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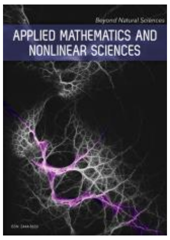

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A Study on Normal Motion of the Torus of Revolution in $\mathbb{R}^3$

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Abstract

In the present research paper, we investigate the motion of surfaces in $\mathbb{R}^3$ according to their curvatures. We study the motion of the torus of revolution via the normal velocity. We consider two cases: when the normal velocity is a function of both the time and the coordinates of the torus, and when it is a function of time only. We also study how the torus moves under different types of curvature flows, such as inverse mean curvature flow, inverse Gaussian curvature flow, and harmonic mean curvature flow. Moreover, we present some new applications of these flows.

Keywords: Evolution of surfaces; mean curvature flow; Gaussian curvature flow; inverse mean curvature flow; inverse Gaussian curvature flow; harmonic mean curvature flow.
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1 Introduction

The curvature flow is a significant type of geometric flow of surfaces. The studying of the normal motion of surfaces in $\mathbb{R}^3$ is governed by a law that the normal velocity is characterized by the mean curvature (MC) and its variants such as inverse mean curvature (IMC). The normal motion with mean curvature flow (MCF) has extensive applications in geometry, physics, and analysis. The (MCF) is a well-known example of hypersurface geometric flow in a Riemannian manifold. If a point on a surface moves according to a normal velocity equal to (MC), then a family of surfaces will evolve by (MC) at different time values. Curvature flow is a classical mathematical study. In addition to resulting in fascinating systems of nonlinear partial differential equations, they enable the accurate mathematical modeling of physical phenomena such as material interface propagation and crystal growth. The topic of the motion of curves and surfaces is a very attractive and essential study in differential geometry, and it is investigated by many researchers:

Leung [1] studied the (MC) evolution of hypersurfaces in Euclidean spaces, Lagrangian (MCF), and the (MCF) of conormal bundles. Yoon et al. [2], obtained the time evolution equations (TEEs) for curvatures of a space-like curve in Minkowski space $\mathbb{R}^{2,1}$. The evolution equations were derived for a time-like ruled surface generated by a time-like normal vector and a space-like binormal vector of a space-like curve. Furthermore, the necessary conditions for the evolution of inelastic surfaces were derived. Several applications of evolution equations of curvatures for space-like curves were provided.

Kaymanli et al. [3], investigated the ruled surfaces with normal and binormal vectors along a time-like space curve in $\mathbb{R}^{2,1}$ by employing the q-frame. The directrices of quasi-normal and quasi-binormally ruled surfaces were used to study their directional evolution. Abd Ellah [4], studied the motion of translation surfaces and their generated curves in $\mathbb{R}^3$ and derived their evolution equations.

Nakayama et al. [5] studied the kinematics of surfaces in $\mathbb{R}^3$ by using differential geometry, and presented some applications of surfaces that are parameterized by the lines of curvature. Smoczyk [6] investigated the regularity of 2-surfaces which are evolving by their (IMC) in an asymptotically $3-$flat Riemannian manifold. Eren et al. [7] employed the modified orthogonal frame to investigate the evolution of space curves and some special ruled surfaces. Yuksel et al. [8], studied the inextensibility of tangential, normal, and binormal ruled surfaces generated by a Salkowski curve. Some theorems related to the inextensibility of ruled surfaces in $\mathbb{R}^3$ were proved. A general approach for the flows of an inextensible curve lying on an oriented surface in $\mathbb{R}^3$ was derived by Yildiz [9] according to the Darboux frame. Furthermore, the necessary and sufficient conditions for the flows of inextensible curves were obtained as partial differential equations (PDEs) involving the geodesic curvatures. Specific cases for inextensible curves were provided.

Hussien and Gaber [10], constructed new surfaces in $\mathbb{R}^3$ by the motion of inextensible Frenet frame curves. Hussien et al. [11], studied the evolution of normal and binormal ruled surfaces generated by the normal and binormal vector fields of a space curve in $\mathbb{R}^3$. Bas et al. [12], investigated the inextensible flows of spacelike curves on oriented space-like surfaces in Minkowski three-space M_1^3. Gaber [13] studied the evolution of curves with type-1 Bishop frame. The (TEEs) of the type-1 Bishop frame and the (TEEs) of Bishop curvatures were derived.

Korpinar et al. [14], studied the inextensible flows for the tangent developable surfaces in $\mathbb{R}^3$ and obtained some novel results related to the minimal tangent developable surfaces. Tapia [15] obtained the evolution equations for the Riemann tensor, the Ricci tensor, and the scalar curvature specified by (MCF). Asil et al. [16], investigated inextensible flows of novel type surfaces generated by the first principle direction curve in $\mathbb{R}^3$. In addition, the (GC) and (MC) of these surfaces were obtained. Yuzbasi, et al. [17], investigated the flows of an inextensible spacelike curve on a lightlike surface in $\mathbb{R}^{2,1}$. A necessary and sufficient condition for the flows to be inextensible was derived as a (PDE) in terms of the curvatures of the curve. Furthermore, the lightlike ruled surfaces in $\mathbb{R}^{2,1}$ were classified. The inextensible evolution of a lightlike curve on the lightlike tangent developable surface was characterized.

The present paper focuses on the study of the normal motion of surfaces by taking the torus of revolution as an application. The evolution of the torus is investigated, and the time evolutions of the first and second fundamental quantities are derived. We give some special cases for normal motion with velocity as a function of the coordinates of the surface and the time parameter. In addition, we study the case for normal motion with velocity as a function of time only. Furthermore, we investigate the classification of the normal motion of the torus by its curvature flows and give some novel applications for the normal motion via the inverse mean curvature flow, the inverse Gaussian curvature flow, and the harmonic mean curvature flow.

This work is outlined as follows: Section 2 presents geometric preliminaries about surfaces and their motion in $\mathbb{R}^3$. In Section 3, we obtain the results and discussion on studying the evolution of surfaces in $\mathbb{R}^3$ and provide the torus of revolution as an application of the normal motion of surfaces. In Section 4 and Section 5, we give some new applications on the normal motion of the torus by its curvature flows. Finally, we present conclusions.

2 Geometric Preliminaries

In this section, we present a short review for some geometric concepts of surfaces and their motions in $\mathbb{R}^3$ [5, 18, 19].

2.1 The Geometry of surfaces in $\mathbb{R}^3$

Let U be an open subset of $\mathbb{R}^2$, and define the function

$$X : U \subset \mathbb{R}^2 \to \mathbb{R}^3$$

Consider a surface $\Sigma : X(u^1, u^2)$ in $\mathbb{R}^3$ evolves with local coordinates (u^1, u^2) at a point q on the surface Σ. Let t be the time parameter and $X(u^1, u^2, t)$ be the position vector at a point q on Σ.

Definition 1. Let X_μ and X_ν be the tangent vectors at a point q on the surface Σ. The metric $g_{\mu\nu}$ on the surface Σ is defined by:

$$g_{\mu\nu} = X_\mu \cdot X_\nu, \qquad \mu, \nu = 1, 2, \tag{1}$$

where X_μ are the tangent vectors and defined by:

$$X_\mu = \frac{\partial X}{\partial u^\mu}, \qquad \mu = 1, 2. \tag{2}$$

Let $g^{\mu\nu}$ be the inverse metric tensors of $g_{\mu\nu}$, where

$$g_{\mu\nu} \cdot g^{\mu k} = \delta^k_\nu = \begin{cases} 1 \text{ if } & k = \nu, \\ 0 \text{ if } & k \neq \nu. \end{cases} \tag{3}$$

Definition 2. The unit normal vector N at regular points is defined by the linear independent tangent vectors X_1 and X_2 as follows:

$$N = \frac{X_1 \times X_2}{|X_1 \times X_2|}. \tag{4}$$

Definition 3. The second fundamental quantities (curvature tensors) $L_{\mu\nu}$ are defined by:

$$L_{\mu\nu} = \langle X_{\mu\nu}, N \rangle, \qquad \mu, \nu = 1, 2. \tag{5}$$

Definition 4. The Gaussian curvature (GC) and the mean curvature (MC) of the surface Σ are defined, respectively by:

$$G = \kappa_1 \cdot \kappa_2 = \frac{L_{11}L_{22} - (L_{12})^2}{g_{11}g_{22} - (g_{12})^2}, \tag{6}$$

$$H = \frac{\kappa_1 + \kappa_2}{2} = \frac{L_{11}g_{22} - 2L_{12}g_{12} + L_{22}g_{11}}{2(g_{11}g_{22} - (g_{12})^2)}, \tag{7}$$

where κ_1 and κ_2 are the principal curvatures at a point q on the surface Σ.

