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Metric and upper dimension of zero divisor graphs associated to commutative rings

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Abstract. Let R be a commutative ring with $Z^*(R)$ as the set of non-zero zero divisors. The zero divisor graph of R, denoted by $\Gamma(R)$, is the graph whose vertex set is $Z^*(R)$, where two distinct vertices x and y are adjacent if and only if $xy = 0$. In this paper, we investigate the metric dimension $\dim(\Gamma(R))$ and upper dimension $\dim^+(\Gamma(R))$ of zero divisor graphs of commutative rings. For zero divisor graphs $\Gamma(R)$ associated to finite commutative rings R with unity $1 \neq 0$, we conjecture that $\dim^+(\Gamma(R)) = \dim(\Gamma(R))$, with one exception that $R \cong \Pi\mathbb{Z}_2^n$, $n \geq 4$. We prove that this conjecture is true for several classes of rings. We also provide combinatorial formulae for computing the metric and upper dimension of zero divisor graphs of certain classes of commutative rings besides giving bounds for the upper dimension of zero divisor graphs of rings.

1 Introduction

Throughout this article, R is assumed to be a commutative ring with unity $1 \neq 0$, unless otherwise stated. Let $Z(R)$ be its set of zero divisors. The zero

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divisor graph of R [2], denoted by $\Gamma(R)$, is defined as an undirected graph associated to a commutative ring R having vertex set $V(\Gamma(R)) = Z^*(R) = Z(R) - \{0\}$, where distinct vertices x and y are adjacent if and only if $xy = 0$. The original definition of zero divisor graph was introduced by Beck [6] and in his work he defined $V(\Gamma(R)) = Z(R)$ and two vertices x and y being adjacent if and only if $xy = 0$. This definition of zero divisor graphs was first introduced in [3]. Recently, Kimball and LaGrange [13] generalized the definition to the idempotent divisor graph of a commutative ring. Besides this the zero divisor graph has also been extended to other algebraic structures like semi rings, Abelian groups, vector spaces, modules etc, (e.g., see the articles such as [5, 8, 9, 23] and references therein).

For any set X, let $|X|$ denote the cardinality of X and X^* denote the set of non-zero elements of X. We denote an empty set by φ. An element x in a ring R is called nilpotent if $x^m = 0$ for some positive integer m. A ring R is called reduced if it has no non-zero nilpotent elements. A ring is called local if it has a unique maximal ideal. An element x in a ring R is called a unit if there exists an element y in R such that $xy = 1$, where 1 is a multiplicative identity in R. The set of all units of a ring R is denoted by $U(R)$. We denote a ring of integers by $\mathbb{Z}$, a ring of integer modulo n by $\mathbb{Z}_n$ and a finite field with q elements by $\mathbb{F}_q$.

This article continues the investigation of zero divisor graphs that have same metric and upper dimension that was started in [17]. Section 2 reviews basic definitions and known results concerning the metric and upper dimension of zero divisor graphs of rings, as well as the results obtained for graphs in general. The rest of the paper focuses on zero-divisor graphs of commutative rings. In Section 3, we show that if either $\Gamma(R)$ (or $\overline{\Gamma}(R)$) is a regular graph, then $dim(\Gamma(R)) = dim^+(\Gamma(R))$. We also characterize certain families of local and reduced artinian rings and show that their zero divisor graphs have same values for these two parameters. Further, we compute the metric and upper dimension formulae for certain other classes of rings and show that the two values are equal. We obtain a lower bound for the upper dimension of zero divisor graph of a finite Boolean ring.

2 Preliminaries and terminology

A graph G with vertex set $V(G) \neq \emptyset$ and edge set $E(G)$ of unordered pairs of distinct vertices is called a simple graph. A graph G is connected if and only if there is path between any two pair of vertices x and y of G. In a graph G, the distance between two vertices x and y is the length of the shortest path between x and y. A subset $\mathfrak{B}$ of $V(G)$ is said to resolve a pair of vertices $\{u, v\} \subset V(G)$, if there exists some $b \in \mathfrak{B}$ such that $d(u, b) \neq d(v, b)$ or equivalently if the metric representations of distinct vertices are distinct, where the metric representation for a vertex $v \in V(G)$ with respect to an ordered set $\mathfrak{B} = \{b_1, b_2, \ldots, b_k\}$ of vertices of G is an ordered k-tuple defined as $r(v \mid \mathfrak{B}) = (d(v, b_1), d(v, b_2), \cdots, d(v, b_k))$. If $\mathfrak{B}$ resolves all the vertices of G, we say $\mathfrak{B}$ is a resolving set of G and $\mathfrak{B}$ is said to be a minimal resolving set if no proper subset of $\mathfrak{B}$ resolves all vertices of G. A minimal resolving set $\mathfrak{B}$ with least number of vertices is called a *metric basis* of G and the cardinality of metric basis is called the *metric dimension* of G, denoted by $\dim(G)$. Also, a minimal resolving set containing the maximum number of vertices is called an *upper basis* of G and the cardinality of the upper basis is called the *upper dimension* of G, denoted by $\dim^+(G)$.

The concept of finding the metric dimension of a graph first appeared in 1970's introduced by Slater [24] and independently by Harary and Melter [11] and the concept of upper dimension of graphs was introduced by Chartrand et al. [7], where they defined the upper dimension to be the order of the minimal resolving set that has the maximum cardinality. Recently, these concepts of metric and upper dimension of graphs were extended to zero divisor graphs of rings, see [17, 20, 21].

For other definitions, terminology and notations of ring theory, we refer to [4, 12] and for graph theory, we refer to [14].

Theorem 1 [Theorem 3.5, [10]] *For every pair* a, b *of integers with* $2 \leq a \leq b$*, there exists a connected graph* G *with* $\dim(G) = a$ *and* $\dim^+(G) = b$

Let $u \leftrightarrow v$ denote that u is adjacent to v and $u \not\leftrightarrow v$ denote that u is not adjacent to v.

Distance similarity. In a connected graph G, two vertices u and v are said to be *distance similar* if for any vertex $x \in V(G) - \{u, v\}$, $d(u, x) = d(v, x)$. The relation of distance similarity is an equivalence relation. Therefore, it partitions the vertex set of a graph into equivalence classes known as *distance similar equivalence classes*.

Example 2 Let G be a book graph (see Figure 1) with 5 pages with corners of the pages as $p_1, p_2, \ldots, p_5$, q_1, q_2, $\ldots$, q_5, p, q and the adjacencies as follows: $p \leftrightarrow p_i$, $q \leftrightarrow q_i$, for all $1 \leq i \leq 5$, $p \leftrightarrow q$ and $p_i \leftrightarrow q_j$, if and only if, $i = j$, $1 \leq i, j \leq 5$, elsewhere non-adjacencies. Then we have $d(p_i, p_j) = 2 = d(q_i, q_j)$, $d(p_i, q_j) = 3$, whenever $i \neq j$. Choose $\mathfrak{B} = \{p_1, p_2, q_3, q_4\}$ and

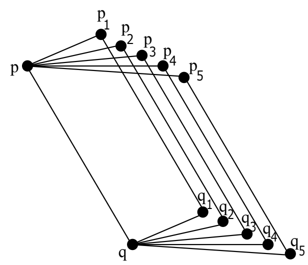

Figure 1: Book graph with 5 pages.

$\mathfrak{B}^* = \{p_1, q_1, q_2, q_3, q_4\}$. Then we see that $\mathfrak{B}$ is a minimal resolving set of minimum order and $\mathfrak{B}^*$ is a minimal resolving set of maximum order and so $\dim(G) = 4$, whereas, $\dim^+(G) = 5$.

