Acta Univ. Sapientiae, Informatica **11,** 1 (2019) 95–108

DOI: 10.2478/ausi-2019-0007

J-coloring of graph operations

Sudev NADUVATH
Department of Mathematics
CHRIST (Deemed to be University
Bangalore, INDIA
email: sudev.nk@christuniversity.in

Johan KOK
Department of Mathematics
CHRIST (Deemed to be University
Bangalore, INDIA
email: jacotype@gmail.com

Abstract. A vertex v of a given graph is said to be in a rainbow neighbourhood of G if every color class of G consists of at least one vertex from the closed neighbourhood $N[v]$. A maximal proper coloring of a graph G is a J-coloring if and only if every vertex of G belongs to a rainbow neighbourhood of G. In general all graphs need not have a J-coloring, even though they admit a chromatic coloring. In this paper, we characterise graphs which admit a J-coloring. We also discuss some preliminary results in respect of certain graph operations which admit a J-coloring under certain conditions.

1 Introduction

For general notations and concepts in graphs and digraphs we refer to [1, 3, 9]. For further definitions in the theory of graph coloring, see [2, 4]. Unless specified otherwise, all graphs mentioned in this paper are simple, connected and undirected graphs.

The degree of a vertex $v \in V(G)$ is the number of edges in G incident with v and is denoted $d_G(v)$ or when the context is clear, simply as $d(v)$. A *pendant vertex* or an *end vertex* of a graph G is a vertex having degree 1. A vertex

Computing Classification System 1998: G.2.2
Mathematics Subject Classification 2010: 05C15, 05C38, 05C75, 05C85.
Key words and phrases: rainbow neighbourhood, J-coloring, J^*-coloring.

which is not a pendant vertex is called an *internal vertex* of G (see [3]). A *pendant edge* of G is an edge incident on a pendant vertex of G. Also, unless mentioned otherwise, the graphs we consider in this paper has the order n and size p with minimum and maximum degree δ and Δ, respectively.

Recall that if $\mathcal{C} = \{c_1, c_2, c_3, \dots, c_\ell\}$ and ℓ sufficiently large, is a set of distinct colors, a proper *vertex coloring* of a graph G is a vertex coloring $\varphi : V(G) \mapsto \mathcal{C}$ such that no two distinct adjacent vertices have the same color. The cardinality of a minimum set of colors which allows a proper vertex coloring of G is called the *chromatic number* of G and is denoted by $\chi(G)$. When a vertex coloring is considered with colors of minimum subscripts, the coloring is called a *minimum parameter coloring*. Unless stated otherwise, all colorings in this paper are minimum parameter color sets.

The number of times a color c_i is allocated to vertices of a graph G is denoted by $\theta(c_i)$ and $\varphi : v_i \mapsto c_j$ is abbreviated, $c(v_i) = c_j$. Furthermore, if $c(v_i) = c_j$ then $\iota(v_i) = j$. The color class of a color c_i, denoted by $\mathcal{C}_i$, is the set of vertices of G having the same color c_i.

We shall also color a graph in accordance with the *rainbow neighbourhood convention* (see [5]), which is stated as follows.

Rainbow neighbourhood convention: ([5]) For a proper coloring $\mathcal{C} = \{c_1, c_2, c_3, \dots, c_\ell\}$, $\chi(G) = \ell$, we always color maximum possible number of vertices with the color c_1, then color the maximum possible number of remaining vertices by the color c_2 and proceeding like this and finally color the remaining vertices by the color c_ℓ. Such a coloring is called a χ^--coloring of a graph.

The inverse to the convention requires the mapping $c_j \mapsto c_{\ell-(j-1)}$. Corresponding to the inverse coloring we define $\iota'(v_i) = \ell - (j-1)$ if $c(v_i) = c_j$. The inverse of a χ^--coloring is called a χ^+-coloring.

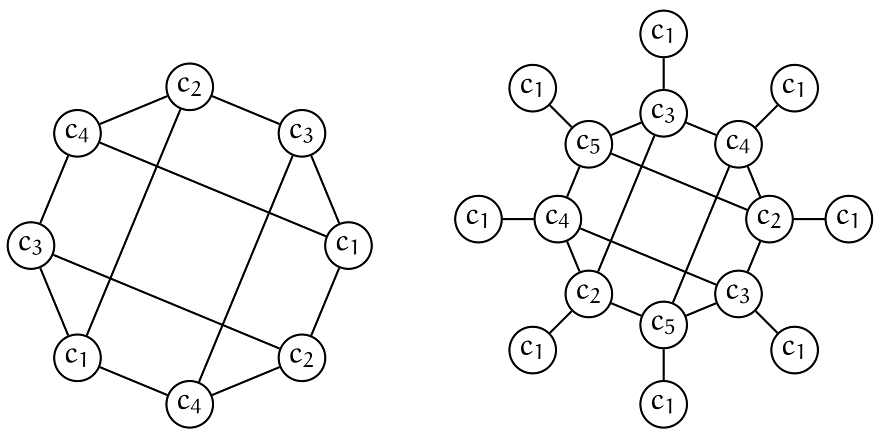
The closed neighbourhood $N[v]$ of a vertex $v \in V(G)$ which contains at least one colored vertex of each color in the chromatic coloring, is called a *rainbow neighbourhood*. That is, a vertex V is said to be in a rainbow neighbourhood if $\mathcal{C}_i \cap N[v] \neq \emptyset$, for all $1 \leq i \leq \chi(G)$. The number of vertices of a graph G, which belong to some rainbow neighbourhoods of G is called the *rainbow neighbourhood number* of G, denoted by $r_{\chi(G)}$ (see [5]). The rainbow neighbourhood number of certain graph classes have been determined in [6, 7].

