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On ordering of minimal energies in bicyclic signed graphs

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Abstract. Let $S = (G, \sigma)$ be a signed graph of order n and size m and let $x_1, x_2, \ldots, x_n$ be the eigenvalues of S. The energy of S is defined as $\mathcal{E}(S) = \sum_{j=1}^{n} |x_j|$. A connected signed graph is said to be bicyclic if $m = n + 1$. In this paper, we determine the bicyclic signed graphs with first 20 minimal energies for all $n \geq 30$ and with first 16 minimal energies for all $17 \leq n \leq 29$.

1 Introduction

Let $S = (G, \sigma)$ be a signed graph, where $G = (V, E)$ is the underlying graph of S and $\sigma : E \to \{-1, 1\}$ is the signing function (or signature). We represent a positive edge by a plain line and a negative edge by a dotted line. The sign of a signed cycle is defined as the product of signs of its edges. A signed cycle

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is said to be positive (resp., negative) if its sign is positive (resp., negative), that is, it contains an even (resp., odd) number of negative edges. A signed graph is said to be balanced if each of its cycle is positive and unbalanced, otherwise. Throughout, by C_n^+, we denote a positive cycle of order n and by C_n^- a negative cycle of order n. A connected signed graph of order n is said to be unicyclic or bicyclic according as the number of its edges is respectively n or $n+1$.

The adjacency matrix of a signed graph S with vertex set $\{v_1, v_2, \ldots, v_n\}$ is the $n \times n$ matrix $A(S) = (a_{ij})$, where $a_{ij} = \sigma(v_i, v_j)$ if v_i and v_j are adjacent and zero, otherwise. The adjacency matrix $A(S)$ is real symmetric and so has real eigenvalues. Let $\psi(S, x)$ denote the characteristic polynomial of the adjacency matrix of S. The eigenvalues of $A(S)$ are called the eigenvalues of S.

Gutman [7] defined the energy of a graph as the sum of the absolute values of eigenvalues of its adjacency matrix. Germina, Hameed and Zaslavsky [6] extended this concept to signed graphs. The energy of a signed graph S with eigenvalues $x_1, x_2, \ldots, x_n$ is defined as $\mathcal{E}(S) = \sum\limits_{j=1}^{n} |x_j|$. Bhat and Pirzada [2] characterized the unicyclic signed graphs with minimal energy. Bhat et al. [4], characterized the bicyclic signed graphs with minimal and second minimal energy. Similar problems for graphs, digraphs, signed graphs and signed digraphs have been studied in [3, 5, 9, 10, 11, 13, 14, 15, 16, 17].

Let S be a signed graph with vertex set V. Switching S by a set $X \subset V$ means reversing the signs of all the edges between X and its complement. Two signed graphs of the same order are said to be switching equivalent if one can be obtained from the other by a switching. Switching equivalence is an equivalence relation on the signings of a fixed graph. For more details about switching see [4, 15]. An equivalence class is called a switching class. Switching a signed graph does not change the sign of cycles (see [15]), switching equivalent signed graphs have the same set of positive cycles, and they are either both balanced or both unbalanced. Also, switching preserves the spectrum. So, as long as spectra is concerned, we use a single signed graph for a switching class and call that the representative of the switching class.

The rest of the paper is organized as follows. In section 2, we give some definitions and state preliminary results, which will be used to prove our main results. All the main results are in section 3. In that section, we compare energy by using integral formula, Descartes' rule of signs, by cut set deletion and energy change techniques.

2 Preliminaries

In this section, we give some notations, definitions and state some of the results which will be used in the sequel. A basic figure is a signed graph whose components are signed cycles or edges or both.