Theorem 3 *(i)* [Theorem A, [7]] *Let G be a connected graph on n vertices. Then $\dim(G) = 1$ if and only if $G \cong P_n$, where P_n denotes the path on n vertices.*
(ii) [Lemma 2.3, [18]] *For a connected graph G of order $n \geq 1$, $\dim^+(G) = 1$ if and only if $G \cong P_2$ or P_3 and for $n \geq 4$ $\dim^+(P_n) = 2$, where P_n denotes the path on n vertices.*

Theorem 4 [Theorem 2.3, [21] and Theorem 2.5, [15]] *Let G be a connected graph of order n. Then $\dim(G) = \dim^+(G) = n - 1$ if and only if $G \cong K_n$.*

For integers $k \geq 2$ and $n_1, n_2, \ldots, n_k$ with $2 \leq n_1 \leq n_2 \leq \cdots \leq n_k$, let $G = K_{n_1,n_2,\ldots,n_k}$ be a complete r-partite graph with $V_1, V_2, \ldots, V_k$ as the partite sets. Let u, v be vertices in some partite set, then $d(u,v) = 2$ and for any other vertex $w \in V(G)$, we have either $d(u,w) = 2$ if and only if w is in the same partite set or otherwise $d(u,w) = 1$. Thus, these partite sets partition the vertex set of G into distance similar equivalence classes and therefore we have the following theorem.

Theorem 5 *(i) For integers $k \geq 2$ and $n_1, n_2, \ldots, n_k$ with $2 \leq n_1 \leq n_2 \leq \cdots \leq n_k$ and $n_1 + n_2 + \cdots + n_k = n$,*

$$\dim(K_{n_1,n_2,\ldots,n_k}) = \dim^+(K_{n_1,n_2,\ldots,n_k}) = n - k$$

(ii) For all positive integers $n \geq 2$, $\dim(K_{1,n}) = \dim^+(K_{1,n}) = n - 1$.

The Cartesian product of two graphs G_1 and G_2, denoted by $G = G_1 \times G_2$, is the graph whose vertex set is $V = V(G_1) \times V(G_2)$ and for any two vertices $w_1 = (u_1, v_1)$ and $w_2 = (u_2, v_2)$ in V with $u_1, u_2 \in V(G_1)$ and $v_1, v_2 \in V(G_2)$, there is an edge $w_1 w_2 \in E(G)$ if and only if

(a) either $u_1 = u_2$ and $v_1 v_2 \in E(G_2)$ or (b) $v_1 = v_2$ and $u_1 u_2 \in E(G_1)$,

Theorem 6 *(i)* [Theorem 3.2, [18]] *For* $n \geq 3$,
$$\dim^+(K_{1,n} \times K_2) = \dim^+(K_{1,n}) + 1 = n.$$
(ii) [Corollary 3.3, [18]] *For* $n \geq 5$, $\dim(K_{1,n} \times K_2) = \dim^+(K_{1,n}) = n - 1$.

Theorem 7 [Theorem 2.8, [17]] *Let* R *be a commutative ring with unity. Then* $\dim^+(\Gamma(R))$ *is finite if and only if* R *is finite (and not a domain).*

Recall that the characteristic of a ring R is a smallest positive integer k such that for every $r \in R$, we have $kr = 0$, and if no such integer exists then the ring R is said to have infinite characteristic.

Theorem 8 [Theorem 3.2, [17]] *Let* R *be a finite commutative ring that is not a field such that* R *has odd characteristic. Then* $\dim^+(\Gamma(R)) = \dim(\Gamma(R))$.

Theorem 9 [Theorem 3.3, [17]] *Let* S *be a finite commutative ring of order* $2k$, *where* k *is an odd integer. Then* $\dim^+(\Gamma(S)) = \dim(\Gamma(S))$.

Theorem 10 [Theorem 6-7, [1]] *Let* R *be a finite commutative ring. If all vertices of* $\Gamma(R)$ *(or* $\overline{\Gamma}(R)$*) have the same degrees, then either* $Z(R)^2 = \{0\}$ *or* $R \cong \mathbb{F} \times \mathbb{F}$, *for some finite field* $\mathbb{F}$.

Theorem 11 [Lemma 2.2, [15]] *If* G *is a connected graph and* $D \subseteq V(G)$ *is a subset of the distance similar vertices with* $|D| \geq 2$, *then every resolving set of* G *contains exactly* $|D| - 1$ *vertices of* D.

3 Main results

Theorem 12 *Let* R *be a commutative ring with unity. Then* $\dim(\Gamma(R)) = \dim^+(\Gamma(R)) = 1$ *if and only if* R *is one of the following rings.*

(i) $\dfrac{\mathbb{Z}_3[x]}{(x^2)}$, $\mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_9$.

(ii) $\mathbb{Z}_6$, $\mathbb{Z}_8$, $\dfrac{\mathbb{Z}_2[x]}{(x^3)}$, $\dfrac{\mathbb{Z}_4[x]}{(2x, x^2 - 2)}$.

Proof. These are the only rings whose zero divisor graph is isomorphic to (i) P_2 or (ii) P_3 and the only connected graphs whose metric and upper dimension is 1 is either P_2 or P_3. Hence the result follows. $\square$

Example 13 *Let* R *be a commutative ring with unity. If* $R \cong \mathbb{Z}_2 \times \mathbb{Z}_4$, $\mathbb{Z}_2 \times \frac{\mathbb{Z}_2[x]}{(x^2)}$, $\mathbb{Z}_2 \times \mathbb{F}_4$, $\frac{\mathbb{Z}_2[x,y]}{(x,y)^2}$, $\frac{\mathbb{Z}_4[x,y]}{(2,x)^2}$, $\frac{\mathbb{F}_4[x]}{(x^2)}$, $\frac{\mathbb{Z}_4[x]}{(x^2+x+1)}$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_3 \times \mathbb{Z}_3$, *then* $\dim(\Gamma(R)) = \dim^+(\Gamma(R)) = 2$.

If $R \cong \mathbb{Z}_2 \times \mathbb{Z}_4$, then the only basis sets are $\{(0,1),(0,2)\}$, $\{(0,3),(0,2)\}$, $\{(0,1),(1,2)\}$, $\{(0,3),(1,2)\}$ or $\{(0,1),(0,3)\}$. A similar list can be constructed for $\mathbb{Z}_2 \times \frac{\mathbb{Z}_2[x]}{(x^2)}$ because $\Gamma\left(\mathbb{Z}_2 \times \frac{\mathbb{Z}_2[x]}{(x^2)}\right) \cong \Gamma(\mathbb{Z}_2 \times \mathbb{Z}_4)$. If $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ then any two elements of $S_1 = \{(1,0,0),(0,1,0),(0,0,1)\}$ or $S_2 = \{(1, 1, 0), (1, 0, 1), (0, 1, 1)\}$ forms a basis. If $R \cong \frac{\mathbb{Z}_2[x,y]}{(x,y)^2}$ or $\frac{\mathbb{Z}_4[x,y]}{(2,x)^2}$, or $\frac{\mathbb{F}_4[x]}{(x^2)}$, or $\frac{\mathbb{Z}_4[x]}{(x^2+x+1)}$ then $\Gamma(R) \cong K_3$ and if $R \cong \mathbb{Z}_2 \times \mathbb{F}_4$, then $\Gamma(R) \cong K_{1,3}$.

Theorem 14 [Theorem 2.8, [18]] *Let* G *be a finite connected graph such that every* $v \in V(G)$ *is distance similar to some vertex* $u \neq v$. *Then* $\dim^+(G) = \dim(G)$.

By $G \vee H$, we shall denote the join of two graphs G and H.