Motivated by these studies, two types of vertex colorings in terms of rainbow neighbourhoods have been introduced in [8] as follows.

Definition 1 [8] A maximal proper coloring of a graph G is a *Johan coloring* of G, or J-*coloring* in short, if and only if every vertex of G belongs to a rainbow neighbourhood of G. The maximum number of colors in a J-coloring is called the J-*chromatic number* of G, denoted by $\mathcal{J}(G)$.

Definition 2 [8] A maximal proper coloring of a graph G is a *modified Johan coloring*, or J*-*coloring* in short, if and only if every internal vertex (a vertex having degree at least 2) of G belongs to a rainbow neighbourhood of G. The maximum number of colors in a J*-coloring is denoted by $\mathcal{J}^*(G)$.

Figure 1 illustrate a J-colorable and a J*-colorable graph.



(a) A graph with a J-coloring. (b) A graph with a J*-coloring.

Figure 1: J-colorable and J* colorable graphs

In this paper, we characterise the graphs which admit J-coloring. We also discuss some preliminary results in respect of certain graph operations which admit a J-coloring under certain conditions.

2 Results and discussions

A *null graph* on n vertices is an edgeless graph and is denoted by $\mathfrak{N}_n$. We follow the convention that $\mathcal{J}(\mathfrak{N}_n) = \mathcal{J}^*(\mathfrak{N}_n) = 1,\ n \in \mathbb{N}$. Also, note that for any graph G which admits a J-coloring, we have $\chi(G) \leq \mathcal{J}(G)$.

Note that if a graph G admits a J-coloring, it also admits a J*-coloring. However, the converse need not be true always. It can also be noted that if graph G has no pendant vertex and it admits a J-coloring, then $\mathcal{J}(G) = \mathcal{J}^*(G)$.

In view of the above mentioned concepts and facts, we have the following theorem.

Theorem 3 *If* G *is a tree of order* $n \geq 2$*, then* $\mathcal{J}(G) < \mathcal{J}^*(G)$.

Proof. A tree G of order $n \geq 2$ has at least two pendant vertices, say u and v. Therefore, the maximum number of colors which will allow both vertices u and v to yield rainbow neighbourhoods is $\chi(G) = 2$. Therefore, G admits a J-coloring and $\mathcal{J}(G) = 2$.

Any internal vertex w of G has $d(w) \geq 2$. Therefore, $\mathcal{J}^*(G) \leq 3$. Consider any diameter path of G say $P_{diam(G)}$. Beginning at a pendant vertex of the diameter path, label the vertices consecutively $v_1, v_2, v_3, \ldots, v_{diam(G)}$. color the vertices consecutively $c(v_1) = c_1$, $c(v_2) = c_2$, $c(v_3) = c_3$, $c(v_4) = c_1$, $c(v_5) = c_2$, $c(v_6) = c_3$ and so on such that

$$c(v_{diam(G)}) = 1; \quad \text{if } diam(G) \equiv 1 (\text{mod } 3) \tag{1}$$
$$c(v_{diam(G)}) = 2; \quad \text{if } diam(G) \equiv 2 (\text{mod } 3) \tag{2}$$
$$c(v_{diam(G)}) = 3; \quad \text{if } diam(G) \equiv 0 (\text{mod } 3). \tag{3}$$

Clearly, in respect of path $P_{diam(G)}$, it is a proper coloring and all internal vertices yield a rainbow neighbourhood on 3 colors. Consider any maximal path starting from, say $v \in V(P_{diam(G)})$. Hence, v is a pendant vertex to that maximal path. color the vertices consecutively from v as follows:

(a) If $c(v) = c_1$ in $P_{diam(G)}$, color as $c_1, c_2, c_3, c_1, c_2, c_3, \ldots, \underbrace{c_1 \text{ or } c_2 \text{ or } c_3}$

(b) If $c(v) = c_2$ in $P_{diam(G)}$, color as $c_2, c_3, c_1, c_2, c_3, c_1, \cdots, \underbrace{c_2 \text{ or } c_3 \text{ or } c_1}$.

(c) If $c(v) = c_3$ in $P_{diam(G)}$, color as $c_3, c_1, c_2, c_3, c_1, c_2, \cdots, \underbrace{c_3 \text{ or } c_1 \text{ or } c_2}$.

It follows from mathematical induction that all maximal branching can receive such coloring which remains a proper coloring with all internal vertices $v \in V(G)$ having $|c(N[v])| = 3$. Furthermore, all nested branching can be colored in a similar way until all vertices of G are colored. Therefore, $\mathcal{J}^*(G) \geq 3$. Hence, $\mathcal{J}(G) < \mathcal{J}^*(G)$. □

An easy example to illustrate Theorem 3 is the star $K_{1,n}$, $n \geq 2$ for which $\mathcal{J}(K_{1,n}) = 2 < n + 1 = \mathcal{J}^*(K_{1,n})$. This example prompts the next results.