Theorem 1 [1] *If S is a signed graph with characteristic polynomial*

$$\psi(S,x) = x^n + a_1(S)x^{n-1} + \cdots + a_{n-1}(S)x + a_n(S),$$

then

$$a_k(S) = \sum_{L\in\mathscr{L}_k} (-1)^{p(L)} 2^{|c(L)|} \prod_{X\in c(L)} s(X),$$

for all $k = 1, 2, \ldots, n$, where $\mathscr{L}_k$ is the set of all basic figures L of S of order k, $p(L)$ denotes number of components of L, $c(L)$ denotes the set of all cycles of L and $s(X)$ the sign of cycle X.

The following is the integral formula for the energy of signed graphs.

Theorem 2 [2] *Let S be a signed graph on n vertices with characteristic polynomial $\psi(S,x) = x^n + a_1(S)x^{n-1} + \cdots + a_{n-1}(S)x + a_n(S)$. Then*

$$\mathcal{E}(S) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{x^2} \log\left[\left(\sum_{k=0}^{\lfloor \frac{n}{2}\rfloor}(-1)^k a_{2k}(S)x^{2k}\right)^2 + \left(\sum_{k=0}^{\lfloor \frac{n}{2}\rfloor}(-1)^k a_{2k+1}(S)x^{2k+1}\right)^2\right] dx.$$

In a signed graph S, if the even and odd coefficients respectively alternate in sign, we have two cases to consider.
Case (i). $(-1)^k a_{2k}(S) \geq 0$ and $(-1)^k a_{2k+1}(S) \leq 0$ for $k \geq 0$.
Case (ii). $(-1)^k a_{2k}(S) \geq 0$ and $(-1)^k a_{2k+1}(S) \geq 0$ for $k \geq 0$.
Put $b_k(S) = |a_k(S)|$, then for a signed graph S with even and odd coefficients alternating, above integral formula takes the form

$$\mathcal{E}(S) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{x^2} \log\left[\left(\sum_{k=0}^{\lfloor \frac{n}{2}\rfloor} b_{2k}(S)x^{2k}\right)^2 + \left(\sum_{k=0}^{\lfloor \frac{n}{2}\rfloor} b_{2k+1}(S)x^{2k+1}\right)^2\right] dx.$$

Let S' and S'' be two signed graphs of same order with even and odd coefficients of their respective characteristic polynomials alternating in sign. If $b_{2k}(S') = b_{2k}(S'')$ and $b_{2k+1}(S') = b_{2k+1}(S'')$ for all $k \geq 0$, then it is clear that $E(S') = E(S'')$. In this case, we write $S' \sim S''$. Further, if $b_{2k}(S') \leq b_{2k}(S'')$ and $b_{2k+1}(S') \leq b_{2k+1}(S'')$ for all $k \geq 0$, we write $S' \preceq S''$ or $S'' \succeq S'$. If $b_{2k}(S') \leq b_{2k}(S'')$ and $b_{2k+1}(S') \leq b_{2k+1}(S'')$ for all $k \geq 0$ and for some k_0, strict inequality holds in one of the two inequalities, then we write $S' \prec S''$ or $S'' \succ S'$. Clearly, $\preceq$ is a transitive relation on the coefficients. Thus, if $S' \preceq S''$, we see that $E(S') \leq E(S'')$. Moreover, if $S' \prec S''$, then $E(S') < E(S'')$.

Lemma 3 [4] *Let* $e = uv$ *be an edge of a signed graph* S. *Then*

$$\begin{aligned}\psi(S,x) &= \psi(S-\{e\},x) - \psi(S-\{u,v\},x) \\ &- 2\left(\sum_{Z\in\mathscr{C}^+_{uv}} \psi(S-V(Z),x) - \sum_{Z\in\mathscr{C}^-_{uv}} \psi(S-V(Z),x)\right).\end{aligned}$$

where $\mathscr{C}^+_{uv}$ *and* $\mathscr{C}^-_{uv}$ *respectively denote the set of positive and negative cycles containing the edge* $e = uv$ *and by* $V(Z)$ *we mean the vertex set of* Z.

From this recurrence relation, it is easy to obtain the following lemma.