Definition 5. The covariant derivatives ∇_ν of the covariant vector X_μ and contravariant vector X^μ are defined, respectively by:

$$\nabla_\nu X_\mu = X_{\nu\mu} - X_k \Gamma^k_{\nu\mu}, \tag{8}$$

$$\nabla_\nu X^\mu = X^\mu_\nu + X^k \Gamma^\mu_{\nu k}, \tag{9}$$

where $\Gamma^k_{\nu\mu}$ are Christoffel's symbols of the second kind and they are defined by:

$$\Gamma^k_{\nu\mu} = \frac{1}{2} g^{kr} \Big(\frac{\partial g_{\mu r}}{\partial u^\nu} + \frac{\partial g_{\nu r}}{\partial u^\mu} - \frac{\partial g_{\mu\nu}}{\partial u^r} \Big). \tag{10}$$

Definition 6. The Gauss Weingarten equations are defined by the form:

$$\begin{aligned} X_{\mu\nu} &= \Gamma^k_{\nu\mu} X_k + L_{\mu\nu} N, \\ N_\nu &= -g^{\mu k} L_{\mu\nu} X_k, \end{aligned} \tag{11}$$

where the vectors $X_{\mu\nu}$ and N_ν are linear combinations of the set of basis vectors X_μ and N. From (8) and (11), then we have,

$$L_{\mu\nu} = \langle \nabla_\nu X_\mu, N \rangle.$$

2.2 The dynamics of surfaces in $\mathbb{R}^3$

Consider the surface Σ with position vector $X(u^1(t), u^2(t), t)$ moves in $\mathbb{R}^3$, where the local coordinates (u^1, u^2) of the surface depend on the time parameter t. The motion of the surface Σ is prescribed by:

$$\frac{\mathrm{d}X}{\mathrm{d}t} = \sigma^\mu X_\mu + \psi N, \quad \mu = 1, 2. \tag{12}$$

where, σ^μ, $\mu = 1, 2$ are the tangential velocities and ψ is the normal velocity.

Theorem 1. *[5] The (TEEs) for the tangent vectors X_ν and normal vector N at a point q on the surface Σ, can be written in a matrix form as follows,*

$$\frac{\partial}{\partial t} \begin{pmatrix} X_\nu \\ N \end{pmatrix} = \begin{pmatrix} \nabla_\nu (\sigma^k - \dot{u}^k) - L^k_\nu \psi & \frac{\partial \psi}{\partial u^\nu} + (\sigma^\mu - \dot{u}^\mu) L_{\mu\nu} \\ -g^{k\nu} \Big(\frac{\partial \psi}{\partial u^\nu} + (\sigma^k - \dot{u}^k) L_{k\nu} \Big) & 0 \end{pmatrix} \begin{pmatrix} X_k \\ N \end{pmatrix} \tag{13}$$

Proof. Since, the local coordinates (u^1, u^2) of the surface depend on the time t, then the total derivative of $X(u^1(t), u^2(t), t)$, with respect to the time parameter t is given by:

$$\frac{\mathrm{d}X}{\mathrm{d}t} = \dot{u}^\mu X_\mu + \frac{\partial X}{\partial t}. \qquad (\dot{\ }) = \frac{d}{\mathrm{d}t} \tag{14}$$

Then, from (12) and (14), we have

$$\frac{\partial X}{\partial t} = (\sigma^\mu - \dot{u}^\mu) X_\mu + \psi N. \tag{15}$$

Using the compatibility condition ($\frac{\partial X_\nu}{\partial t} = \frac{\partial}{\partial u^\nu}(\frac{\partial X}{\partial t})$) and using (11) and (15), then we get the following the (TEEs):

$$\begin{aligned} \frac{\partial X_\nu}{\partial t} &= \frac{\partial}{\partial u^\nu} \Big((\sigma^\mu - \dot{u}^\mu) X_\mu + \psi N \Big) \\ &= \Big(\frac{\partial}{\partial u^\nu} (\sigma^\mu - \dot{u}^\mu) \Big) X_\mu + (\sigma^\mu - \dot{u}^\mu) \Big(\Gamma^k_{\nu\mu} X_k + L_{\mu\nu} N \Big) + \frac{\partial \psi}{\partial u^\nu} N + \psi \big(-g^{\mu k} L_{\mu\nu} X_k \big) \\ &= \Big(\frac{\partial}{\partial u^\nu} (\sigma^k - \dot{u}^k) + \Gamma^k_{\nu\mu} (\sigma^\mu - \dot{u}^\mu) - \psi g^{k\mu} L_{\mu\nu} \Big) X_k + \Big(\frac{\partial \psi}{\partial u^\nu} + (\sigma^\mu - \dot{u}^\mu) L_{\mu\nu} \Big) N. \end{aligned} \tag{16}$$

By using (9), then we get the (TEEs) for the tangent vectors at a point q on the surface Σ:

$$\frac{\partial X_\nu}{\partial t} = \Big(\nabla_\nu(\sigma^k - \dot{u}^k) - L^k_\nu \psi\Big) X_k + \Big(\frac{\partial \psi}{\partial u^\nu} + (\sigma^\mu - \dot{u}^\mu) L_{\mu\nu}\Big) N. \tag{17}$$

Since $\langle X_\nu, N\rangle = 0$, by taking the t derivative of this equation, then we have the following compatibility condition:

$$\Big\langle X_\nu, \frac{\partial N}{\partial t}\Big\rangle = -\Big\langle \frac{\partial X_\nu}{\partial t}, N\Big\rangle. \tag{18}$$

Substituting from (17) into (18), then we get the (TEE) for the normal vector field N as follows:

$$\frac{\partial N}{\partial t} = -g^{\mu\nu}\Big(\frac{\partial \psi}{\partial u^\nu} + (\sigma^\mu - \dot{u}^\mu) L_{\mu\nu}\Big) X_\mu. \tag{19}$$

Theorem 2. *The (TEEs) for the metric tensors $g_{\mu\nu}$ are given by:*

$$\frac{\partial g_{\mu\nu}}{\partial t} = \nabla_\mu(\sigma_\nu - \dot{u}_\nu) + \nabla_\nu(\sigma_\mu - \dot{u}_\mu) - 2\psi L_{\mu\nu}. \tag{20}$$

Proof. Since

$$\frac{\partial g_{\mu\nu}}{\partial t} = \frac{\partial}{\partial t}\Big\langle X_\mu, X_\nu\Big\rangle.$$

Using (17), then we have:

$$\begin{aligned}\frac{\partial g_{\mu\nu}}{\partial t} &= \Big\langle \frac{\partial X_\mu}{\partial t}, X_\nu\Big\rangle + \Big\langle X_\mu, \frac{\partial X_\nu}{\partial t}\Big\rangle \\ &= \Big\langle (\nabla_\mu(\sigma^k - \dot{u}^k) - L^k_\mu \psi) X_k, X_\nu\Big\rangle + \Big\langle X_\mu, (\nabla_\nu(\sigma^k - \dot{u}^k) - L^k_\nu \psi) X_k\Big\rangle \\ &= (\nabla_\mu(\sigma^k - \dot{u}^k) - L^k_\mu \psi) g_{k\nu} + (\nabla_\nu(\sigma^k - \dot{u}^k) - L^k_\nu \psi) g_{k\mu}.\end{aligned} \tag{21}$$

Hence, we get

$$\frac{\partial g_{\mu\nu}}{\partial t} = \nabla_\mu(\sigma_\nu - \dot{u}_\nu) + \nabla_\nu(\sigma_\mu - \dot{u}_\mu) - 2\psi L_{\mu\nu} \tag{22}$$