Theorem 15 *Let* R *be a commutative ring with unity* $1 \neq 0$, *(not a domain).*

(1) If $|R| = p^2$, *where* p *is prime, then* $\dim(\Gamma(R)) = \dim^+(\Gamma(R))$.
(2) If R *is local with order* p^3, *then* $\dim(\Gamma(R)) = \dim^+(\Gamma(R)$.

Proof. (1). If R is local, then either R is isomorphic to $\mathbb{Z}_{p^2}$ or $\frac{\mathbb{Z}_p[x]}{(x^2)}$ and in either case $\Gamma(R)$ is complete with order $p-1$. If R is reduced, then R is isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$, so $\Gamma(R)$ is complete bipartite. Therefore, the result follows.

(2). If R is a local ring of order p^3, then R is isomorphic to one of the following rings; $\frac{\mathbb{F}_p[x,y]}{(x,y)^2}$, $\frac{\mathbb{F}_p[x]}{(x^3)}$, $\frac{\mathbb{Z}_{p^2}[x]}{(px,x^2)}$, or $\frac{\mathbb{Z}_{p^2}[x]}{(px,x^2-\bar{s}p)}$, where $\bar{s}$ is a non-square element in $\mathbb{Z}_p$. If $R \cong \frac{\mathbb{F}_p[x,y]}{(x,y)^2}$, then $Z^*(R) = \{ux\} \cup \{uy\} \cup \{ux + u'y\}$, where $u, u' \in \mathbb{F}_p - \{0\}$. Thus, $\left|\Gamma\left(\frac{\mathbb{F}_p[x,y]}{(x,y)^2}\right)\right| = p^2 - 1$, and for all $u, v \in Z^*(R)$, we have $uv = 0$. Therefore, $\dim(\Gamma(R)) = \dim^+(\Gamma(R) = p^2 - 2$. Also, if $R \cong \frac{\mathbb{Z}_{p^2}[x]}{(px, x^2)}$, then $Z^*(R) = \{up\} \cup \{ux\}$, where $u \in \mathbb{Z}_p - \{0\}$, so $\Gamma(R) \cong K_{p^2-1}$. Next, if $R \cong \frac{\mathbb{F}_p[x]}{(x^3)}$, then $Z^*(R)$ can be partitioned into two subsets; $Z_1 = \{ux^2 | u \in \mathbb{F}_p - \{0\}\}$ and $Z_2 = \{ax + bx^2 | a \in \mathbb{F}_p - \{0\}, b \in \mathbb{Z}_p\}$. Then Z_1 induces a clique on $p-1$ vertices and Z_2 is an independent subset. Also, for all $z_1 \in Z_1$ and $z_2 \in Z_2$, we have $z_1 z_2 = 0$. Thus, $\Gamma\left(\frac{\mathbb{F}_p[x]}{(x^3)}\right) \cong K_{p-1} \vee T_{p^2-p}$. Let $\{u_1, u_2, \cdots, u_{p-1}\}$ be the set of units of $\mathbb{F}_p$ and let $z_2, z_2' \in Z_2$, then for $1 \leq i \neq j \leq n-1$, we have $d(u_i x, z_2) = d(u_j x, z_2) = 1$ and $d(u_i x, z_1) = d(u_i x, z_2') = 1$, but however $d(z_2, z_2') = 2$, therefore, the sets Z_1 and Z_2 partition the vertex set of $\Gamma\left(\frac{\mathbb{F}_p[x]}{(x^3)}\right)$ into distance similar equivalence classes. Therefore, the result follows by Theorem 14. Finally, if $R \cong \frac{\mathbb{Z}_{p^2}[x]}{(px, x^2 - \bar{s}p)}$, where $\bar{s}$ is a non-square element in $\mathbb{Z}_p$, then we partition the vertex set of $\Gamma\left(\frac{\mathbb{Z}_{p^2}[x]}{(px, x^2 - \bar{s}p)}\right)$ into the subsets $S_1 = \{up \mid u \in \mathbb{Z}_p - \{0\}\}$ and $S_2 = \{ux\} \cup \{up + u'x \mid u, u' \in \mathbb{Z}_p - \{0\}\}$. Then for all $s_1, s_1' \in S_1$ and $s_2, s_2' \in S_2$, we have $s_1 s_1' = 0$, $s_1 s_2 = 0$ and $s_2 s_2' \neq 0$. In fact, the collection $\{S_1, S_2\}$ partitions the vertex set of $\Gamma\left(\frac{\mathbb{Z}_{p^2}[x]}{(px, x^2 - \bar{s}p)}\right)$ into distance similar classes, so the result follows by Theorem 14. □

Corollary 16 *The metric and the upper dimension of zero divisor graph of* $\frac{\mathbb{F}_p[x, y]}{(x, y)^2}$ *and* $\frac{\mathbb{Z}_{p^2}[x]}{(px, x^2)}$ *are equal to* $p^2 - 2$ *and for the zero divisor graph of* $\frac{\mathbb{F}_p[x]}{(x^3)}$ *and* $\frac{\mathbb{Z}_{p^2}[x]}{(px, x^2 - \bar{s}p)}$, *these two values are both equal to* $p^2 - 3$.

Proof. This can be obtained using Theorem 11 in part (2) of Theorem 15 along with Theorem 14. □

In the following theorem, the metric and upper dimension of a class of local rings is characterized.

Theorem 17 [Theorem 2.9 [16]] *Let* R *be a ring (local) isomorphic to* $\mathbb{Z}_{p^n}$, *then* $\dim(\Gamma(R)) = \dim^+(\Gamma(R)) = p^{n-1} - n$.

Theorem 18 *Let* R *be a finite commutative ring. If either* $\Gamma(R)$ *(or* $\overline{\Gamma}(R)$*) is a regular graph, then either* $\dim(\Gamma(R)) = \dim^+(\Gamma(R)) = Z^*(R) - 1$ *or* $\dim(\Gamma(R)) = \dim^+(\Gamma(R)) = Z^*(R) - 2$.

Proof. If either of the graphs $\Gamma(R)$ or its complement is regular, then by Theorem 10, either $Z(R)^2 = 0$ or there is a field $\mathbb{F}$ such that $R \cong \mathbb{F} \times \mathbb{F}$. Therefore, either $\Gamma(R)$ is complete or a complete bipartite graph. Hence the result follows by Theorem 5. $\square$

Resolving sets for zero-divisor graphs have previously been studied in [17, 19] and [21]. In these articles, it was noted that distance similarity was a key factor in determining resolving sets. The following results illustrate this connection between concepts.

Theorem 19 [Theorem 2.1 [19]] *Let G be a connected graph. Suppose G is partitioned into k distinct distance similar classes $V_1, V_2, \dots, V_k$ (that is, $x, y \in V_i$ if and only if $d(x, a) = d(y, a)$ for all $a \in V(G) - \{x, y\}$).*

- *(i) Any resolving set W for G contains all but at most one vertex from each V_i.*
- *(ii) Each V_i induces a complete subgraph or a graph with no edges.*
- *(iii) $\dim(G) \geq |V(G)| - k$.*
- *(iv) There exists a minimal resolving set W for G such that if $|V_i| > 1$, at most $|V_i| - 1$ vertices of v_i are elements of W.*
- *(v) If m is the number of distance similar classes that consist of a single vertex, then $|V(G)| - k \leq \dim(G) \leq |V(G)| - k + m$.*

Theorem 20 *Let R be a reduced Artinian ring with unity (not a domain) containing no factor isomorphic to $\mathbb{Z}_2$. Then $\dim^+(\Gamma(R)) = \dim(\Gamma(R))$.*

Proof. It is a well known fact that every reduced Artinian ring is a direct product of fields. Therefore, we can write $R \cong R_1 \times R_2 \times \cdots \times R_m$, for some positive integer m, where each R_i, $1 \leq i \leq m$ is a field.