Lemma 4 *If* S *is a signed graph with even and odd coefficients alternating in sign and if* (u,v) *is the pendent edge of* S *with pendent vertex* v, *then*

$$b_i(S) = b_i(S-v) + b_{i-2}(S-v-u).$$

It is well known that there are three classes of bicyclic signed graphs, which are defined as follows.

(1). For positive integers p and q with $p, q \geq 3$ and $6 \leq p+q \leq n$, we denote by $CC[n,p,q]$, the class of bicyclic signed graphs of order n and having two vertex disjoint cycles of length p and q.

(2). For positive integers p and q with $p, q \geq 3$ and $6 \leq p+q \leq n+1$, we denote by $\infty(n,p,q)$, the class of bicyclic signed graphs of order n with two cycles of lengths p and q such that these cycles have exactly one vertex in common.

(3). For positive integers p, q and r with $(p-r) \geq r$, $(q-r) \geq r$, $p, q \geq 3, r \geq 1$ and $6 \leq p+q \leq n-r+1$, we denote by $\theta(n,p,q,r)$, the class of bicyclic

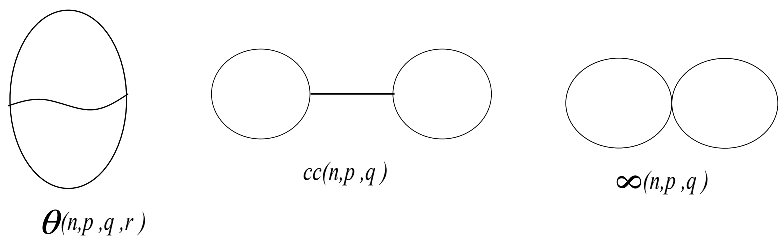

Figure 1: Three classes of bicyclic signed graphs

signed graphs on n vertices with three cycles; one has length p, and the other has length q and two cycles share r edges so that the third cycle has $p+q-2r$ edges. For illustration, see Figure 1, where we have not shown non-cyclic edges which can be present.

Lemma 5 [4] *Let C be a cut set of a signed graph S. Then $\mathcal{E}(S-C) \leq \mathcal{E}(S)$. Moreover, if C is a single edge, then $\mathcal{E}(S-C) < \mathcal{E}(S)$.*

Given a signed star S_n on n vertices, let $S_{n,n}$ denote the collection of unicyclic signed graphs on n vertices such that each element of $S_{n,n}$ is obtained from S_n by adding a single signed edge between any two non adjacent vertices. Then there are two switching classes in $S_{n,n}$, one containing unicyclic signed graphs with positive cycle C_3^+ and other containing unicyclic signed graphs with negative cycle C_3^-. In $S_{n,n}$, if a unicyclic signed graph contains C_3^+, we denote it by $S^1_{n,n}$ and if it contains C_3^-, we denote it by $S^2_{n,n}$. We denote a signed path on n vertices by P_n.

The following result characterizes unicyclic signed graphs with minimal energy [2].

Lemma 6 *Among all unicyclic signed graphs with $n \geq 3$ vertices, $n \neq 4, 5$, each signed graph in $S_{n,n}$ has the minimal energy. Moreover, for $n = 4$, C_4^+ has the minimal energy. Further, for $n = 5$, the signed graph S as shown in Figure 5 has the minimal energy.*

Consider the graph $K_4 - e$ and nonnegative integer $0 \leq k \leq n-4$. Let $G(K_4 - e, n, k)$ be the graph obtained from $K_4 - e$ by respectively identifying the centers of the stars S_{k+1} and S_{n-k-3} to two vertices of degree 3. Let $S^k_{n,n+1}$ denote the collection of bicyclic signed graphs on n vertices obtained from