Corollary 4 *For any graph* G *which admits a* J^**-coloring, we have* $\mathcal{J}^*(G) \leq \Delta(G) + 1$.

Corollary 5 *If $\mathcal{J}^*(G) > \mathcal{J}(G)$ for a graph G, then G has at least one pendant vertex.*

Proof. Since all $v \in V(G)$ are internal vertices and any vertex u for which $d(u) = \delta(G)$ must yield a rainbow neighbourhood, it follows that any maximal proper coloring $\mathcal{C}$ are bound to $|\mathcal{C}| = |N[u]| = \delta(G) + 1$. Therefore, if $\mathcal{J}^*(G) > \mathcal{J}(G)$, then G has at least one pendant vertex. □

In [5], the rainbow neighbourhood number $r_\chi(G)$ is defined as the number of vertices of G which yield rainbow neighbourhoods. It is evident that not all graphs admit a J-coloring. Then, we have

Lemma 6 *(i) A maximal proper coloring $\varphi : V(G) \mapsto \mathcal{C}$ of a graph G which satisfies a graph theoretical property, say $\mathfrak{P}$, can be minimised to obtain a minimal proper coloring which satisfies $\mathfrak{P}$.*

(ii) A minimal proper coloring $\varphi : V(G) \mapsto \mathcal{C}$ of a graph G which satisfies a graph theoretical property, say $\mathfrak{P}$, can be maximised to obtain a maximal proper coloring which satisfies $\mathfrak{P}$.

Proof.

(i) Consider a maximal proper coloring $\varphi : V(G) \mapsto \mathcal{C}$ of a graph G which satisfies a graph theoretical property say, $\mathfrak{P}$. If a minimum color set $\mathcal{C}'$, with $|\mathcal{C}'| < |\mathcal{C}|$, such that a minimal proper coloring $\varphi' : V(G) \mapsto \mathcal{C}'$ which satisfies the graph theoretical property $\mathfrak{P}$ cannot be found, then $|\mathcal{C}|$ is minimum.

(ii) Consider a minimal proper coloring $\varphi : V(G) \mapsto \mathcal{C}$ of a graph G which satisfies a graph theoretical property say, $\mathfrak{P}$. If a maximum color set $\mathcal{C}'$, $|\mathcal{C}'| > |\mathcal{C}|$, such that a maximal proper coloring $\varphi' : V(G) \mapsto \mathcal{C}'$ which satisfies the graph theoretical property $\mathfrak{P}$ cannot be found, then $|\mathcal{C}|$ is maximum.

□

The following theorem characterises those graphs which admit a J-coloring.

Theorem 7 *A graph G of order n admits a J-coloring if and only if $r_\chi(G) = n$.*

Proof. If $r_\chi(G) = n$, then every vertex of G belongs to a rainbow neighbourhood. Hence, either the chromatic coloring $\varphi : V(G) \mapsto \mathcal{C}$ is maximal or a maximal coloring $\varphi' : V(G) \mapsto \mathcal{C}'$ exists.

An immediate consequence of Definition 1 is that if graph G admits a J-coloring then each vertex $v \in V(G)$ yields a rainbow neighbourhood. This consequence also follows from the the result that for any connected graph G, $\mathcal{J}(G) \leq \delta(G) + 1$ (see [8]). Hence, from Lemma 6 it follows that either the J-coloring is minimal or a minimal coloring $\varphi' : V(G) \mapsto \mathcal{C}'$ exists such that $r_\chi(G) = n$. □

The following theorem establishes a necessary and sufficient condition for a graph G to have a J-coloring with respect to a χ^--coloring of G.

Theorem 8 *A graph G admits a J-coloring if and only if each $v \in V(G)$ yields a rainbow neighbourhood with respect to a χ^--coloring of G.*

Proof. If in a χ^--coloring of G, each $v \in V(G)$ yields a rainbow neighbourhood it follows from the second part of Lemma 6 that the corresponding proper coloring can be maximised to obtain a J-coloring.

Conversely, assume that a graph G admits a J-coloring. Then, it follows from Lemma 6(i) that the corresponding proper coloring can be minimised to obtain a minimal proper coloring for which each $v \in V(G)$ yields a rainbow neighbourhood. Let the aforesaid set of colors be $\mathcal{C}'$. Assume that a minimum set of colors $\mathcal{C}$ exists which is a χ^--coloring of G and $|\mathcal{C}| < |\mathcal{C}'|$. It implies that there exists at least one vertex $v \in V(G)$ for which at least one distinct pair of vertices, say $u, w \in N(v)$ exists such that u and v are non-adjacent. Furthermore, $c(u) = c(w)$ under the coloring $\varphi : V(G) \mapsto \mathcal{C}$.