Therefore, $V(\Gamma(R)) = \{(r_1, r_2, \dots, r_m)$: $r_i \in R_i$, with $r_i \neq 0$ for some i and $r_j = 0$ for some j, $1 \leq i, j \leq m\}$ and two vertices $x = (x_1, x_2, \dots, x_m)$ and $y = (y_1, y_2, \dots, y_m)$ are adjacent if and only if either $x_i = 0$ or $y_i = 0$. Assume that $|R_i| > 2$ for each i and choose $x_i \in R_i^*$. Consider the set $E = \{(x_1, 0, \dots, 0)$, $(x_1, x_2, 0, \dots, 0)$, $\dots$, $(x_1, x_2, \dots, x_{m-1}, 0)\}$. Then clearly, no two vertices are adjacent in E. Now, for $1 \leq i \leq m-1$, let E_i denote the collection of those vertices of $\Gamma(R)$ having exactly i non-zero coordinates. In particular, let

$$E_1 = \{(x_1, 0, \dots, 0), (0, x_2, 0, \dots, 0), \dots, (0, 0, \dots, 0, x_m)\},$$
$$E_2 = \{(x_1, x_2, 0, \dots, 0), (x_1, 0, x_3, 0, \dots, 0), \dots, (x_1, 0, \dots, 0, x_m), (0, x_2, x_3, 0, \dots, 0),$$
$$(0, x_2, 0, x_4, 0 \dots, 0), \dots, (0, x_2, 0, \dots, x_m), (0, \cdots, 0, x_{m-1}, x_m)\},$$
$$\vdots$$
$$E_{m-1} = (x_1, \dots, x_{m-1}, 0), (x_1, \dots, x_{m-2}, 0, x_{m-1}), \dots, (0, x_2, \dots, x_m).$$

It is easy to see that the collection $E_1, E_2, \cdots, E_{m-1}$ partitions the vertex set of $\Gamma(R)$ and for $x_i \in R_i^*$, each E_i, $1 \le i \le m-1$, defines a distance similar equivalence class. Also, as $|R_i| > 2$ for each i, therefore each E_i has at least two vertices. Therefore, the result follows by Theorem 14. □

Remark 21 $\dim^+(\mathbb{Z}_2 \times \mathbb{Z}_2) = \dim(\mathbb{Z}_2 \times \mathbb{Z}_2) = 1$, $\dim^+(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2) = \dim(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2) = 2$. If $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, then $\dim(\Gamma(R)) = 3$, with $\mathfrak{B}$ ={(1,1,1,0), (1,1,0,1), (1, 0, 1, 1)} an example of a minimal resolving set, whereas with $\mathfrak{B}'$ ={(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1)} as an example of a minimal resolving set, we have $\dim^+(\Gamma(R))$= 4. Notice that a vertex z of $\Pi\mathbb{Z}_2^n$ is a pendent vertex if and only if z has exactly one zero coordinate and for $n = 4$, any three (i.e $n-1$) pendent vertices of $\Pi\mathbb{Z}_2^4$ form a metric basis. However, the same does not follow if $n \ge 5$ as can be seen in Theorem 28 to the end of this section.

Theorem 22 *Let R_1 be a finite commutative ring with unity and R_2 be an integral domain.*

(i) If $R_1 \cong \mathbb{F}_1 \times \mathbb{F}_2 \times \cdots \times \mathbb{F}_k$, where each $\mathbb{F}_i$ is a field and $\mathbb{F}_i \ne \mathbb{Z}_2$ for each i, $1 \le i \le k$ and if $R_2 \ne \mathbb{Z}_2$, then $\dim^+(\Gamma(R_1 \times R_2)) = \dim(\Gamma(R_1 \times R_2))$.
(ii) If $R_1 \cong \mathbb{Z}_{p^k}$, where $k \ge 2$,, then $\dim^+(\Gamma(R_1 \times R_2)) = \dim(\Gamma(R_1 \times R_2))$.
(iii) If R_1 is a local ring other than $\mathbb{Z}_{p^k}$, such that $\Gamma(R_1)$ is a complete graph, then $\dim^+(\Gamma(R_1 \times R_2)) = \dim(\Gamma(R_1 \times R_2))$

Proof. (i) Let $R_1 \times R_2 \cong S_1 \times S_2 \times \dots S_m$, where $|S_i| \ne 2$ for each i, $1 \le i \le m$. Then consider the following partition of $V(\Gamma(R_1 \times R_2)$.

$$A_1^{(1)}, A_1^{(2)}, \dots, A_1^{(m)}, A_2^{(1,2)}, A_2^{(1,3)}, \dots, A_2^{(1,m)}, A_2^{(2,3)}, A_2^{(2,4)}, \dots, A_2^{(2,m)},$$
$$\dots, A_2^{(m-1,m)}, \dots, A_{m-1}^{(1,2,\dots,m-1)}, A_{m-1}^{(1,2,\dots,m-2,m)}, \dots, A_{m-1}^{(2,3,\dots,m)},$$

where $A_i^{(s)}$ denotes the subset of $Z^*(R_1 \times R_2)$ having i non-zero coordinates and the non-zero positions are given by the string (s). For example, let $s_i \in S_i^*$, then $A_1^{(2)} = \{(0, s_2, 0 \ldots, 0)\}$, $A_3^{(124)} = \{(s_1, s_2, 0, s_4, 0, \ldots, 0)\}$, etc. The above partition of $V(\Gamma(R_1 \times R_2))$ is obtained by an equivalence relation " $\sim$ " defined in the following way: let $S = (s_1, s_2, \ldots, s_m)$ and $S' = (s_1', s_2', \ldots, s_m')$, then $S \sim S'$ if and only if whenever $s_i = 0$, then $s_i' = 0$. Also, the number of equivalence classes is equal to $2^m - 2$. As $|S_i| > 2$ for each $i, 1 \leq i \leq m$, we have each $A_i^{(s)}$ induces a subgraph of order at least 2 and size 0 and it is not difficult to see that each $A_i^{(s)}$ is a distance similar equivalence class. Therefore, the result follows by Theorem 14 $\big($in fact by Theorem 11, since every basis misses exactly one vertex from each equivalence class, we have $\dim^+(\Gamma(R_1 \times R_2)) = \dim(\Gamma(R_1 \times R_2)) = |Z^*(R_1 \times R_2)| - (2^m - 2)\big)$.

(ii) First assume that $R_1 \cong \mathbb{Z}_4$. Then $R_1 \times R_2 \cong \mathbb{Z}_4 \times \mathbb{F}$, where $\mathbb{F} = \{f_0, f_1, \ldots, f_t\}$ is an integral domain and f_0 is the zero element of $\mathbb{F}$. If $\mathbb{F} = \mathbb{Z}_2$, then $R_1 \times R_2 \cong \mathbb{Z}_4 \times \mathbb{Z}_2$ and the result is true in this case. So assume $|\mathbb{F}| > 2$. Consider the vertex set $Z^*(R_1 \times R_2)$ of $\Gamma(R_1 \times R_2)$ and let $A = \{1, 3\} \times \{0\} = \{(1, 0), (3, 0)\}$, $B = \{0\} \times \{f_1, f_2, \ldots, f_t\}$, $C = \{(2, 0)\}$ and $D = \{2\} \times \{f_1, f_2, \ldots, f_t\}$, where $t \geq 2$. Then the sets A, B, C and D partition the vertex set of $\Gamma(\mathbb{Z}_4 \times \mathbb{F})$ into the distance similar equivalence classes with $|A| > 1, |B| > 1, |D| > 1$ and $|C| = 1$. Thus, every basis contains all elements of A, B and D but one element from each set, by Theorem 11. Without loss of generality, let $\mathfrak{B} = \{(1, 0)\} \cup \big(B - \{(0, f_1)\}\big) \cup \big(D - \{(2, f_1)\}\big)$. Then $r((3, 0)|\mathfrak{B}) = (2, 1, \ldots, 1, 2, \ldots, 2)$, $r((0, f_1)|\mathfrak{B}) = (1, 2, \ldots, 2)$, $r((2, f_1)|\mathfrak{B}) = (2, 2, \ldots, 2)$ and $r((2, 0)|\mathfrak{B}) = (2, 1, \ldots, 1)$. Therefore, $\mathfrak{B}$ is the basis. Consequently, the only element of C does not belong to any basis.