Theorem 3. *The (TEEs) for the curvature tensors $L_{\mu\nu}$ are given by:*

$$\frac{\partial L_{\mu\nu}}{\partial t} = \nabla_\mu \nabla_\nu \psi + \nabla_\mu\Big((\sigma^\mu - \dot{u}^\mu) L_{\mu\nu}\Big) + L_{\mu k}\nabla_\nu(\sigma^k - \dot{u}^k) - \psi L^k_\nu L_{\mu k}. \tag{23}$$

Proof. Since

$$L_{\mu\nu} = \langle \nabla_\mu X_\nu, N\rangle. \tag{24}$$

By taking the $t-$derivative of (24) and using (8), (11), and (13), then we get the (TEEs) for curvatures as follows:

$$\begin{aligned}\frac{\partial L_{\mu\nu}}{\partial t} &= \Big\langle \nabla_\mu \frac{\partial X_\nu}{\partial t}, N\Big\rangle + \Big\langle \nabla_\mu X_\nu, \frac{\partial N}{\partial t}\Big\rangle = \Big\langle \nabla_\mu \frac{\partial X_\nu}{\partial t}, N\Big\rangle \\ &= \Big\langle \nabla_\mu\Big((\nabla_\nu(\sigma^k - \dot{u}^k) - L^k_\nu \psi) X_k + (\frac{\partial \psi}{\partial u^\nu} + (\sigma^\mu - \dot{u}^\mu) L_{\mu\nu}) N\Big), N\Big\rangle \\ &= (\nabla_\nu(\sigma^k - \dot{u}^k) - L^k_\nu \psi)\langle \nabla_\mu X_k, N\rangle + \nabla_\nu(\frac{\partial \psi}{\partial u^\nu} + (\sigma^\mu - \dot{u}^\mu) L_{\mu\nu})\langle N, N\rangle.\end{aligned} \tag{25}$$

Then

$$\frac{\partial L_{\mu\nu}}{\partial t} = \nabla_\mu \nabla_\nu \psi + \nabla_\mu\Big((\sigma^\mu - \dot{u}^\mu) L_{\mu\nu}\Big) + L_{\mu k}\nabla_\nu(\sigma^k - \dot{u}^k) - \psi L^k_\nu L_{\mu k}. \tag{26}$$

3 Results and Discussion

Since the tangential velocities σ^{μ}, $\mu = 1,2$ doesn't affect the shape of the surface, so assume that $\sigma^{\mu} = 0$, then we can study the normal motion of the surface. Also, assume that u^{μ} and t are independent, so, $\dot{u}^{\mu} = 0$. Hence, the evolution equation of the surface is described by:

$$\frac{\partial X}{\partial t} = \psi N. \tag{27}$$

Lemma 4. *Assume that the surface moves according to the (TEE) given by (27). Then the (TEEs) of the tangent vectors X_{ν} and normal vector N at a point q on the surface Σ can be written in a matrix form as follows:*

$$\frac{\partial}{\partial t}\begin{pmatrix} X_{\nu} \\ N \end{pmatrix} = \begin{pmatrix} -L_{\nu}^{k}\psi & \frac{\partial \psi}{\partial u^{\nu}} \\ -g^{k\nu}\left(\frac{\partial \psi}{\partial u^{\nu}}\right) & 0 \end{pmatrix}\begin{pmatrix} X_{k} \\ N \end{pmatrix} \tag{28}$$

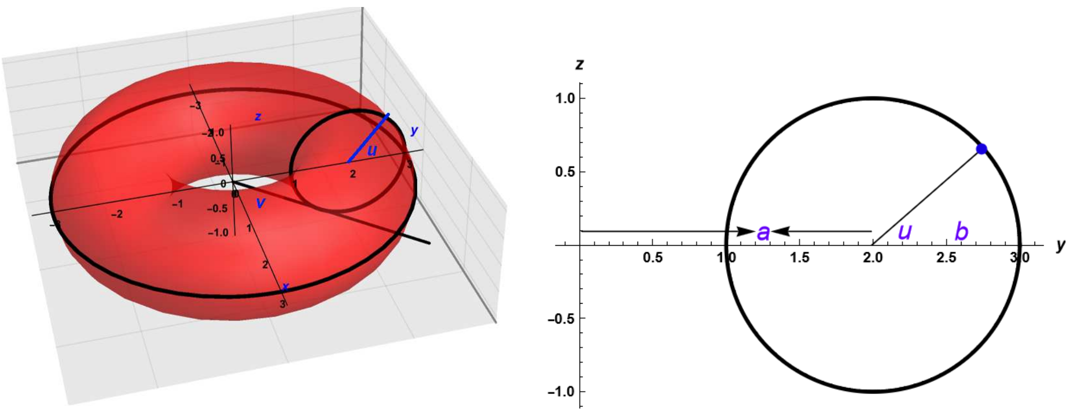
Lemma 5. *Consider the normal motion of the surface that is described by (27). Then the (TEEs) for the metric tensors $g_{\mu\nu}$ and curvature tensors $L_{\mu\nu}$ are given by:*

$$\begin{aligned} \frac{\partial g_{\mu\nu}}{\partial t} &= -2\psi L_{\mu\nu}\ , \\ \frac{\partial L_{\mu\nu}}{\partial t} &= \nabla_{\mu}\nabla_{\nu}\psi - \psi L_{\nu}^{k} L_{\mu k}. \end{aligned} \tag{29}$$

3.1 Normal motion for the torus of revolution

Definition 7. [18, 19] A torus of revolution is a surface obtained by a circle C (the profile curve). Assume that C is the circle in the $yz-$ plane with radius $b > 0$ and center $(0, a, 0)$, $a > b$. The torus of revolution is formed by rotating this circle around the z-axis (Figure 1). For the motion of the torus of revolution, assume the radii a and b depend on the time parameter t. Let $u^1 = u$ and $u^2 = v$ are the local coordinates of the torus, then the parametrization is defined by:

$$X(u,v,t) = \Big(\big(a(t) + b(t)\cos u\big)\cos v, \big(a(t) + b(t)\cos u\big)\sin v, b(t)\sin u\Big), \quad 0 \leq u, v \leq 2\pi \tag{30}$$



(a) (b)

Fig. 1 The torus of revolution with $a = 2$ and $b = 1$ for $0 \leq u, v \leq 2\pi$ and at $t = 0$.

Lemma 6. *The tangent and normal vectors are:*

$$\begin{aligned} X_u &= \Big(-b(t)\sin u\cos v, -b(t)\sin u\sin v, b(t)\cos u\Big),\\ X_v &= \Big(-\big(a(t)+b(t)\cos u\big)\sin v, \big(a(t)+b(t)\cos u\big)\cos v, 0\Big),\\ N &= \Big(-\cos u\cos v, -\cos u\sin v, -\sin u\Big). \end{aligned} \tag{31}$$

Lemma 7. *The metric tensors $g_{\mu\nu}$ and the curvature tensors $L_{\mu\nu}$ are given by:*

$$g_{\mu\nu} = \begin{pmatrix} b(t)^2 & 0 \\ 0 & \big(a(t)+b(t)\cos u\big)^2 \end{pmatrix} \quad , \quad L_{\mu\nu} = \begin{pmatrix} b(t) & 0 \\ 0 & \big(a(t)+b(t)\cos u\big)\cos u \end{pmatrix} \tag{32}$$

Lemma 8. *The (MC) and (GC) of the torus are:*

$$G = \frac{\cos u}{b(t)\big(a(t)+b(t)\cos u\big)} \quad , \quad H = \frac{a(t)+2b(t)\cos u}{2b(t)\big(a(t)+b(t)\cos u\big)}. \tag{33}$$

Lemma 9. *The (TEEs) for the metric tensors $g_{\mu\nu}$ and the curvature tensors $L_{\mu\nu}$ for the torus can be obtained from (29) and (32) by the following ordinary differential equations:*

$$\begin{aligned} \frac{\mathrm{d}b(t)}{\mathrm{d}t} &= -\psi(u,v,t),\\ \frac{\mathrm{d}a(t)}{\mathrm{d}t} &= 0, \end{aligned} \tag{34}$$

and the normal velocity $\psi(u,v,t)$ is given as the following (PDEs):

$$\psi_{uu} = 0 \ \ , \psi_{vv} = 0. \tag{35}$$

Lemma 10. *The radii $a(t)$ and $b(t)$ of the circles of the torus have the following solutions:*

$$\begin{aligned} b(t) &= b_0 - \int_0^t \psi(u,v,t)dt, \quad b(0) = b_0,\\ a(t) &= const = a_0. \end{aligned} \tag{36}$$