Assume that there is exactly one such v and exactly one such vertex pair $u, w \in N(v)$. But then both u and w yield rainbow neighbourhoods in G under the proper coloring $\varphi : V(G) \mapsto \mathcal{C}$, which is a contradiction to the minimality of $\mathcal{C}'$. By mathematical induction, similar contradictions arise for all vertices similar to v. This completes the proof. □

3 Analysis for certain graphs

Note that we have two types of operations related to graphs, that is: operations on a graph G and operations between two graphs G and H. Operations on a graph G result in a well defined derivative of G. Examples are the complement graph G^c, the line graph $L(G)$, the middle graph $M(G)$, the central graph $C(G)$, the jump graph $J(G)$ and the total graph $T(G)$ and so on. Recall that the jump graph $J(G)$ of a graph G of order $n \geq 3$ is the complement graph of the line graph $L(G)$. Also, note that the line graph is the graphical realisation of edge adjacency in G and the jump graph is the graphical realisation of edge independence in G. Some other graph derivative operations are edge deletion, vertex deletion, edge contraction, thorning a graph by pendant vertex addition and so on.

Examples of operations between graphs G and H are, the corona between G and H denoted, $G \circ H$, the join denoted, $G + H$, the disjoint union denoted, $G \cup H$, the Cartesian product denoted, $G \Box H$ and so on.

3.1 Operations between certain graphs

The following result establishes a necessary and sufficient condition for the corona of two graphs G and H to admit a J-coloring.

Theorem 9 *If graphs G and H admit J-colorings, then $G \circ H$ admits a J-coloring if and only if either $G = K_1$ or $\mathcal{J}(G) = \mathcal{J}(H) + 1$.*

Proof. *Part 1:* If $G = K_1$ assume $\mathcal{C} = \{c_1, c_2, c_3, \dots, c_{\mathcal{J}(H)}\}$ provides a J-coloring of H. color K_1 the color $c_{\mathcal{J}(H)+1}$. Clearly, $\mathcal{C}' = \mathcal{C} \cup \{c_{\mathcal{J}(H)+1}\}$ is a J-coloring of $K_1 \circ H$.

Part 2: If $G \neq K_1$ and $\mathcal{J}(G) = \mathcal{J}(H) + 1$ let $\mathcal{C} = \{c_1, c_2, c_3, \dots, c_\ell,\ \ell = \mathcal{J}(G)\}$ and $\mathcal{C}' = \{c_1, c_2, c_3, \dots, c_{\ell-1},\ \ell = \mathcal{J}(G)\}$ provide the J-colorings of G and H, respectively. Assume that $v \in V(G)$ has $c(v) = c_i$ then color all $u \in V(H)$ for the copy of H corona'd to v for which $c(u)_{(\text{in } H)} = c_i$, $1 \leq i \leq \ell$, to be $c_{\ell+1}$. Clearly every vertex $v \in V(G) \cup V(H)$ yields a rainbow neighbourhood and $|\mathcal{C}|$ is maximal.

Conversely, let $G \circ H$ admit a J-coloring. Then, for any vertex $v \in V(G)$ the subgraph $v \circ H$ holds the condition $c(v) \neq c(u),\ \forall u \in V(H)$. Therefore, either $G = K_1$ or $\mathcal{J}(G) = \mathcal{J}(H) + 1$. □

The next corollary requires no proof as it is a direct consequence of Theorem 9.

Corollary 10 *If $G \circ H$ admits a J-coloring then: $\mathcal{J}(G \circ H) = \mathcal{J}(G)$.*

The following theorem discusses the admissibility of J-coloring by the join of two graphs.

Theorem 11 *The join $G + H$ admits a J-coloring if and only if both graphs G and H admit a J-coloring.*

Proof. Assume that both G and H admit a J-coloring. Without loss of generality, let $\mathcal{J}(G) \leq \mathcal{J}(H)$. Assume that $\varphi : V(G) \mapsto \mathcal{C}$, $\mathcal{C} = \{c_1, c_2, c_3, \dots, c_\ell\}$

and $\varphi' : V(H) \mapsto \mathcal{C}'$, $\mathcal{C}' = \{c_1, c_2, c_3, \dots, c_{\ell'}\}$ is a J-coloring of G and H, respectively. For each $v \in V(G)$, $c(v) = c_i$ recolor $c(v) \mapsto c_{i+\ell'}$. Denote the new color set by $\mathcal{C}_{i+\ell'}$. Clearly, each vertex $v \in V(G)$ is adjacent to at least one of each color in $G + H$ hence, each such vertex yields a rainbow neighbourhood in $G + H$. Similarly, each vertex $u \in V(H)$ is adjacent to at least one of each color in $G + H$ and hence each such vertex yields a rainbow neighbourhood in $G + H$. Furthermore, since both $|\mathcal{C}|$, $|\mathcal{C}'|$ is maximal color sets, the set $|\mathcal{C}_{i+\ell'} \cup \mathcal{C}'|$ is maximal. Therefore, $G + H$ admits a J-coloring.

The converse follows trivially from the fact that the additional edges between G and H as defined for join form an edge cut in $G + H$. □

The following result discusses the existence of a J-coloring for the Cartesian product of two given graphs.

Theorem 12 *If graphs* G *and* H *of order* n *and* m *respectively admit a* J-*coloring, then*

(i) $G\square H$ *admits a* J-*coloring.*

(ii) $\mathcal{J}(G\square H) = \max\{\mathcal{J}(G), \mathcal{J}(H)\}$

Proof.