Hence, $\dim^+(\Gamma(\mathbb{Z}_4 \times \mathbb{F})) = \dim(\Gamma(\mathbb{Z}_4 \times \mathbb{F})) = 2|\mathbb{F}| - 3$. Now, assume that $R_1 \cong \mathbb{Z}_{p^n}$, where $p^n \neq 4$ and R_2 is a domain. We partition the vertex set of $\Gamma(\mathbb{Z}_{p^n})$ into $n - 1$ disjoint subsets of the form $V_1, V_2, \ldots, V_{n-1}$, where $V_i = \{k_i p^i : p \nmid k_i\}$, $1 \leq i \leq n-1$. We see that $|V_i| = (p-1)p^{n-i-1}$, $1 \leq i \leq n-1$ and that $|\Gamma(\mathbb{Z}_{p^n})| = p^{n-1} - 1$. In fact the sets $V_1, V_2, \ldots, V_{n-1}$ gives the partition of $V(\Gamma(\mathbb{Z}_{p^n}))$ into distance similar equivalence classes of cardinality at least 2 except for the case that $|V_{n-1}| = 1$, when $p = 2$.

Define the sets $A = U(R_1) \times \{0\}$, $B = \{0\} \times R_2^*$, $C_i = V_i \times \{0\}$ and $D_i = V_i \times R_2^*$, where $1 \leq i \leq n - 1$. The collection $\mathcal{P} = \{A, B, C_1, C_2, \ldots, C_{n-1}, D_1, D_2, \ldots, D_{n-1}\}$ gives the partition of vertex set of $\Gamma(R_1 \times R_2)$ into distance similar equivalence classes. Notice that $|B| = 1$ if and only if $R_2 \cong \mathbb{Z}_2$, $|C_i| = 1$ if and only if $p = 2$ and $i = n - 1$, and $|D_i| = 1$ if and only if $p = 2$, $i = n - 1$ and $R_2 \cong \mathbb{Z}_2$. So first assume that $p > 2$. Then if $R_2 \ncong \mathbb{Z}_2$, then the collection $\mathcal{P}$

gives the partition in which each set has cardinality at least 2. Therefore, the result follows by Theorem 14. Now, if $p > 2$ and $R_2 \cong \mathbb{Z}_2$, then $|R_1 \times R_2| = 2k$, where k is an odd integer. Therefore, the result follows by Theorem 9.

Finally, assume that $R_1 \cong \mathbb{Z}_{2^n}$, $n \geq 3$. If $R_2 \ncong \mathbb{Z}_2$, then each set in the collection $\mathcal{P}$ has at least 2 elements except C_{n-1}. Without loss of generality, by Theorem 11, we construct the set $\mathfrak{B}'$ which takes all elements but one from each element of $\mathcal{P} - C_{n-1}$. Now, it can be easily seen that the set $\mathfrak{B}'$ gives distinct representations to each vertex of $\Gamma(R_1 \times R_2)$. Therefore, there is a basis (which is both a metric as well as an upper basis) which does not contain the only element of C_{n-1}. Hence, the result follows by Theorem 14 and 11. Lastly, if $R_2 \cong \mathbb{Z}_2$, then the sets in the collection $\mathcal{P}$ that have only one element are B, C_{n-1} and D_{n-1}. Utilizing Theorem 11, we construct $\mathfrak{B}''$ by taking all elements but one from each element of $\mathcal{P} - \{B,\ C_{n-1},\ D_{n-1}\}$. The set $\mathfrak{B}''$ so constructed gives distinct representations to all the vertices of $\Gamma(R_1 \times R_2)$. Hence, $\mathfrak{B}''$ is a resolving set and so $\dim^+(\Gamma(\mathbb{Z}_{2^n} \times \mathbb{Z}_2)) = \dim(\Gamma(\mathbb{Z}_{2^n} \times \mathbb{Z}_2))$ by Theorem 14 and 11.

(iii) Since R_1 is local finite commutative ring with unity, therefore $|R_1| = p^n$ for some prime p and a positive integer n. Let $Z^*(R_1) = \{r_1, r_2, \ldots, r_t\}$ be the set of all non-zero zero divisors of R_1 and $U(R_1)$ be the set of units of R_1. We partition the vertex set of $\Gamma(R_1 \times R_2)$ as follows:

$$X = Z^*(R_1) \times \{0\},\ Y = \{0\} \times R_2^*$$

$$Z = U(R_1) \times \{0\},\ X_i = \{r_i\} \times R_2^*,\ 1 \leq i \leq t.$$

If $R_1 \cong \dfrac{\mathbb{Z}_2[x]}{(x^2)}$, then the proof follows similarly as in the case when $R_1 \cong \mathbb{Z}_4$. Hence, we assume that $R_1 \ncong \dfrac{\mathbb{Z}_2[x]}{(x^2)}$ for the rest of the proof. Assume that $\Gamma(R_1)$ is a complete graph, therefore the set X induces a clique. Now, first consider the case, when $R_2 \ncong \mathbb{Z}_2$. In this case, $|X| > 1$, $|Y| > 1$, $|Z| > 1$ and $|X_i| > 1$ for each i, $1 \leq i \leq t$ and each of the sets X, Y, Z and X_i, $1 \leq i \leq t$ defines a distance similar equivalence class. Therefore, the result follows by Theorem 14. Now, let $R_2 \cong \mathbb{Z}_2$ and $|R_1| = p^n$. If $p > 2$, then $|R_1 \times R_2| = 2k$, where k is an odd integer and therefore the result follows by Theorem 9. Finally, let $|R_1| = 2^n$ and $R_2 = \mathbb{Z}_2$. In this case, we partition the vertex set of $\Gamma(R_1 \times R_2)$ into the sets,

$$X = Z^*(R_1) \times \{0\},\ Y = \{0\} \times R_2^* = \{(0, 1)\}$$
$$Z = U(R_1) \times \{0\},\ \text{and}\ \widehat{Z} = Z^*(R_1) \times R_2^* = Z^*(R_1) \times \{1\}.$$

Notice that the set X induces a clique and each of the sets $Y, Z, \widehat{Z}$ is independent. Each $x \in X$ is adjacent to $(0,1)$ and $\widehat{z}$ for all $\widehat{z} \in \widehat{Z}$. The only element of Y i.e., $(0,1)$ is also adjacent to each $z \in Z$ and each element of Z is a pendent vertex. These are the only adjacencies in $\Gamma(R_1 \times \mathbb{Z}_2)$, where R_1 is local other that $\mathbb{Z}_{p^n}$ and not a domain.