The normal velocity ψ that satisfies the (PDEs) given by (35), has the following solutions:

$$\psi(u,v,t) = \psi(u,t) = u\,g_1(t) + f_1(t), \tag{37}$$

or,

$$\psi(u,v,t) = \psi(v,t) = v\,g_2(t) + f_2(t). \tag{38}$$

4 Applications on the normal motion of the torus via the normal velocity $\psi(u,v,t) = \psi(u,t)$

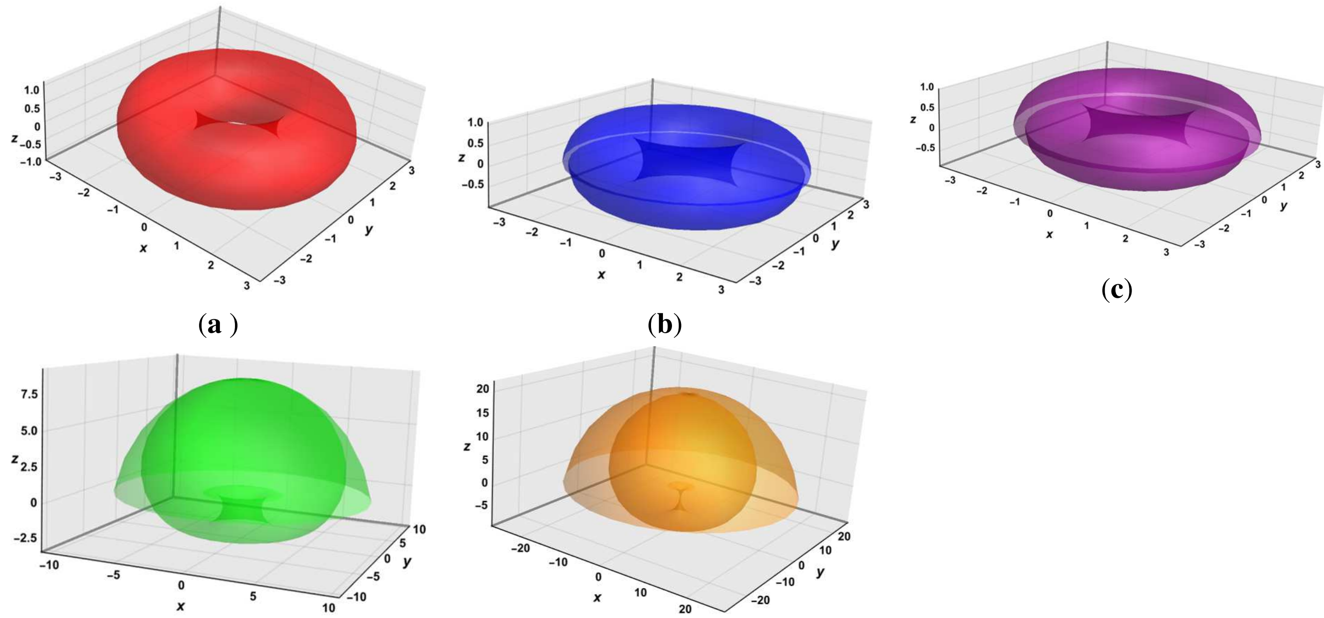
Consider the normal motion of the torus, where the normal velocity is given by (37) and assume that $f_1(t) = 0$, then $\psi(u,v,t) = \psi(u,t) = u\,g_1(t)$. We give the following cases:

4.1 Case 1: $g_1(t) = t$

If $g_1(t) = t$, then the normal velocity $\psi(u,t) = ut$. By substituting in (36), then we have:

$$
\begin{aligned}
b(t) &= \frac{1}{2}(2b_0 - t^2 u), \\
a(t) &= const = a_0.
\end{aligned}
\tag{39}
$$

Substituting from (39) into (30), then we get the evolution of the torus of revolution on which a point q on the surface evolves with the normal velocity given by $\psi(u,t) = ut$. It is illustrated in Figure 2 and Figure 3 at different values of t. Also, the normal motion of $u-$curves at the same values of t is illustrated by Figure 4.



(a) (b) (c) (d) (e)

Fig. 2 The evolution of the torus with $\psi(u,t) = ut$ for $a_0 = 2$, $b_0 = 1$, $0 \le u, v \le 2\pi$ at $t = 0, 0.21, 0.3, 2, 3$, respectively .

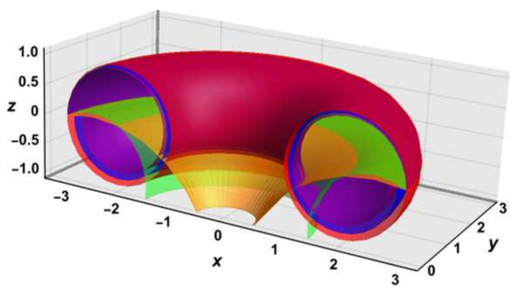
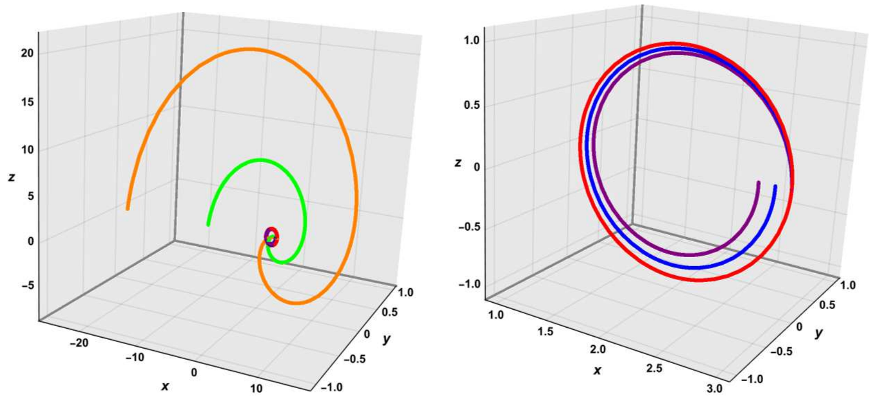


Fig. 3 The red, blue, purple, green, and orange colors depict the evolution of torus with $\psi(u,t) = ut$ for $a_0 = 2$, $b_0 = 1$, $0 \le u \le 2\pi$, and $0 \le v \le \pi$ at $t = 0, 0.21, 0.3, 2, 3$, respectively.



(a) (b)

