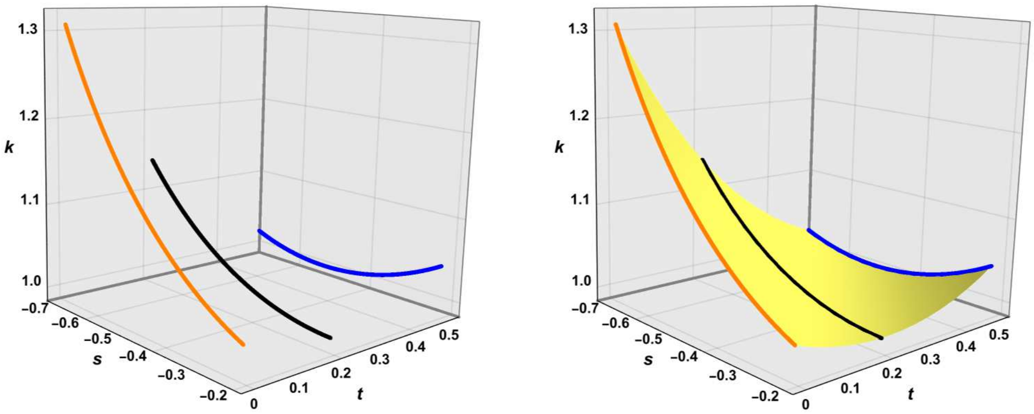
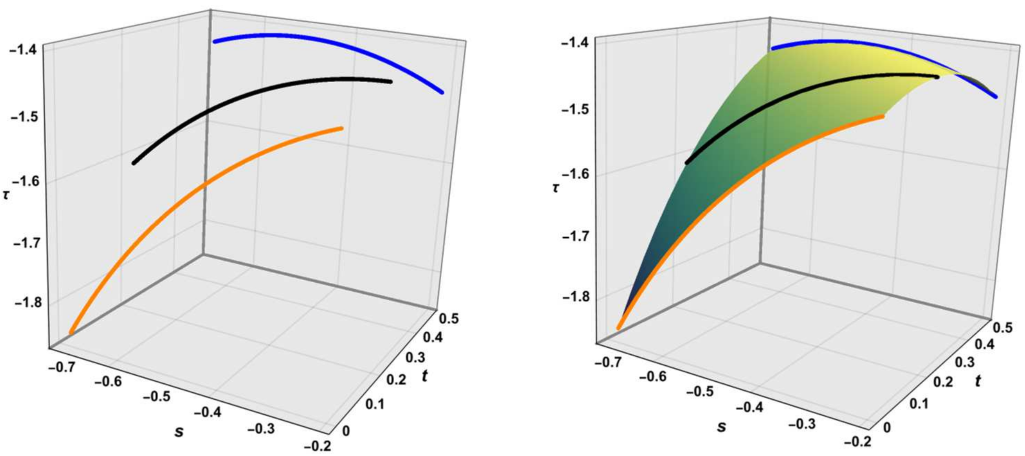


Fig. 14: The orange, black, and blue curves represent the flows of curvature and torsion of the (SPC) for $s \in$[- 0.7,-0.21] at $t = 0, 0.2, 0.5$, respectively.



(a) (b)

Fig. 15: The flows of curvature for the (SPC) $k(s,t) = \sec(s+t)$ for $s \in$[- 0.7,-0.21] and $t \in [0, 0.5]$. The orange, black, and blue curves represent the flows at $t = 0, 0.2, 0.5$, respectively.



(a) (b)

Fig. 16: The flows of torsion for the (SPC) $\tau(s,t) = -\sqrt{2}\sec(s+t)$. for $s \in$[- 0.7,-0.21] and $t \in [0, 0.5]$. The orange, black, and blue curves represent the flows at $t = 0, 0.2, 0.5$, respectively.