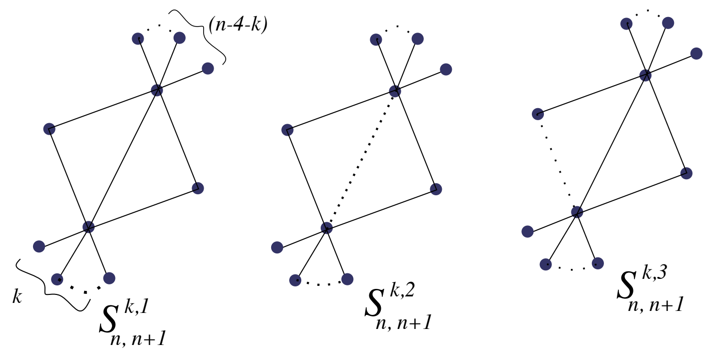

Figure 2: Three switching classes in $S^k_{n,n+1}$

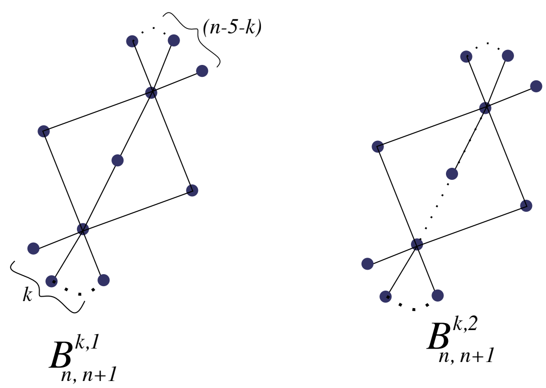

Figure 3: Two switching classes in $B^k_{n,n+1}$

$G(K_4 - e, n, k)$. There are three switching classes in $S^k_{n,n+1}$. We use $S^{k,1}_{n,n+1}$, $S^{k,2}_{n,n+1}$ and $S^{k,3}_{n,n+1}$ as representative of these three switching classes as shown in Figure 2. Note that $S^{k,1}_{n,n+1}$ is balanced, $S^{k,2}_{n,n+1}$ contains two negative cycles of length 3 and a positive cycle of length 4 while as $S^{k,3}_{n,n+1}$ has one positive cycle of length 3, one negative cycle of lengths 3 and one negative cycle of length 4. The following result characterizes bicyclic signed graphs with minimal and second minimal energy[4].

Lemma 7 *Among all bicyclic signed graphs with* $n \geq 12$ *vertices,* $S^{0,1}_{n,n+1}$ *and* $S^{0,2}_{n,n+1}$ *have minimal energy and* $S^{0,3}_{n,n+1}$ *has the second minimal energy.*

3 Main results

Given a complete bipartite graph $K_{2,3}$ and nonnegative integer $0 \leq k \leq n-5$, let $G(K_{2,3}, n, k)$ be the graph obtained by respectively identifying the centers of the stars S_{k+1} and S_{n-k-4} to two vertices of degree 3. Let $B^{k}_{n,n+1}$ denote the collection of bipartite bicyclic signed graphs on n vertices obtained from $G(K_{2,3}, n, k)$. There are two switching classes in $B^{k}_{n,n+1}$. We use $B^{k,1}_{n,n+1}$ and $B^{k,2}_{n,n+1}$ as representative of these two switching classes, for illustration, see Figure 3. $B^{k,1}_{n,n+1}$ contains three positive cycles of length 4 and $B^{k,2}_{n,n+1}$ contains two negative cycles of length 4 and one positive cycle of length 4. With these notations, we have the following observation.

Lemma 8 (i) *For all* $n \geq 6$ *and* $1 \leq k \leq 3$, $\mathcal{E}(B^{k-1,1}_{n,n+1}) < \mathcal{E}(S^{k,1}_{n,n+1}) = \mathcal{E}(S^{k,2}_{n,n+1})$.
(ii) *For all* $n \geq 6$ *and* $k \geq 0$, $\mathcal{E}(S^{k,1}_{n,n+1}) = \mathcal{E}(S^{k,2}_{n,n+1}) < \mathcal{E}(S^{k,3}_{n,n+1})$.
(iii) *For all* $n \geq 6$ *and* $k \geq 1$, $\mathcal{E}(S^{k,3}_{n,n+1}) < \mathcal{E}(B^{k-1,2}_{n,n+1})$.
(iv) *For all* $n > 2k+12$ *and* $k \geq 1$, $\mathcal{E}(B^{k-1,2}_{n,n+1}) < \mathcal{E}(B^{k,1}_{n,n+1})$.
(v) *For all* $n \geq 12$, $\mathcal{E}(S^{0,3}_{n,n+1}) < \mathcal{E}(B^{0,1}_{n,n+1})$.