The collection $\mathcal{P} = \{X, Y, Z, \widehat{Z}\}$ is the partition of $V(\Gamma(R_1 \times R_2))$ into distance similar equivalence classes. Without loss of generality, using Theorem 11, we form a set $\mathfrak{B}''' = \big(X - \{x\}\big) \cup \big(Z - \{z\}\big) \cup \big(\widehat{Z} - \{\widehat{z}\}\big)$ for some $x \in X$, $z \in Z$ and $\widehat{z} \in \widehat{Z}$. But however each vertex of $\Gamma(R_1 \times R_2)$ has a unique representation with respect to $\mathfrak{B}'''$, therefore $\mathfrak{B}'''$ forms a resolving set. Consequently, the unique element of Y does not belong to any metric basis (and upper basis). Therefore, by Theorem 14 and 11, $\dim^+(\Gamma(R_1 \times R_2)) = \dim(\Gamma(R_1 \times R_2))$ and in fact this number equals $(|X| - 1) + (|Z| - 1) + (|\widehat{Z}| - 1) = (|Z^*(R_1)| - 1) + (|U(R_1)| - 1) + (|Z^*(R_1)| - 1)$. Since R_1 is a finite commutative ring with unity and so each element is either a unit or a zero divisor, therefore $\dim^+(\Gamma(R_1 \times R_2)) = \dim(\Gamma(R_1 \times R_2)) = |R_1| + |Z^*(R_1)| - 4$. This completes the proof. □

Remark 23 Let $R = \{x_0, x_1, \ldots, x_n, \ldots\}$, where x_0 is the zero element of R, be an integral domain, the vertex set of the zero divisor graph $\Gamma(\mathbb{Z}_4 \times R)$ can always be partitioned into four distance similar equivalence classes namely, $A = \{1,3\} \times \{0\} = \{(1,0),(3,0)\}$, $B = \{0\} \times \{x_1, x_2, \ldots, x_n, \ldots\}$, $C = \{(2,0)\}$ and $D = \{2\} \times \{x_1, x_2, \ldots, x_n, \ldots\}$. Choose a vertex $a \in A$, $b \in B$, $c \in C$ and $d \in D$. Then for all a, b, c and d, we have $deg(a) = |R| - 1, deg(d) = 1,\ deg(b) = 3$ and $deg(c) = 2|R| - 2$. For an integral domain R, the zero divisor graphs associated to $\mathbb{Z}_4 \times R$ have similar shape except to the number of vertices in the partitions B and D, and degrees of vertices in the partitions A, C and D.

Example 24 Let $R_1 = \mathbb{Z}_4$ or $\dfrac{\mathbb{Z}_2[x]}{(x^2)}$ and $R_2 = \mathbb{F}_{16}$. Then, by Theorem 22, we have $\mathfrak{Dim}^+(\Gamma(\mathbb{Z}_4 \times \mathbb{F}_{16})) = \mathfrak{Dim}(\Gamma(\mathbb{Z}_4 \times \mathbb{F}_{16})) = 2|\mathbb{F}_{16}| - 3 = 29$. From the zero divisor graph of $\Gamma(\mathbb{Z}_4 \times \mathbb{F}_{16})$ (see Figure 2), we notice that for a field $\mathbb{F}$, a metric basis (or upper basis) for $\Gamma(\mathbb{Z}_4 \times \mathbb{F})$ can be formed by taking one element from $A = \{(1,0),(3,0)\}$ and any $|\mathbb{F}| - 1$ elements from each of the sets $B = \{0\} \times U(\mathbb{F})$ and $D = \{2\} \times U(\mathbb{F})$. Therefore, $\mathfrak{Dim}^+(\Gamma(\mathbb{Z}_4 \times \mathbb{F})) = \mathfrak{Dim}(\Gamma(\mathbb{Z}_4 \times \mathbb{F})) = 2(|\mathbb{F}| - 2) + 1 = 2|\mathbb{F}| - 3$.

Now, if $R_1 = \mathbb{Z}_{16}$ and $R_2 = \mathbb{Z}_2$, then under the notations of Theorem 22, we have

$$A = U(R_1) \times \{0\} = \{(1,0),(3,0),(5,0),(7,0),(9,0),(11,0),(13,0),(15,0)\}$$

$$B = \{0\} \times \{1\} = \{(0,1)\},\ C_1 = V_1 \times \{0\} = \{(2,0),(6,0),(10,0),(14,0)\}$$

$$C_2 = V_2 \times \{0\} = \{(4,0),(12,0)\},\ C_3 = V_3 \times \{0\} = \{(8,0)\}$$

$$D_1 = V_1 \times \{1\} = \{(2,1),(6,1),(10,1),(14,1)\}$$

$$D_2 = V_2 \times \{1\} = \{(4,1),(12,1)\},\ D_3 = V_3 \times \{1\} = \{(8,1)\}$$

Therefore, by Theorem 22, $\mathfrak{Dim}^+(\Gamma(\mathbb{Z}_{16} \times \mathbb{Z}_2)) = \mathfrak{Dim}(\Gamma(\mathbb{Z}_{16} \times \mathbb{Z}_2)) = (|A|-1)$ $+(|C_1|-1)$ $+(|C_2|-1)$ $+(|D_1|-1)$ $+(|D_2|-1) = 7+3+1+3+1 = 15$.

In general, we see that for any positive integer $n \geq 3$,

$$\begin{aligned}
\mathfrak{Dim}^+(\Gamma(\mathbb{Z}_{2^n} \times \mathbb{Z}_2)) = \mathfrak{Dim}(\Gamma(\mathbb{Z}_{2^n} \times \mathbb{Z}_2))& \\
&= (|U(R_1)|-1) + \sum_{i=1}^{n-2}(|C_i|-1) + \sum_{i=1}^{n-2}(|D_i|-1) \\
&= (2^{n-1}-1) + 2\sum_{i=1}^{n-2}(|V_i|-1) \\
&= (2^{n-1}-1) + 2\big(|V_1| + |V_2| + \cdots + |V_{n-2}| - (n-2)\big) \\
&= (2^{n-1}-1) + 2(|\Gamma(\mathbb{Z}_{2^n})| - n + 1) \\
&= 2^n + 2^{n-1} - 2n - 1.
\end{aligned}$$

Now, let $S_1 = \dfrac{\mathbb{F}_4[y]}{(y^2)}$, where $\mathbb{F}_4 = \dfrac{\mathbb{Z}_2[x]}{(1+x+x^2)}$ is a field with four elements. Then S_1 is a local ring such that $\Gamma(S_1)$ is a complete graph ($\cong K_3$) and let $S_2 = \mathbb{Z}_2$. Then in the notations of Theorem 22, the distance similar equivalence partition for $\Gamma\left(\dfrac{\mathbb{F}_[y]}{(y^2)} \times \mathbb{Z}_2\right)$ is given as (see Figure 2, $\Gamma\left(\dfrac{\mathbb{F}_4[y]}{(y^2)} \times \mathbb{Z}_2\right)$,

$$X = \{(y,0),(xy,0),(xy+y,0)\},\quad Y = \{(0,1)\},\quad \widehat{Z} = \{(y,1),(xy,1),(xy+y,1)\}$$

$$\begin{aligned}
Z = \{&(1,0),(x,0),(1+x,0),(1+y,0),(x+y,0),(1+x+y,0),(xy+1,0),(x+xy,0),\\
&(1+x+xy,0),(1+y+xy,0),(x+y+xy,0),(1+x+y+xy,0)\}.
\end{aligned}$$

The set of pendant vertices in $\Gamma\left(\dfrac{\mathbb{F}_4[y]}{(y^2)} \times \mathbb{Z}_2\right)$ is the set of vertices given by $Z = U(R_1) \times \{0\}$. Therefore, by Theorem 22, $\mathfrak{Dim}^+\left(\Gamma\left(\dfrac{\mathbb{F}_4[y]}{(y^2)} \times \mathbb{Z}_2\right)\right) = \mathfrak{Dim}\left(\Gamma\left(\dfrac{\mathbb{F}_4[y]}{(y^2)} \times \mathbb{Z}_2\right)\right) = \left|\dfrac{\mathbb{F}_4[y]}{(y^2)}\right| + \left|Z^*\left(\dfrac{\mathbb{F}_4[y]}{(y^2)}\right)\right| - 4 = 16 + 3 - 4 = 15.$