Graphic interpretations on application 4

- *Figure 13(b) represents the (3-D) graph of the flows of the (SPC) (46) for $s \in [-0.7, -0.21]$ and $t \in [0, 0.5]$ The orange, black, and blue curves in Figure 13(a), and Figure 13(b), represents the flows at $t = 0, 0.2, 0.5$, respectively.*

- *Figure 14(a) and Figure 14(b) represent the (2-D) graphs of the flows of curvature $k(s)$ and torsion $\tau(s)$ for $s \in [-0.7, -0.21]$ and $t = 0, 0.2, 0.5$ with orange, black, and blue curves, respectively.*

- *In Figure 15(a) and Figure 15(b), the color curves (orange, black, and blue curves) represent the flows of the curvature* $k(s,t) = \sec(s+t)$ *for* $s \in [-0.7, -0.21]$*, and* $t = 0, 0.2, 0.5$*, respectively. Figure 15(b) represents the (3-D) graph of the flows of the curvature for* $s \in [-0.7, -0.21]$ *and* $t \in [0, 0.5]$.

- *In Figure 16(a) and 16(a), the color curves (orange, black, and blue curves) represent the flows of the torsion* $\tau(s,t) = -\sqrt{2}\sec(s+t)$ *for* $s \in [-0.7, -0.21]$ *, and* $t = 0, 0.2, 0.5$*, respectively. Figure 16(b) represents the (3-D) graph of the flows of the torsion for* $s \in [-0.7, -0.21]$ *and* $t \in [0, 0.5]$.

5 Conclusions and discussion

In this work, the flows of inextensible (SPC) and (TIC) specified by the acceleration functions are investigated according to the equation of motion $\frac{\partial^2 \beta}{\partial t^2} = E\,T + F\,N + G\,B$. This work is restricted to the study of the flows of inextensible (SPC) with the (SPNV) that is described by the normal acceleration $F = k$. We obtained some new results, listed as follows:

1. The relationship between the motions according to velocity fields and acceleration fields is obtained (**Lemma 4**).

2. The second (TEEs) of the Frenet frame are obtained (**Theorem 5**).

3. The flows of (SPC) with (SPNV) described by a normal acceleration that equals the curvature of the curve are investigated by (21) and the position vector of the (SPC) satisfied the one-dimensional wave equation (24).

4. We present four novel applications to discuss the flows of (SPC) with (SPNV) and we graph the flows of the (SPC) and the flows of its curvatures.

5. In application 1, the flows of an inextensible (SPC) is given by the parametrization $\beta(s,t) = \big(s, (s+t)\sin(\log(s+t)), (s+t)\cos(\log(s+t))\big)$, with Frenet frame (35). We obtained the flows of curvatures $k(s,t) = \frac{\sqrt{2}}{s+t}$, and $\tau(s,t) = \frac{1}{(s+t)}$.

6. In application 2, the flows of the inextensible (SPC) are given by the parametrization $\beta(s,t) = \big(s, \sqrt{2}\log\cosh(s+t), 2\sqrt{2}\arctan(\tanh(\frac{s+t}{2}))\big)$ with Frenet frame (39). We obtained the flows of curvatures are $k(s,t) = \sqrt{2}\text{sech}\,(s+t)$, and $\tau(s,t) = \text{sech}\,(s+t)$.

7. In application 3, the flows of the inextensible (SPC) are given by the parametrization $\beta(s,t) = \big(s, \frac{2}{3}(s+t)^{\frac{3}{2}}, -\frac{2}{3}(2-s-t)^{\frac{3}{2}}\big)$, with Frenet frame (43). The flows of curvatures are obtained by $k(s,t) = \frac{1}{\sqrt{2(s+t)(2-s-t)}}$, and $\tau(s,t) = \frac{1}{2\sqrt{(s+t)(2-s-t)}}$.

8. In application 4, the flows of the inextensible (SPC) are given by the parametrization $\beta(s,t) = \big(\log(\sec(s+t) + \tan(s+t)), \sqrt{2}s, \log(\sec(s+t))\big)$, with Frenet frame (47). The flows of curvatures are obtained by $k(s,t) = \sec(s+t)$, and $\tau(s,t) = -\sqrt{2}\sec(s+t)$.

Abbreviations

The abbreviations used in this manuscript are listed as follows:

IFC	Inextensible Flows of Curve
IFSPC	Inextensible Flows of Space-like Curve
2-D	two-dimensional
3-D	three-dimensional
NSC(s)	necessary and sufficient conditions
ODE	Ordinary Differential Equation(s)
PDE(s)	Partial Differential Equation(s)
SPC	Space-like Curve
SPNV	Space-like Principal Normal Vector
TIC	Time-like Curve
TEEs	Time Evolution Equations

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