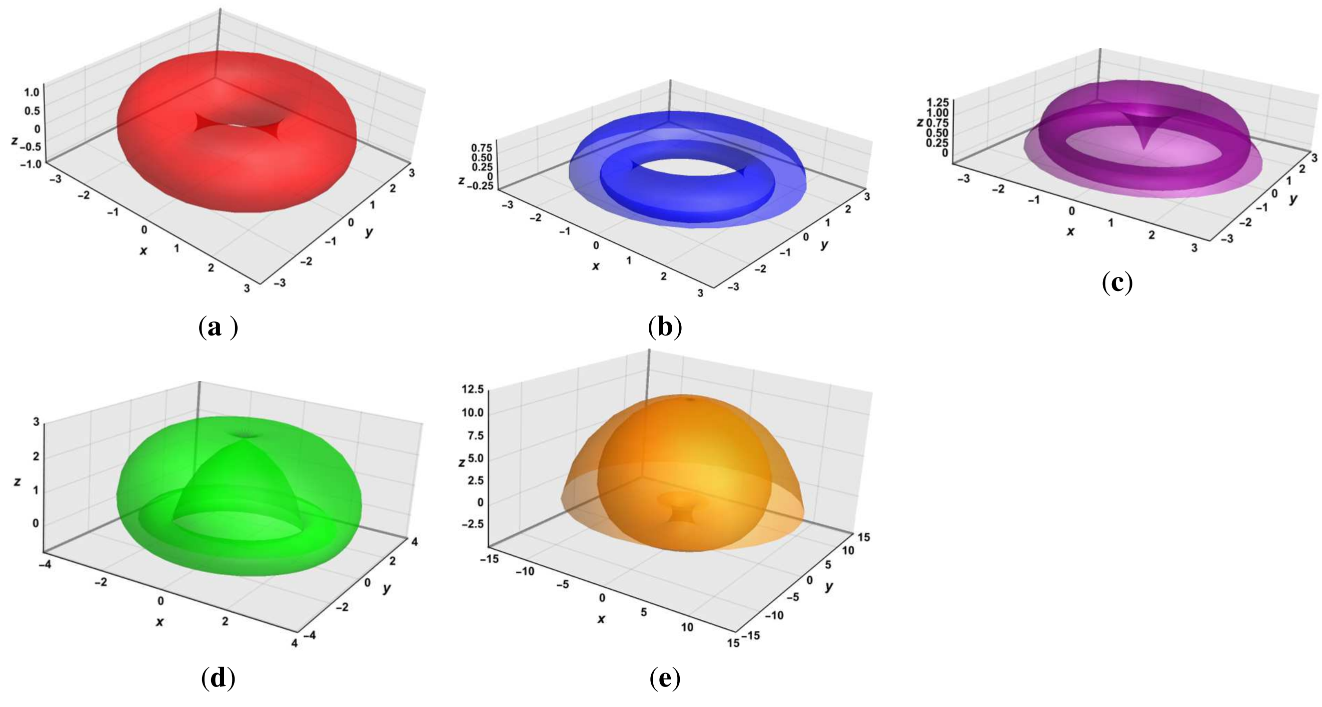
Fig. 4 The normal motion of u-curves of torus with $\psi(u,t) = ut$, for $a_0 = 2$, $b_0 = 1$, $0 \le u \le 2\pi$, and $v = 0$: **(a)** Curves in red, blue, purple, green, and orange colors depict the u-curves at $t = 0, 0.21, 0.3, 2, 3$, respectively. **(b)** Close up: Curves in red, blue, and purple colors depict the u-curves at $t = 0, 0.21, 0.3$, respectively.

4.2 Case 2: $g_1(t) = \tan t$

If $g_1(t) = \tan t$, then the normal velocity $\psi(u,t) = u \tan t$. By substituting in (36), then we have:

$$\begin{aligned} b(t) &= b_0 + u \log(\cos t), \\ a(t) &= const = a_0. \end{aligned} \tag{40}$$

Substituting from (40) into (30), then the evolution of the torus of revolution on which a point q on the surface evolves with the normal velocity given by $\psi(u,t) = u \tan t$ is illustrated in Figure 5 and Figure 6 at different values of t. Also, the normal motion of $u-$curves at the same values of t is illustrated in Figure 7.



(a) (b) (c)

(d) (e)

Fig. 5 The evolution of the torus with $\psi(u,t) = u \tan t$, for $a_0 = 2$, $b_0 = 1$, and $0 \le u, v \le 2\pi$ at $t = 0, 0.5, 0.9, 1.1, 1.5$, respectively.

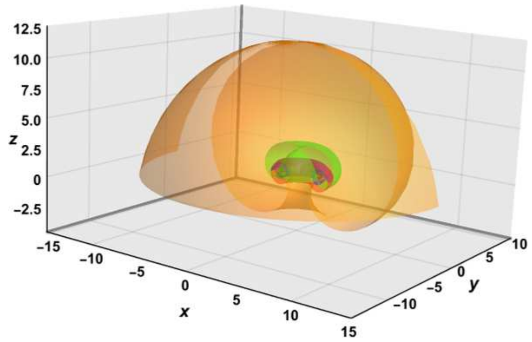
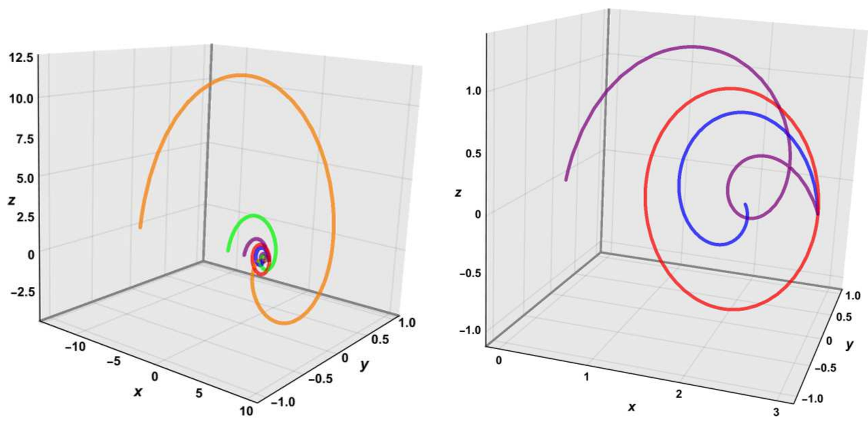


Fig. 6 The red, blue, purple, green, and orange colors depict the evolution of torus with $\psi(u,t) = u\tan t$, for $a_0 = 2$, $b_0 = 1$, and $0 \le u \le 2\pi$, $0 \le v \le \pi$ at $t = 0, 0.5, 0.9, 1.1, 1.5$, respectively.



(**a**) (**b**)

Fig. 7 The evolution of u-curves of the torus with $\psi(u,t) = u\tan t$, for $a_0 = 2$ and $b_0 = 1$ at $v = 0$ and $0 \le u \le 2\pi$: **(a)** Curves in red, blue, purple, green, and orange colors depict the u-curves at $t = 0, 0.5, 0.9, 1.1, 1.5$, respectively. **(b)** Close up: Curves in red, blue, and purple colors depict the u-curves at $t = 0, 0.5, 0.9$, respectively.

4.3 Case 3: $g_1(t) = \operatorname{sech}^2 t$

If $g_1(t) = \operatorname{sech}^2 t$, then the normal velocity $\psi(u,t) = u\operatorname{sech}^2 t$. By substituting in (36), then we have:

$$\begin{aligned} b(t) &= b_0 - u\tanh t, \\ a(t) &= const = a_0. \end{aligned} \tag{41}$$

Substituting from (41) into (30), then we obtain the evolution of the torus of revolution on which a point q on the surface evolves with the normal velocity given by $\psi(u,t) = u\operatorname{sech}^2 t$. This evolution is illustrated in Figure 8 and Figure 9 at different values of t. Also, the normal motion of $u-$curves at the same values of t is illustrated in Figure 10.

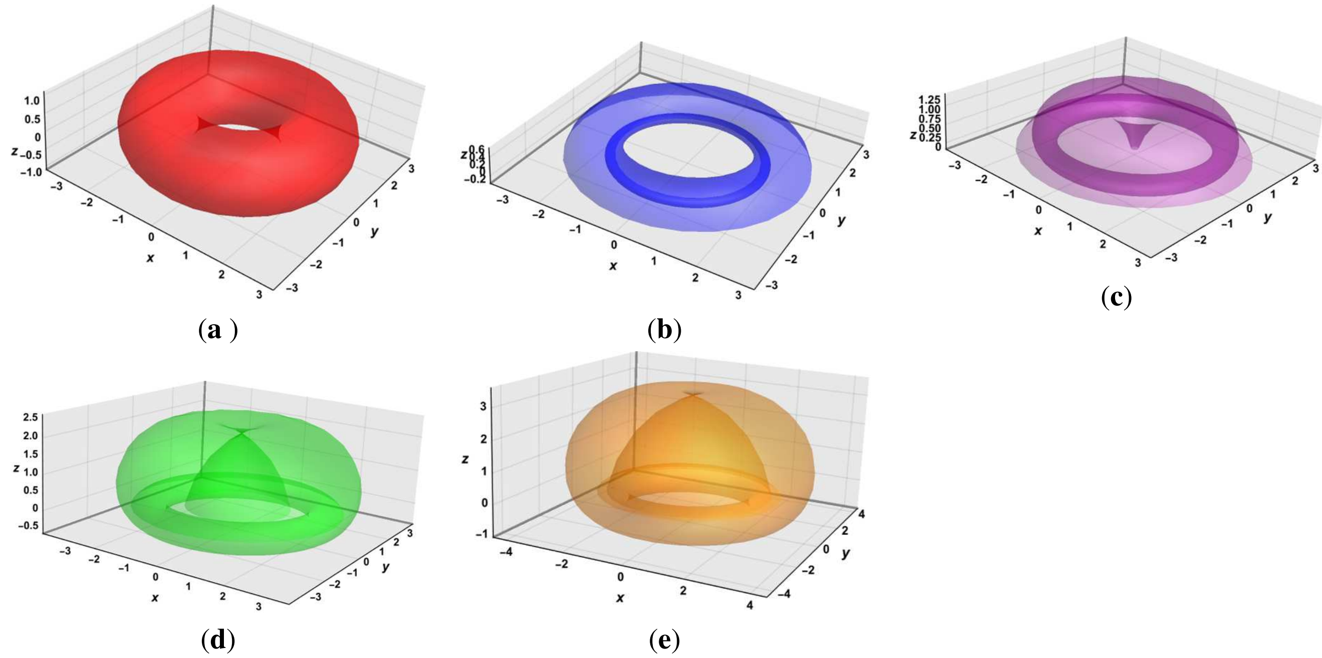


Fig. 8 The evolution of the torus with $\psi(u,t) = u\,\mathrm{sech}^2 t$, for $a_0 = 2$, $b_0 = 1$, and $0 \leq u, v \leq 2\pi$ at $t = 0, 0.2, 0.5, 0.9, 1.5$, respectively.