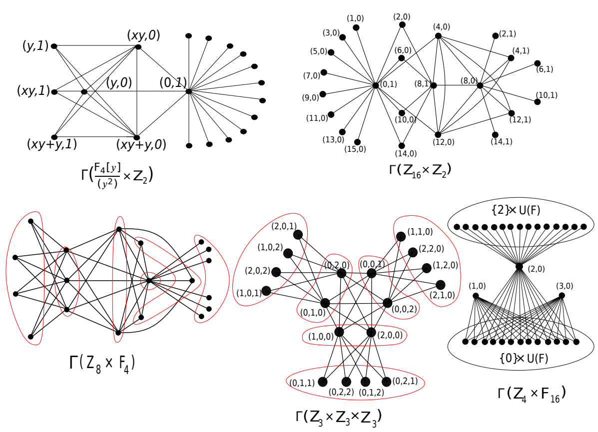

Figure 2:

A similar partition of $\Gamma(\mathbb{Z}_8 \times \mathbb{F}_4)$ and $\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3)$ into the distance similar equivalence classes is displayed in Figure 2.

We also notice that the graphs given in Figure 2 are either symmetric with respect to horizontal or vertical axis. This symmetry is achieved easily with the help of partition given in Theorem 22.

A (connected) graph G is said to be Hamiltonian if it contains a cycle that traverses every vertex of G. In the following theorem, $R^{\times}$ shall denote the set of units of the ring R.

Theorem 25 *Let* $R \cong R_1 \times R_2$ *be a commutative ring such that* $\dim^+(\Gamma(R)) < \infty$. *Then if* $\Gamma(R)$ *is Hamiltonian,* $\dim^+(\Gamma(R)) = \dim(\Gamma(R))$.

Proof. As $\dim^+(\Gamma(R)) < \infty$, therefore by Theorem 7, R is finite. We claim that if $\Gamma(R)$ has to be Hamiltonian then both R_1 and R_2 must be integral domains. Assume to the contrary and define $X = \{0\} \times Z^*(R_2)$ and $Y = \big(R_1 - Z(R_1)\big) \times Z^*(R_2)$. Then there is $x \in X$ and $y \in Y$ such that $xy = 0$ and for every $y_1, y_2 \in Y$, we have $y_1 y_2 \neq 0$, i.e., Y is an independent subset of $V(\Gamma(R))$. Now, by definition, a Hamiltonian cycle in $\Gamma(R)$ contains all vertices of Y and therefore contains a matching between X and Y. As the set Y is an independent subset of vertices, it follows that $|Y| \leq |X|$. But this implies that $|R_1 - Z(R_1)| \leq 1$ whence it follows that identity element is the only unit in R_1. Therefore, $R_1 \cong \Pi \mathbb{Z}_2^k$ for some positive integer k. Let $z_1 = (1, 1, \cdots, 1, 0) \in R_1$, then the vertex $(z_1, 1) \in V(\Gamma(R_1 \times R_2))$ is only adjacent to $z_2 = (0, 0, \cdots, 0, 1, 0)$, which is a contradiction to the fact that $\Gamma(R)$ is Hamiltonian. Thus, both R_1 as well as R_2 are integral domains, therefore the vertex set of $\Gamma(R)$ can be partitioned into two distance similar equivalence classes $V_1 = R_1^{\times} \times \{0\}$ and $V_2 = \{0\} \times R_2^{\times}$ (of orders $|R_1| - 1$ and $|R_2| - 1$). Now, if either $|R_1| = 2$ or $|R_2| = 2$. then $\Gamma(R)$ is a star graph, otherwise the result follows by Theorem 14. □

In the following theorem, we give a formula for computing the metric and upper dimension of zero divisor graph of a class of rings given by $\Gamma(R) \times \Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2)$ and prove that the two are equal.

Theorem 26 *Let* R *be a finite commutative ring with unity such that* $xy = 0$ *for all* $x, y \in Z^*(R)$*. Then* $\dim(\Gamma(R) \times \Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2)) = \dim^+(\Gamma(R) \times \Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2)) = |Z^*(R)| - 1$*, where* $|Z^*(R)| \geq 3$.

Proof. Let $G = \Gamma(R) \times \Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2)$. Assume the two copies of $\Gamma(R)$ in G be denoted by Γ_1 and Γ_2. Let $V(G) = \{x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n\}$, where $|Z^*(R)| = n$, such that $x_i's$ and $y_j's$, $1 \leq i,\ j \leq n$, are vertices of Γ_1 and Γ_2 respectively and suppose, without loss of generality, the adjacencies between Γ_1 and Γ_2 be $x_i \sim y_j$ if and only if $i = j$.

For n =3, it is easily verified that any two vertex subset of Γ_i, $i = 1, 2$, is a metric and an upper basis for G. Note that in this case, the basis sets of $\Gamma(R)$ are the only basis sets of $\Gamma(R) \times \Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2)$. So assume $n \geq 4$.

Let $\mathfrak{B}$ be a minimal resolving set for G. If $x_i, x_j \in V(G) - \mathfrak{B}$ with $i \neq j$ such that $y_i \notin \mathfrak{B}$ and $y_j \notin \mathfrak{B}$, then $r(x_i|\mathfrak{B}) = r(x_j|\mathfrak{B})$, since $d(x_t, x_i) = d(x_t, x_j) = 1$ for all $x_t \in \mathfrak{B}$ and $d(y_s, x_i) = d(y_s, x_j) = 2$ for all $y_s \in \mathfrak{B}$. Hence, $|\mathfrak{B}| \geq n - 1$.

For an example of a minimal resolving set of order $n-1$, consider $\mathfrak{B}_0 = \{x_1,$ $x_2, \ldots, x_{n-1}\}$. Note that $r(x_n|\mathfrak{B}_0) = (1, 1, \ldots, 1)$, $r(y_n|\mathfrak{B}_0) = (2, 2, \ldots, 2)$ and, for each $1 \leq i < n$, $r(y_i|\mathfrak{B}_0)$ is the vector with 1 in the i^{th} coordinate and 2 in all other coordinates. With a similar argument, it can be shown that every subset $\mathfrak{B}_1$ of order $n - 1$ for which $\mathfrak{B}_1 \cap \{x_i, y_i\} = \varphi$ for only one index i and $|\mathfrak{B}_1 \cap \{x_j, y_j\}| = 1$ for all $j \neq i$ is a minimal resolving set.

Next, assume $\mathfrak{B}_2$ is a minimal resolving set with $|\mathfrak{B}_2| \geq n$. Then $\mathfrak{B}_2$ cannot contain a subset of the type described in the previous paragraph. Hence, there must be some k such that $x_k \in \mathfrak{B}_2$ and $y_k \in \mathfrak{B}_2$.

Consider $\mathfrak{B}_3 = \mathfrak{B}_2 - \{x_k\}$. We will show that $\mathfrak{B}_3$ is a resolving set. Suppose $a, b \in V(G) - \mathfrak{B}_2$ with $a \neq b$ and $r(a|\mathfrak{B}_3) = r(b|\mathfrak{B}_3)$ but $r(a|\mathfrak{B}_2) \neq r(b|\mathfrak{B}_2)$. This means, without loss of generality, $d(a, x_k) = 1$ and $d(b, x_k) = 2$. Thus $a = x_r$ for some $r \neq k$ and $b = y_q$ for some $q \neq k$. But then $d(a, y_k) = 2$ and $d(b, y_k) = 1$, contradicting $r(a|\mathfrak{B}_3) = r(b|\mathfrak{B}_3)$. Hence, if $a, b \in V(G) - \mathfrak{B}_2$ with $a \neq b$, then $r(a|\mathfrak{B}_3) \neq r(b|\mathfrak{B}_3)$.