Proof. (i). By Sach's theorem, we have

$$\psi(B^{k-1,1}_{n,n+1}, x) = x^{n-4}\{x^4 - (n+1)x^2 + [(k+2)(n-k-4) + 3k - 3]\},$$

$$\psi(S^{k,1}_{n,n+1}, x) = x^{n-4}\{x^4 - (n+1)x^2 - 4x + [(k+2)(n-k-4) + 2k]\}$$

and

$$\psi(S^{k,2}_{n,n+1}, x) = x^{n-4}\{x^4 - (n+1)x^2 + 4x + [(k+2)(n-k-4) + 2k]\}.$$

It is clear that $B^{k-1,1}_{n,n+1} \prec S^{k,1}_{n,n+1}, S^{k,2}_{n,n+1}$ for all $1 \leq k \leq 3$ and $S^{k,1}_{n,n+1} \sim S^{k,2}_{n,n+1}$, therefore
$\mathcal{E}(B^{k-1,1}_{n,n+1}) < \mathcal{E}(S^{k,1}_{n,n+1}) = \mathcal{E}(S^{k,2}_{n,n+1})$ for all $n \geq 6$ and $1 \leq k \leq 3$.
(ii). The characteristic polynomial of $S^{k,r}_{n,n+1}$ for $r = 1, 2, 3$ are given by

$$\psi(S^{k,1}_{n,n+1}, x) = x^{n-4}\{x^4 - (n+1)x^2 - 4x + [(k+2)(n-k-4) + 2k]\},$$

$$\psi(S^{k,2}_{n,n+1}, x) = x^{n-4}\{x^4 - (n+1)x^2 + 4x + [(k+2)(n-k-4) + 2k]\}$$

and

$$\psi(S^{k,3}_{n,n+1}, x) = x^{n-4}\{x^4 - (n+1)x^2 + [(k+2)(n-k-4) + 2k + 4]\}.$$

Clearly, $S^{k,1}_{n,n+1} \sim S^{k,2}_{n,n+1}$ and so $\mathcal{E}(S^{k,1}_{n,n+1}) = \mathcal{E}(S^{k,2}_{n,n+1})$. Therefore, to compare the energy of $S^{k,r}_{n,n+1}$ for $r = 1, 2$ and $S^{k,3}_{n,n+1}$, it is enough to compare the energy of $S^{k,1}_{n,n+1}$ and $S^{k,3}_{n,n+1}$. We see that even and odd coefficients of $S^{k,r}_{n,n+1}$ for $r = 1, 2, 3$, alternate in sign but coefficients are not quasi-order comparable for $r = 1$ or 2 and $r = 3$. We will compare energy using Coulson's integral formula by directly solving the integrals. We have $\mathcal{E}(S^{k,3}_{n,n+1}) - \mathcal{E}(S^{k,1}_{n,n+1})$

$$= \frac{1}{\pi}\int_0^\infty \ln \frac{\{1 + (n+1)x^2 + [(k+2)(n-k-4) + 2k + 4]x^4\}^2}{\{1 + (n+1)x^2 + [(k+2)(n-k-4) + 2k]x^4\}^2 + 16x^6} dx.$$

Put

$$\alpha_1(x) = \{1 + (n+1)x^2 + [(k+2)(n-k-4) + 2k + 4]x^4\}^2$$

and

$$\beta_1(x) = \{1 + (n+1)x^2 + [(k+2)(n-k-4) + 2k]x^4\}^2 + 16x^6.$$

Since $n \geq k+4$, we get $\alpha_1(x) - \beta_1(x) = 8x^4 + 8(n-1)x^6 + 8[(k+2)(n-k-4) + 2k + 2]x^8 > 0$ for $n \geq 6$ and $x > 0$. Thus, $\mathcal{E}(S^{k,3}_{n,n+1}) > \mathcal{E}(S^{k,1}_{n,n+1})$.