(i) Without loss of generality assume $\mathcal{J}(H) \geq \mathcal{J}(G)$. Also, assume that $V(G) = \{v_i : 1 \leq i \leq n\}$ and $V(H) = \{u_i : 1 \leq i \leq m\}$. From the definition of $G\square H$ it follows that $V(G\square H) = \{(v_i, u_j) : 1 \leq i \leq n, 1 \leq j \leq m\}$. For $i = 1$, if $u_j \sim u_k$ in H, where $\sim$ denotes the adjacency, then $(v_1, u_j) \sim (v_1, u_k)$ and hence we obtain an isomorphic copy of H. Such a copy admits a J-coloring identical to that of H in respect of the vertex elements $u_1, u_2, u_3, \dots, u_m$. Now obtain the disjoint union with the copies of H corresponding to $i = 2, 3, 4, \dots, n$. Apply the definition of $G\square H$ for u_1 and if $v_i \sim v_j$ in G, then $(v_i, u_1) \sim (v_j, u_1)$. An interconnecting copy of G is obtained which result in the first iteration connected graph. Similarly, this copy of G admits a J-coloring identical to that of G in respect of the vertex elements $v_1, v_2, v_3, \dots, v_n$. Proceeding iteratively to add all copies of G for $i = 2, 3, 4, \dots, n$ in terms of the definition of $G\square H$, clearly shows that a J-coloring is admitted.

(ii) The second part of the result follows from the similar reasoning used to prove and hence, $\chi(G\square H) = \max\{\chi(G), \chi(H)\}$.

□

3.2 Operations on certain graphs

Recall that for any connected graph G, $\mathcal{J}(G) \le \delta(G)+1$ (see [8]) and for $n \ge 2$, $\mathcal{J}(P_n) = 2$ and $\mathcal{J}^*(P_n) = 3$. In view of these results, we have the following results in respect of certain operations on paths and cycles.

Proposition 13 *For a path P_n, $n \ge 2$ with edge set consecutively labeled as $e_1, e_2, e_3, \dots, e_{n-1}$ and the corresponding line graph vertices consecutively labeled as $u_1, u_2, u_3, \dots, u_{n-1}$. We have*

(i) $\mathcal{J}(L(P_n)) = 2$ *and* $\mathcal{J}^*(L(P_n)) = 3$.

(ii) $\mathcal{J}(M(P_2)) = 2$ *and* $M(P_n)$ $n \ge 3$ *does not admit a* J*-coloring and* $\mathcal{J}^*(M(P_n)) = 3$.

(iii) $\mathcal{J}(T(P_n)) = \mathcal{J}^*(T(P_n)) = 3$.

(iv) *For connectivity, let* $n \ge 5$. *Then* $\mathcal{J}(J(P_5)) = 3$ *and* $\mathcal{J}^*(J(P_5)) = 3$ *and for* $n \ge 6$,

$$\mathcal{J}(J(P_n)) = \mathcal{J}^*(J(P_n)) = \begin{cases} \frac{n}{2} & n \text{ is even} \\ \lfloor \frac{n}{2} \rfloor & n \text{ is odd.} \end{cases}$$

(v) $\mathcal{J}(C(P_n)) = \mathcal{J}^*(C(P_n)) = 3$.

Proof.

(i) Since $L(P_n) = P_{n-1}$, the result follows from the result that for any connected graph G, $\mathcal{J}(G) \le \delta(G)+1$.

(ii) Since $M(P_2) = P_3$ the result follows from the result that for any connected graph G, $\mathcal{J}(G) \le \delta(G)+1$. For $n \ge 3$, the middle graph contains a triangle hence, $\mathcal{J}(M(P_n)) \ge \chi(M(P_n)) \ge 3$. Also $M(P_n)$ has two pendant vertices therefore $r_\chi(M(P_n)) \ne n$. So $M(P_n)$, $n \ge 3$ does not admit a J-coloring. The derivative graph $G' = M(P_n) - \{v_1, v_n\}$ contains a triangle and $\delta(G') = 2$. Therefore, $\mathcal{J}^*(M(P_n)) = 3$.

(iii) Since $\mathcal{J}(T(P_n)) \le \delta(\mathcal{J}(T(P_n)) + 1 = 3$ and $T(P_n)$ contains a triangle, $\mathcal{J}(T(P_n)) = 3$. As $T(P_n)$ has no pendant vertex and contains an odd cycle C_3, the result $\mathcal{J}^*(T(P_n)) = 3$ is immediate.

(iv) For P_5 we have $J(P_5) = P_4$. Hence, the result follows from for any connected graph G, $\mathcal{J}(G) \le \delta(G)+1$. For a path P_n, $n \ge 6$ and edge set consecutively labeled as $e_1, e_2, e_3, \dots, e_{n-1}$ and the corresponding line graph vertices consecutively labeled as $u_1, u_2, u_3, \dots, u_{n-1}$, we have the consecutive vertex χ^--coloring sequence of $J(P_n)$ is given by $c_1, c_1, c_2, c_2, c_3, c_3, \dots, c_{\frac{n}{2}}$ if n is even and $c_1, c_1, c_2, c_2, c_3, c_3, \dots, c_{\lfloor \frac{n}{2} \rfloor}, c_{\lfloor \frac{n}{2} \rfloor}$ if n is odd. Since the vertices u_i, u_{i+1}, $1 \leq i \leq n-2$ are pairwise not adjacent, the χ^--coloring is maximal as well. Clearly, every vertex u_i yields a rainbow neighbourhood. Therefore, the result follows.