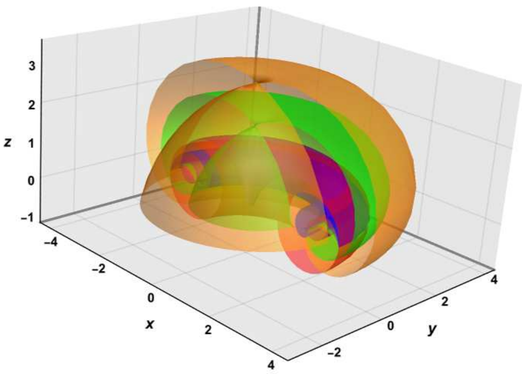
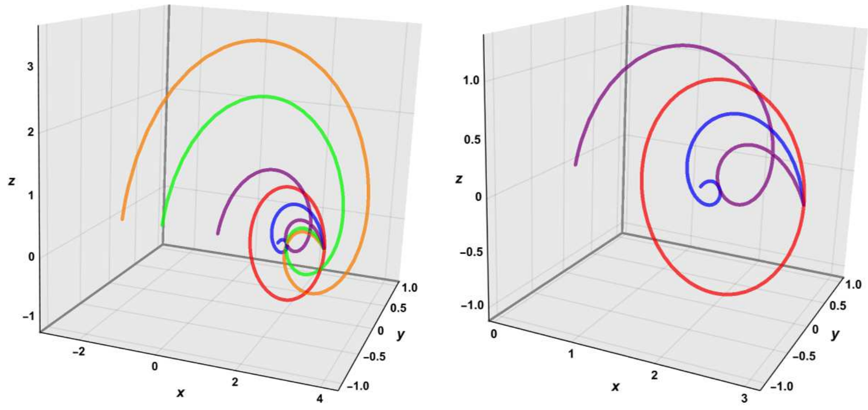


Fig. 9 The red, blue, purple, green, and orange colors depict the evolution of torus with $\psi(u,t) = u\,\mathrm{sech}^2 t$, for $a_0 = 2$, $b_0 = 1$, and $0 \leq u \leq 2\pi$, $0 \leq v \leq \pi$ at $t = 0, 0.2, 0.5, 0.9, 1.5$, respectively.



(a) (b)

Fig. 10 The evolution of u-curves of the torus with $\psi(u,t) = u\,\mathrm{sech}^2 t$, for $a_0 = 2$ and $b_0 = 1$ at $v = 0$ and $0 \leq u \leq 2\pi$: **(a)** Curves in red, blue, purple, green, and orange colors depict the u-curves at $t = 0, 0.2, 0.5, 0.9, 1.5$, respectively. **(b)** Close up: Curves in red, blue, and purple colors depict the u-curves at $t = 0, 0.2, 0.5$, respectively.

5 Applications on the normal motion of the torus via the normal velocity $\psi(u,v,t) = \psi(t)$

Consider the normal motion of the torus via the normal velocity that is given by (37) and by taking $u = 0$ on the external circle of the torus on $yz-$plane, then the normal velocity is given by: $\psi(u,v,t) = \psi(t) = f_1(t)$. Then the radii $a(t)$ and $b(t)$ of the circles of the torus are given by:

$$\begin{aligned} b(t) &= b_0 - \int_0^t \psi(t)\, dt, \quad b(0) = b_0. \\ a(t) &= const = a_0. \end{aligned} \tag{42}$$

Also, for ($u = 0$), the (MC), and (GC) satisfy the following equations:

$$\begin{aligned} G = G_0 &= \frac{1}{b(t)\big(a_0 + b(t)\big)}, \\ H = H_0 &= \frac{a_0 + 2b(t)}{2b(t)\big(a_0 + b(t)\big)}. \end{aligned} \tag{43}$$

5.1 The normal motion of the torus by inverse mean curvature flow

Definition 8. A family of surfaces Σ_t evolves by mean curvature flow (MCF) if the normal velocity ψ of which a point q on the surface moves by (MC) is given by $\psi = H(u,v,t)$. The normal motion is called normal motion by mean curvature flow (NMMCF).

Definition 9. A family of surfaces Σ_t evolves by inverse mean curvature flow (IMCF) if the normal velocity ψ of which a point q on the surface moves by inverse mean curvature (IMC) ($\psi = \frac{1}{H}, H \neq 0$). The normal motion is called normal motion by inverse mean curvature flow (NMIMCF).

Lemma 11. *If the torus of revolution evolves according to (NMIMCF) with normal velocity $\psi = f_1(t) = \frac{1}{H_0}$, $H_0 \neq 0$, then the radius $b(t)$ has the following explicit form:*

$$b(t) = \frac{1}{2}\Big(-a_0 + \sqrt{a_0^2 + 4(a_0 b_0 + b_0^2)e^{-2t}}\,\Big). \tag{44}$$

The (NMIMCF) of the torus is illustrated by Figure 11, Figure 12, and the normal motion of $u-$curves at the same values of t is illustrated in (Figure 13).

Lemma 12. *According to the (NMIMCF) for the torus, the radius $b(t)$ converges to zero as the time t approaches infinity.*

$$\lim_{t\to\infty} b(t) = 0. \tag{45}$$

So, by increasing the time values, the radius $b(t)$ of the circle of the torus is shrinking until the shape of the tours looks like the shape of a thin ring (Figure 11(e)).

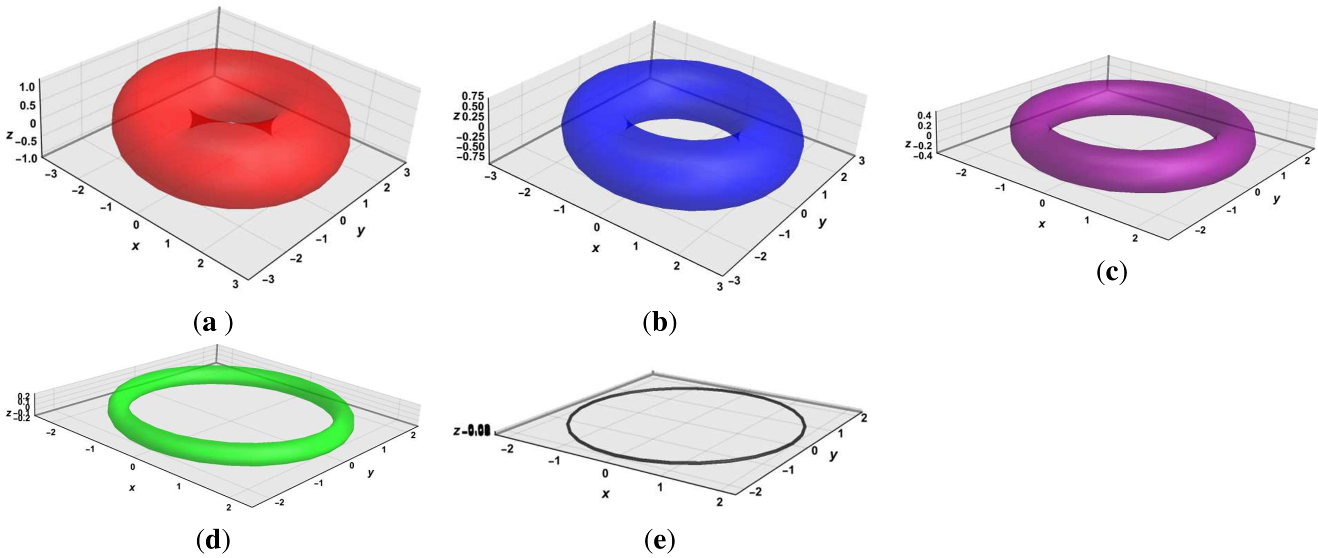


Fig. 11 The evolution of the torus by (NMIMCF) for $a_0 = 2$, $b_0 = 1$, $0 \le u, v \le 2\pi$ at $t = 0, 0.2, 0.6, 1, 2$, respectively.