Finally, assume $c \in V(G) - \mathfrak{B}_2$ such that $r(c|\mathfrak{B}_3) = r(x_k|\mathfrak{B}_3)$. Since this implies $d(c, y_k) = d(x_k, y_k) = 1$, $c = y_p$ for some $p \neq k$. If there is some

$y_m \in \mathfrak{B}_3$ with $m \notin \{k,p\}$, then $d(x_k, y_m) = 2$ and $d(c, y_m) = 1$. If there is no such $y_m \in W_3$, since $|\mathfrak{B}_3| \geq n-1 \geq 3$, there must be some $x_g \in \mathfrak{B}_3$ with $g \notin \{k,p\}$. Then, $d(x_k, x_g) = 1$ and $d(c, x_g) = 2$. In all possible cases, $r(x_k|\mathfrak{B}_3) \neq r(c|\mathfrak{B}_3)$. Thus W_3 is a resolving set, showing that $\mathfrak{B}_2$ was not a minimal resolving set. Hence, any minimal resolving set must have $n-1$ elements. □

Note. If R is a finite commutative ring with $|Z^*(R)| = 2$, then $\Gamma(R) \cong K_2$ so that $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, or Z_9 or $\dfrac{Z_2[x]}{(x^2)}$ and so in this case $\Gamma(R) \times \Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2) \cong K_{2,2}$, therefore $\dim(\Gamma(R) \times \Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2)) = \dim^+(\Gamma(R) \times \Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2)) = 2$.

Corollary 27 *Let $\mathbb{F}$ be a finite field and let $R = \dfrac{\mathbb{F}[x_1, x_2, \cdots, x_n]}{I}$, where I is the ideal generated by the set $\{x_i x_j | 1 \leq i \leq n, 1 \leq j \leq n\}$. Then $\dim^+(\Gamma(R) \times \Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2)) = \dim(\Gamma(R) \times \Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2)) = |\mathbb{F}|^n - 2$.*

Proof. We write $R = \dfrac{\mathbb{F}[x_1, x_2, \cdots, x_n]}{I} = \{a_0 + a_1 x_1 + \cdots + a_n x_n : \ a_i \in \mathbb{F}\}$. Thus, $Z(R) = \{a_1 x_1 + \cdots + a_n x_n : \ a_i \in \mathbb{F}, \ i = 1, 2, \ldots, n\}$. Hence $Z^*(R) = |\mathbb{F}|^n - 1$. Clearly, the product of any two elements of $Z(R)$ is zero. Hence the result follows. □

Theorem 28 *If $n \geq 4$ is a positive integer, then $\dim^+(\Pi\mathbb{Z}_2^n) \geq n$.*

Proof. Assume $n \geq 4$, and choose a subset $\mathfrak{B} = \{e_1, e_2, \ldots, e_n\} \subset V(\Gamma(\Pi\mathbb{Z}_2^n))$, where e_i has i^{th} coordinate as non-zero and all other coordinates zero. Put $e = e_1 + e_2 + \cdots + e_n$, and let z be any vertex in $V(\Pi\mathbb{Z}_2^n) - \mathfrak{B}$. Then $d(v, e_i) = 1$ if and only if i^{th} coordinate of v is zero, otherwise $d(v, e_i) = 2$. Therefore, $r(v|\mathfrak{B}) = v + e$ and so $\mathfrak{B}$ forms a resolving set for $\Gamma(\Pi\mathbb{Z}_2^n)$. Further, one can see that $\mathfrak{B}$ forms a minimal resolving set, as by removing e_i from $\mathfrak{B}$ to obtain $\mathfrak{B}_i$, $1 \leq i \leq n-1$, the vertices $x = e_1 + e_2 + \cdots + e_{n-1}$ and $y = e_1 + e_2 + \cdots + e_{n-1} - e_i$ have the same representations with respect to $\mathfrak{B} - \{e_i\}$. Also, with respect to $\mathfrak{B}_n = W - \{e_n\}$, the vertices $x' = e_2 + e_3 + \cdots + e_n$ and $y' = x' - e_n$ have the same representations. This shows that B forms an upper basis for $\Pi\mathbb{Z}_2^n$. Consequently it follows that $\dim^+(\Pi\mathbb{Z}_2^n) \geq n$. □

Remark 29 As an illustration to Theorem 28, we choose an example of a minimal resolving set for $\Pi\mathbb{Z}_2^5$ as $\mathfrak{B}$ =(1, 0, 0, 0, 0), (0, 1, 0, 0, 0), (0, 0, 1, 0, 0), (0, 0, 0, 1, 0), (0, 0, 0, 0, 1), where by removing $(1,0,0,0,0)$ from $\mathfrak{B}$ to obtain $\mathfrak{B}_1$, we have $r((1, 1, 1, 1, 0)| \mathfrak{B}_1) = r((0, 1, 1, 1, 0)$

$| \mathfrak{B}_1)$ =(2, 2, 2, 1), removing (0,1,0,0,0) to get $\mathfrak{B}_2$ gives $r((1,1,1,1,0) \mid \mathfrak{B}_2) = r((1,0,1,1,0)|\mathfrak{B}_2) = (2,2,2,1)$, removing $(0,0,1,0,0)$ to obtain $\mathfrak{B}_3$ gives $r((1,1,1,1,0) \mid \mathfrak{B}_3) = r((1,1,0,1,0) \mid \mathfrak{B}_3) = (2,2,2,1)$, removing (0, 0, 0, 1, 0) to get $\mathfrak{B}_4$ gives $r((1,1,1,1,0) \mid \mathfrak{B}_4) = r((1,1,1,0,0) \mid \mathfrak{B}_4) = (2,2,2,1)$ and removing (0, 0, 0, 0, 1) to get $\mathfrak{B}_5$ gives $r((0,1,1,1,1) \mid \mathfrak{B}_5) = r((0,1,1,1,0) \mid \mathfrak{B}_5) = (1,2,2,2)$.

While examining the metric dimension (and upper dimension) of zero divisor graphs of small finite commutative rings R with $|V(\Gamma(R))| \leq 14$, we found that there is only one ring i.e., $R \cong \Pi\mathbb{Z}_2^n$, $n \geq 4$ for which $\dim^+(\Gamma(R)) \neq \dim(\Gamma(R))$. It has been earlier shown in Remark 21 that $\dim(\Gamma(\Pi\mathbb{Z}_2^4)) = 3$, whereas $\dim^+(\Gamma(\Pi\mathbb{Z}_2^4)) = 4$. It is also not difficult to check that a set of $\frac{n(n-1)}{2} - 1$ elements of $\Pi\mathbb{Z}_2^n$, $n \geq 4$, that have exactly two non-zero coordinates forms a minimal resolving set for $\Gamma(\Pi\mathbb{Z}_2^n)$. A complete list of rings with 14 or fewer vertices with given metric dimension can be found in [19] and the zero divisor graphs of such rings can be found in [22].

Unlike Theorem 1 for graphs in general, with the results obtained in this paper and the observations made during the work and by the inspection of the zero divisor graphs of rings, there is a reason to believe that the metric dimension and the upper dimension of zero divisor graph of a ring R is always same, unless $R \cong \Pi\mathbb{Z}_2^n$, $n \geq 4$. We conclude the paper with the following open problem.

Conjecture 30 *Let R be a finite commutative ring with unity $1 \neq 0$, then $\dim^+(\Gamma(R)) = \dim(\Gamma(R))$, unless $R \cong \Pi\mathbb{Z}_2^n$, where $n \geq 4$.*

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