(v) Since $C(P_n)$ has no pendant vertex and contains an odd cycle C_5, the result is immediate.

□

Next, we consider cycles $C_n,\ n \geq 3$. In [8], it is proved that

Theorem 14 *[8] If C_n admits a J-coloring then:*

$$\mathcal{J}(C_n) = \begin{cases} 3 & \text{if } n \equiv 0 \pmod 3 \\ 2 & \text{if } n \equiv 0 \pmod 2 \text{ and } n \not\equiv 0 \pmod 3. \end{cases}$$

Analogous to the proof of Theorem 2.7 in [8], we now establish the corresponding results for the derivatives of cycle graphs in the following proposition.

Proposition 15 *For a cycle $C_n, n \geq 3$ and edge set consecutively labeled as $e_1, e_2, e_3, \dots, e_n$ and the corresponding line graph vertices consecutively labeled as $u_1, u_2, u_3, \dots, u_n$, we have*

(i) $\mathcal{J}(L(C_n)) = \mathcal{J}^*(L(C_n)) = 2$ *if and only if* $n \equiv 0 \pmod 2$ *and* $n \not\equiv 0 \pmod 3$, *and* $\mathcal{J}(L(C_n)) = \mathcal{J}^*(L(C_n)) = 3$ *if and only if* $n \equiv 0 \pmod 3$, *else,* $L(C_n)$ *does not admit a* J*-coloring.*

(ii) For $n \geq 3$, $\mathcal{J}(M(C_n)) = \mathcal{J}^*(M(C_n)) = 3$ *if* $n \equiv 0 \pmod 3$, *or if,* $M(C_n)$ *for* $n \not\equiv 0 \pmod 3$, *and without loss of generality admits the coloring:* $c(v_1) = c_1$, $c(u_1) = c_2$, $c(v_2) = c_3$, $c(u_2) = c_1$, $c(v_3) = c_2$, $c(u_3) = c_3, \dots, c(v_{n-1}) = c_1, c(u_{n-1}) = c_2, c(v_n) = c_1, c(u_n) = c_3$, *else,* $M(C_n)$ *does not admit a* J*-coloring.*

(iii) $\mathcal{J}(T(C_n)) = \mathcal{J}^*(T(C_n)) = 4$ *if and only if* n *is even, else,* $T(C_n)$ *does not admit a* J*-coloring.*

(iv) For $n \geq 6$, $\mathcal{J}(J(C_n)) = \mathcal{J}^*(J(C_n)) = \begin{cases} \frac{n}{2} & n \text{ is even} \\ \lfloor \frac{n}{2} \rfloor & n \text{ is odd.} \end{cases}$

(v) $\mathcal{J}(C(C_n)) = \mathcal{J}^*(C(C_n)) = 3.$

Proof. (i) Because $L(C_n) = C_n$ the result follows from Corollary 3.6. Also because $L(C_n)$ has no pendant edges, $\mathcal{J}(L(C_n)) = \mathcal{J}^*(L(C_n))$.

(ii) If $M(C_n)$ admits a J-coloring then $\mathcal{J}(M(C_n)) \leq \delta(\mathcal{J}(M(C_n)) + 1 = 3$. For $n \equiv 0 \,(\mathrm{mod}\ 3)$, consider the coloring: $c(v_1) = c_1$, $c(u_1) = c_2$, $c(v_2) = c_3$, $c(u_2) = c_1$, $c(v_3) = c_2$, $c(u_3) = c_3, \ldots, c(u_{n-1}) = c_1, c(v_n) = c_2, c(u_n) = c_3$.

From the definition of the middle graph, we know that $M(C_n)$ has n triangles stringed so clearly the proper coloring is maximum and all vertices yield a rainbow neighbourhood. Part 2 follows by similar reasoning and hence the result follows. Also, since $M(C_n)$ has no pendant edges, $\mathcal{J}(M(C_n)) = \mathcal{J}^*(M(C_n))$. In all other cases, $\chi((M(C_n)) = 4$ and a J-coloring does not exist.

(iii) Note that $\mathcal{J}(T(C_n)) \leq \delta(\mathcal{J}(T(C_n)) + 1 = 5$. Since $T(C_n)$ contains a triangle, $\mathcal{J}(T(C_n)) \geq 3$. Furthermore, $\chi((T(C_n)) = 4$ if and only if $n \equiv 0 \,(\mathrm{mod}\ 2)$ and $n \not\equiv 0 \,(\mathrm{mod}\ 3)$, and all vertices yield a rainbow neighbourhood. Also, for any set of vertices $V' = \{v_i, v_{i+1}, v_{i+2}, v_{i+2}, v_{i+3}, v_{i+4}\} \mapsto \{v_i v_j : 1 \leq i \leq n,\ 0 \leq j \leq 4,\ \mathrm{and}\ (i+j) \mapsto (i+j)\ (\mathrm{mod}\ 6)\}$, the induced subgraph $\langle V' \rangle \neq K_5$. Therefore, $\mathcal{J}(T(C_n)) = 4$. Also because $T(C_n)$ has no pendant edges, $\mathcal{J}(T(C_n)) = \mathcal{J}^*(T(C_n))$. Otherwise, $\chi((T(C_n)) = 5$, and not all vertices yield a rainbow neighbourhood and hence a J-coloring is not obtained.