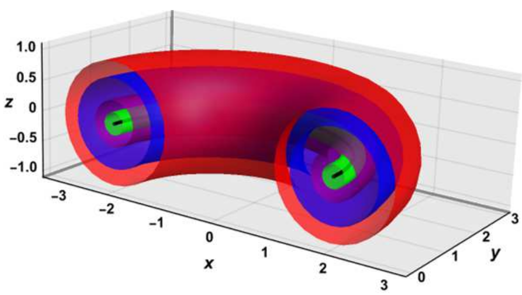


Fig. 12 The $(NMIMCF)$ for the torus $a_0 = 2$, $b_0 = 1$, $0 \le u \le 2\pi$, and $0 \le v \le \pi$. at $t = 0, 0.2, 0.6, 1, 2$, respectively from outside to inside.

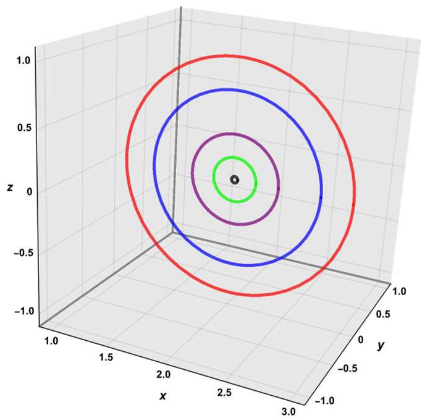


Fig. 13 The $(NMIMCF)$ of u-curves of the torus for $a_0 = 2$, $b_0 = 1$, $0 \leq u \leq 2\pi$, and $v = 0$ at $t = 0$, 0.2, 0.6, 1, 2, respectively from outside to inside.

5.2 The normal motion of the torus of revolution by inverse Gaussian curvature flow

Definition 10. A family of surfaces Σ_t evolves by Gaussian curvature flow (GCF) if the normal velocity ψ of which a point q on the surface moves by Gaussian curvature (GC), so $\psi = G(u,v,t)$. The normal motion is called normal motion by the Gaussian curvature flow (NMGCF).

Definition 11. A family of surfaces Σ_t evolves by the inverse Gaussian curvature flow (IGCF) if the normal velocity ψ of which a point q on the surface moves by inverse Gaussian curvature (IGC), so $\psi = \frac{1}{G}$, $G \neq 0$. The normal motion is called normal motion by inverse Gaussian curvature flow (NMIGCF).

Lemma 13. *If the torus of revolution evolves according to (NMIGCF) with normal velocity $\psi = f_1(t) = \frac{1}{G_0}$, $G_0 \neq 0$, then we obtain:*

$$b(t) = \frac{a_0 b_0 e^{-a_0 t}}{a_0 + b_0 - b_0 e^{-a_0 t}}. \tag{46}$$

The (NMIGCF) of the torus is illustrated in Figure 14, Figure 15, and the normal motion of $u-$curves at the same values of t is illustrated in (Figure 16).

Lemma 14. *According to the (NMIGCF) for the torus, the radius $b(t)$ converges to zero as the time t approaches infinity:*

$$\lim_{t \to \infty} b(t) = 0, \; a_0 > b_0 > 0. \tag{47}$$

So, by increasing the time values, the radius $b(t)$ of the circle of the torus is shrinking until the shape of the tours looks like a thin ring (Figure 14(e)).

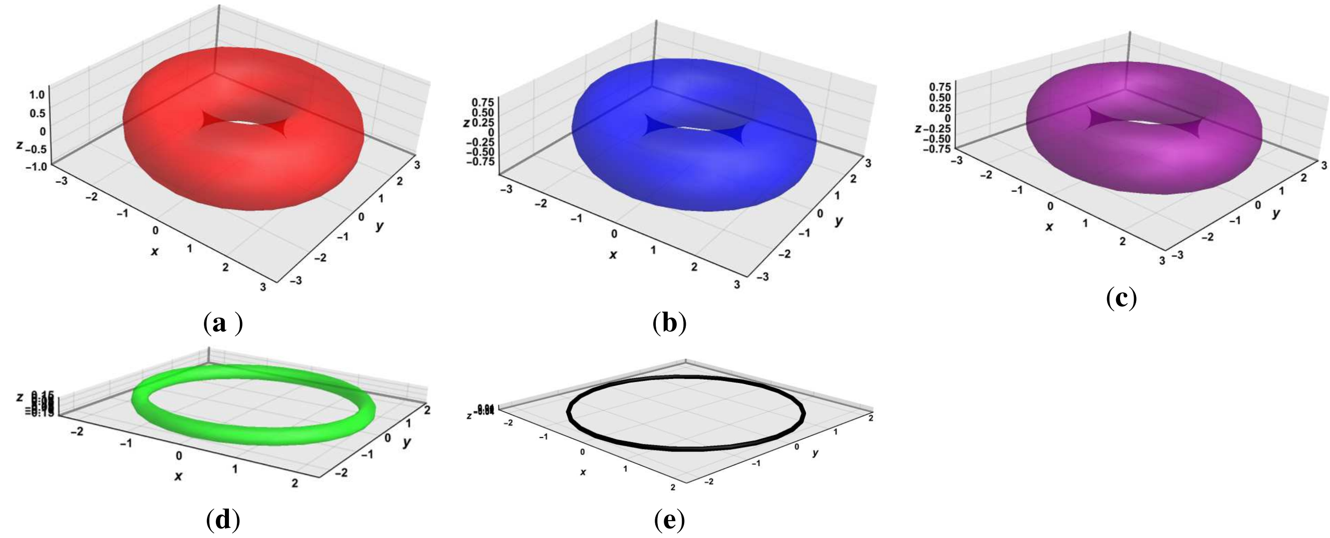


Fig. 14 The evolution of the torus by (NMIGCF) for $a_0 = 2$, $b_0 = 1$, and $0 \leq u, v \leq 2\pi$ at $t = 0, 0.04, 0.1, 0.8, 1.5$, respectively.

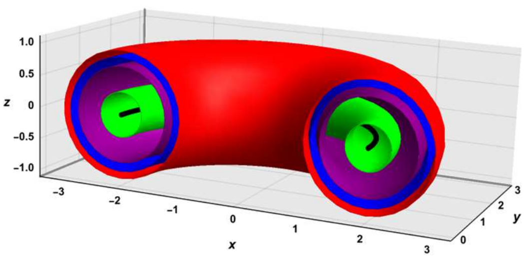


Fig. 15 The (NMIGCF) of the torus for $a_0 = 2$, $b_0 = 1$, $0 \leq u \leq 2\pi$, and $0 \leq v \leq \pi$ at $t = 0, 0.04, 0.1, 0.8, 1.5$, respectively from outside to inside.

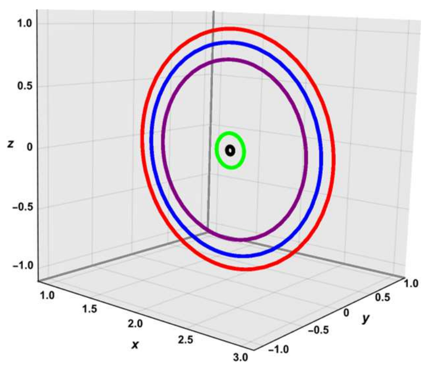


Fig. 16 The (NMIGCF) of u-curves of the torus for $a_0 = 2$, $b_0 = 1$, $0 \leq u \leq 2\pi$, and $v = 0$ at $t = 0, 0.04, 0.1, 0.8, 1.5$, respectively from outside to inside.