(iv) For $n = 5$, $J(C_5) = C_5$ and thus, does not admit a J-coloring. For a path C_n, $n \geq 6$ and edge set consecutively labeled as $e_1, e_2, e_3, \ldots, e_{n-1}$ and the corresponding line graph vertices consecutively labeled as $u_1, u_2, u_3, \ldots, u_{n-1}$, we have the consecutive vertex χ^--coloring sequence of $J(C_n)$ is given by $c_1, c_1, c_2, c_2, c_3, c_3, \ldots, c_{\frac{n}{2}}$ if n is even and $c_1, c_1, c_2, c_2, c_3, c_3, \ldots, c_{\lfloor \frac{n}{2} \rfloor}, c_{\lfloor \frac{n}{2} \rfloor}$ if n is odd ($n-1$ entries). As the vertices u_i, u_{i+1}, $1 \leq i \leq n-2$ are pairwise not adjacent, the χ^--coloring is maximal as well. Clearly, every vertex u_i yields a rainbow neighbourhood. Therefore, the result follows.

(v) The result is trivial for $C(C_3)$. For $n \geq 4$, $\mathcal{J}(C(C_n)) \leq \delta(\mathcal{J}(C(C_n)) + 1 = 3$. Since $\chi((C(C_n)) = 3$ and all vertices yield a rainbow neighbourhood and $C(C_n)$ contains a cycle C_5, the result $\mathcal{J}(C(C_n)) = 3$ holds immediately. Also, since $C(C_n)$ has no pendant edges, $\mathcal{J}(C(C_n)) = \mathcal{J}^*(C(C_n))$. □

4 Extremal results for certain graphs

For a graph G of order $n \geq 1$, which admits a J-coloring the minimum (or maximum) number of edges in a subset $E'_k \subseteq E(G)$ whose removal ensures that $\mathcal{J}(G - E'_k) = k$, $1 \leq k \leq \mathcal{J}(G)$, is discussed in this section. These extremal variables are called the minimum (or maximum) rainbow bonding variables and are denoted $r^-_k(G)$ and $r^+_k(G)$, respectively. A graph G which does not admit a J-coloring has $r_k^-(G)$ and $r_k^+(G)$ undefined. For such aforesaid graph it is always possible to remove a minimal set of edges, E'', which is not necessarily unique such that $G - E''$ admits a J-coloring. This is formalised in the next result.

Lemma 16 *For any connected graph G which does not admit a J-coloring, a minimal set of edges, E'' which is not necessarily unique, can be removed such that $G - E''$ admits a J-coloring.*

Proof. Since any connected graph G of order n and size $\varepsilon(G) = p$ has a spanning subtree and any tree admits a J-coloring, at most $p - (n - 1)$ edges must be removed from G. Therefore, if $p - (n - 1)$ is not a minimal number of edges to be removed then a minimal set of edges E', $|E'| < p - (n - 1)$ must exist whose removal results in a spanning subgraph G' which allows a J-coloring. □

It is obvious from Lemma 16 that the restriction of connectedness can be relaxed if $G = \bigcup H_i,\ 1 \leq i \leq t$ and it is possible that $\mathcal{J}(H_i - E"_i)_{\forall i} = k$, k some integer constant.

It is obvious that for a complete graph K_n, $\mathcal{J}(K_n) = n$. To ensure $\mathcal{J}(K_n) = n$, no edges can be removed. Therefore, $r_n^-(K_n) = r_n^+(K_n) = 0$.

Theorem 17 *For a complete graph K_n, $n \geq 1$ we have*

(i) For n is even and $\frac{n}{2} \leq k \leq n$ and $\mathcal{J}(K_n - E'_k) = k$, then $r_k^-(K_n) = n - k$.

(ii) For n is odd and $\lceil \frac{n}{2} \rceil \leq k \leq n$, and $\mathcal{J}(K_n - E'_k) = k$, then $r_k^-(K_n) = n - k$.

(iii) For $n \in \mathbb{N}$ and $1 \leq k \leq n$, and $\mathcal{J}(K_n - E'_k) = k$, then $r_k^+(K_n) = \frac{1}{2}(n + 1 - k)(n - k)$.

Proof. (i) For n is even and $\frac{n}{2} \leq k \leq n$, exactly 0 or 1 or 2 or 3 or $\cdots$ or $\frac{n}{2}$ edges between distinct pairs of vertices can be removed to obtain $\mathcal{J}(K_n - E'_k) = n, n-1, n-2, \ldots, \frac{n}{2}$. Hence, $r_k^-(K_n) = 0, 1, 2, 3, \ldots, \frac{n}{2}$. In other words $r_k^-(K_n) = n - k$, $\frac{n}{2} \leq k \leq n$.

(ii) The result follows through similar reasoning as that in (i).