5.3 The normal motion of the torus of revolution by the harmonic mean curvature flow

Definition 12. A family of surfaces Σ_t evolves by harmonic mean curvature flow (HMCF), if the normal velocity ψ at a point q on the surface moves by the harmonic mean curvature (HMC) of the surface ($\psi = \frac{G}{H}$), $H \neq 0$. The normal motion is called normal motion by the harmonic mean curvature flow (NMHMCF).

Lemma 15. *If the torus of revolution evolves by (NMHMCF) with normal velocity ($\psi = f_1(t) = \frac{G_0}{H_0}$, $H_0 \neq 0$), then we have:*

$$\psi = \frac{G_0}{H_0} = \frac{2}{a_0 + 2b(t)}. \tag{48}$$

Hence, we obtain the explicit formula of the radius $b(t)$:

$$b(t) = \frac{1}{2}\big(-a_0 + \sqrt{(a_0+2b_0)^2 - 8t}\ \big). \tag{49}$$

The (NMHMCF) of the torus is illustrated in Figure 17, Figure 18, and the normal motion of $u-$curves at the same values of t is illustrated in Figure 19.

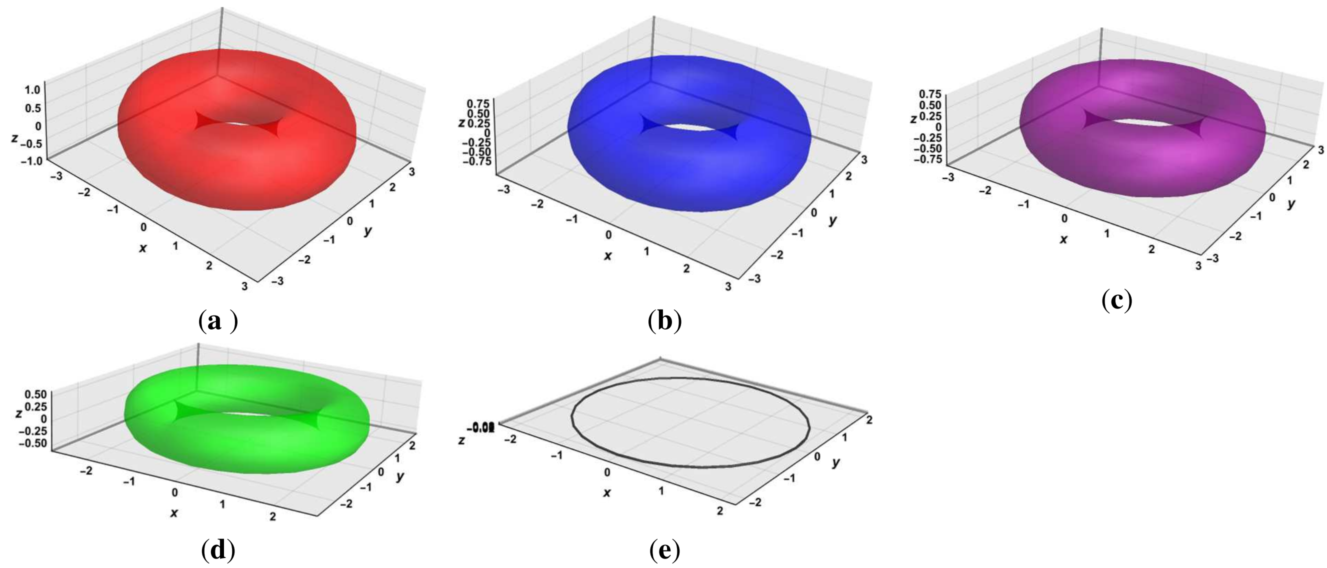


Fig. 17 The (NMHMCF) of the torus with normal velocity $\psi = \frac{G_0}{H_0}$, for $a_0 = 2$, $b_0 = 1$, $0 \leq u, v \leq 2\pi$ at $t = 0$, 0.2, 0.5, 0.8, 1.48, respectively.

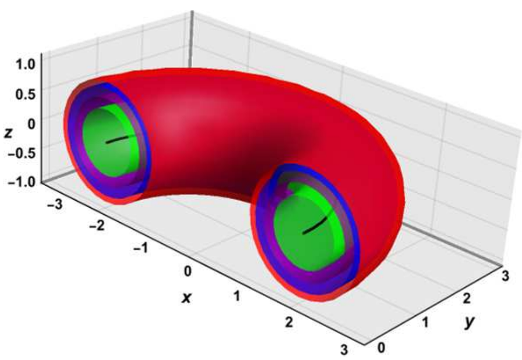


Fig. 18 The (NMHMCF) of the torus with $\psi = \frac{G_0}{H_0}$ for $a_0 = 2$, $b_0 = 1$, $0 \leq u \leq 2\pi$ and $0 \leq v \leq \pi$ at $t = 0, 0.2$, 0.5, 0.8, 1.48 respectively, from outside to inside .

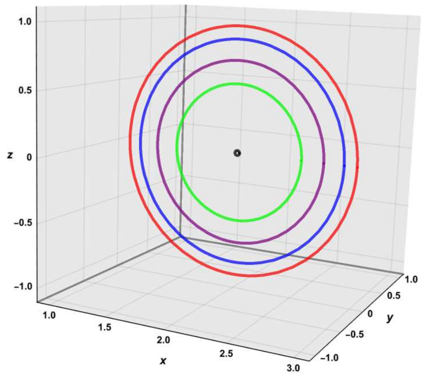


Fig. 19 The (NMHMCF) of u-curves of the torus for $a_0 = 2$, $b_0 = 1$, $0 \leq u \leq 2\pi$ and $v = 0$ at $t = 0, 0.2$, 0.5, 0.8, 1.48, respectively from outside to inside.

6 Conclusion

The present work concerns the study of the normal motion of surfaces. The results of this work can be summarized as follows:

1. The time evolution equations (TEEs) for the normal motion of surfaces are derived.

2. The evolution of the torus of revolution is studied and plotted by the normal velocity that is given as $\psi(u,v,t) = ut$ in Figure 2, and Figure 3. Also, as $\psi(u,v,t) = u\tan t$ in Figure 5 and Figure 6. In addition, as $\psi(u,v,t) = u\,\mathrm{sech}^2 t$ in Figure 8 and Figure 9.

3. The evolution of the torus of revolution is studied and plotted under the normal velocity that is given as a function of the time only $\psi(u,v,t) = \psi(t)$ as the following cases:

   a The evolution of the torus of revolution by (NMIMCF) is studied, by increasing the time values, the radius $b(t)$ of the circle of the torus is shrinking until the shape of the tours looks like a thin ring (Figure 11(e)). The radius $b(t)$ converges to zero as the time t approaches infinity, and

   b The evolution of the torus of revolution by (NMIGCF) is studied, by increasing the time values, the radius $b(t)$ of the circle of the torus is shrinking until the shape of the tours looks like a thin ring (Figure 14(e)). The radius $b(t)$ of the circle of the torus converges to zero as the time t approaches infinity,

   c The evolution of the torus of revolution by (NMHMCF) with normal velocity $\psi = \frac{G_0}{H_0}$, $H_0 \neq 0$ is studied and plotted by Figure 17 and Figure 18.

Data availability

No data were used to support this study.

Conflicts of interest

The authors declared no conflicts of interest.

Abbreviations

The following abbreviations are used in this manuscript:

GC	Gaussian Curvature.
GCF	Gaussian Curvature Flow.
IGC	Inverse Gaussian Curvature.
IGCF	Inverse Gaussian Curvature Flow.
IMC	Inverse Mean Curvature.
IMCF	Inverse Mean Curvature Flow.
MC	Mean Curvature.
MCF	Mean Curvature Flow.
NMMCF	Normal motion according to Mean Curvature Flow.
NMIMCF	Normal Motion according to Inverse Mean Curvature Flow.
NMGCF	Normal Motion according to Gaussian Curvature Flow.
NMIGCF	Normal Motion according to Inverse Gaussian Curvature Flow.
NMHMCF	Normal Motion according to Harmonic Mean Curvature Flow.
PDE(s)	Partial Differential Equation(s).
TEE(s)	Time Evolution Equation(s).

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