(iii) In any clique of order t, the removal of the $\frac{1}{2}t(t-1)$ edges is the maximum number of edges whose removal renders $\mathcal{J}(\mathfrak{N}_t) = 1$ hence, all vertices can be colored say, c_1. Through immediate mathematical induction it follows that we iteratively remove the maximum number of edges $r_k^+(K_n) = 0, 1, 3, 6, 10, \ldots, \frac{1}{2}(n + 1 - k)(n - k)$, $1 \leq k \leq n$ of cliques $K_1, K_2, K_3, \ldots, K_n$ to obtain $\mathcal{J}(K_n - E'_k) = n, n-1, n-2, \ldots, 1$. Hence, the result follows. □

Theorem 18 *A graph* G *of order* n *which allows a* J*-coloring, has* $r^-_k(G) = r^+_k(G)$ *if and only if* $\mathcal{J}(G) = 2$.

Proof. If $\mathcal{J}(G) = 2$ then all edges are incident with colors c_1, c_2. Therefore all edges must be removed to obtain the null graph $\mathfrak{N}_0$ for which $\mathcal{J}(\mathfrak{N}_0) = 1$. Hence, $r^-_k(G) = r^+_k(G)$.

Conversely, let $r^-_k(G) = r^+_k(G)$. Then, assume that at least one edge say, e is incident with color c_3. It implies that G contains at least a triangle or an odd cycle. Therefore, $\varepsilon(G) \geq 3$. To ensure a proper coloring on removing edge e the color c_3 must change to either c_1 or c_2 which is always possible. If $\mathcal{J}(G-e) = 2$ then $r^+_k(G) = 1$ which is a contradiction because any one additional edge may have been removed, implying $r^+_k(G) \geq 2$. For colors $c_4, c_5, c_6, \ldots, \mathcal{J}(G)$, similar contradictions follows through immediate induction. Therefore, if $r^-_k(G) = r^+_k(G)$ then, $\mathcal{J}(G) = 2$. □

5 Conclusion

Clearly the cycles for which the the middle graphs admit a J-coloring in accordance with the second part of Proposition 13(ii) require to be characterised if possible. It follows from Theorem 18 that for the cases n is even and $1 \leq k < \frac{n}{2}$, or n is odd and $1 \leq k < \lceil \frac{n}{2} \rceil$, determining $r^-_k(K_n)$ remains open. It is suggested that an algorithm must be described to obtain these values.

Example 19 For the complete graph K_9 with vertices $v_1, v_2, v_3, \ldots, v_9$, Theorem 17(ii) admits the minimum removal of $r^-_{n,k}(K_n) = 4$ edges to obtain $\mathcal{J}(K_n - E'_k) = 5$. Without loss of generality say the edges were. v_1v_2, v_3v_4, v_6v_6, v_7v_8. To obtain $\mathcal{J}(K_n - E'_k) = 4$ we only remove without loss of generality say, the edges v_7v_9, v_8v_9. To obtain $\mathcal{J}(K_n - E'_k) = 3$ we only remove without loss of generality say, the edges v_1v_3, v_1v_4, v_2v_3, v_2v_4. To obtain $\mathcal{J}(K_n - E'_k) = 2$ we only remove without loss of generality say, the edges v_5v_7, v_5v_8, v_5v_9, v_6v_7, v_6v_8, v_6v_9. To obtain $\mathcal{J}(K_n - E'_k) = 1$ we remove all remaining edges. It implies that as $\mathcal{J}(K_n - E'_k)$ iteratively ranges through the values $5, 4, 3, 2, 1$ the value of $r^-_k(K_9)$ ranges through, $4, 6, 10, 16, 36$.

Determining the range of minimum (maximum) rainbow bonding variables for other classes of graphs is certainly worthy research. For a graph G which does not allow a J-coloring it follows from Lemma 16 that a study of $r^-_k(G')$ and $r^+_k(G')$ with G' a maximal spanning subgraph of G which does allow a J-coloring, is open.

References

[1] J. A. Bondy and U. S. R. Murty, *Graph Theory*, Springer, New York, 1976. ⇒95

[2] G. Chartrand and P. Zhang, *Chromatic Graph Theory*, CRC Press, 2009. ⇒95

[3] F. Harary, *Graph Theory*, New Age International, New Delhi, 2001. ⇒ 95, 96

[4] T. R. Jensen and B. Toft, *Graph Coloring Problems*, John Wiley & Sons, 1995. ⇒95

[5] J. Kok, N. K. Sudev and M. K. Jamil, Rainbow neighbourhood number of graphs, *Proy. J. Math.*, **38,** 3 (2019) 471–487. ⇒96, 99

[6] S. Naduvath, S. Chandoor, S.J. Kalayathankal, J Kok, A note on the rainbow neighbourhood number of graphs, *Nat. Acad. Sci. Letters*, **42,** 2 (2019) 135–138. ⇒96

[7] S. Naduvath, S. Chandoor, S. J. Kalayathankal, J Kok, Some new results on the rainbow neighbourhood number of graphs, *Nat. Acad. Sci. Letters*, **42,** 3 (2019) 249–252. ⇒96

[8] N. K. Sudev, On certain J-coloring parameters of graphs, *Nat. Acad. Sci. Letters*, (2019), in press. ⇒96, 97, 100, 103, 104

[9] D. B. West, *Introduction to Graph Theory*, Pearson Education Inc., Delhi, 2001. ⇒95

Received: July 4, 2019 • *Revised: August 6, 2019*