(iii). The characteristic polynomial of $B^{k-1,2}_{n,n+1}$ and $S^{k,3}_{n,n+1}$ are given by

$$\psi(B^{k-1,2}_{n,n+1}, x) = x^{n-4}\{x^4 - (n+1)x^2 + [(k+2)(n-k-4) + 3k + 5]\}$$

and

$$\psi(S^{k,3}_{n,n+1}, x) = x^{n-4}\{x^4 - (n+1)x^2 + [(k+2)(n-k-4) + 2k + 4]\}.$$

Clearly, $S^{k,3}_{n,n+1} \prec B^{k-1,2}_{n,n+1}$ for all $k \geq 1$ and therefore $\mathcal{E}(S^{k,3}_{n,n+1}) < \mathcal{E}(B^{k-1,2}_{n,n+1})$ for all $n \geq 6$ and $k \geq 1$.

(iv). Again, the characteristic polynomial of $B^{k-1,2}_{n,n+1}$ and $B^{k,1}_{n,n+1}$ are respectively, given by

$$\psi(B^{k-1,2}_{n,n+1}, x) = x^{n-4}\{x^4 - (n+1)x^2 + [(k+2)(n-k-4) + 3k + 5]\}$$

and

$$\psi(B^{k,1}_{n,n+1}, x) = x^{n-4}\{x^4 - (n+1)x^2 + [(k+3)(n-k-5) + 3k]\}.$$

Clearly, $B^{k-1,2}_{n,n+1} \prec B^{k,1}_{n,n+1}$ for all $n > 2k+12$. Therefore, $\mathcal{E}(B^{k-1,2}_{n,n+1}) < \mathcal{E}(B^{k,1}_{n,n+1})$ for all $n > 2k+12$.

(v). The characteristic polynomial of $S^{0,3}_{n,n+1}$ and $B^{0,1}_{n,n+1}$ are given by

$$\psi(S^{0,3}_{n,n+1}, x) = x^{n-4}\{x^4 - (n+1)x^2 + [2(n-4)+4]\}$$

and

$$\psi(B^{0,1}_{n,n+1}, x) = x^{n-4}\{x^4 - (n+1)x^2 + [3(n-5)]\}.$$

Clearly, $S^{0,3}_{n,n+1} \prec B^{0,1}_{n,n+1}$ for all $n \geq 12$ and therefore $\mathcal{E}(S^{0,3}_{n,n+1}) < \mathcal{E}(B^{0,1}_{n,n+1})$ for all $n \geq 12$. □

Let $Q^{r,1}_{n,n+1}$, $r = 1,2,3$ and $H^1_{n,n+1}$ be the graphs as shown in Figure 4. Then it is easy to see that there are two switching classes on the signings of $Q^{1,1}_{n,n+1}$. Let $Q^{1,1}_{n,n+1}$ and $Q^{1,2}_{n,n+1}$ be the representative for these two switching classes, where $Q^{1,1}_{n,n+1}$ contains C_4^+, C_4^+, C_4^+ and $Q^{1,2}_{n,n+1}$ contains C_4^-, C_4^- and C_4^+. There are four switching classes on the signings of $Q^{2,1}_{n,n+1}$. Let $Q^{2,1}_{n,n+1}$, $Q^{2,2}_{n,n+1}$, $Q^{2,3}_{n,n+1}$ and $Q^{2,4}_{n,n+1}$, respectively be the representative for these four switching classes, where $Q^{2,1}_{n,n+1}$ contains C_3^+, C_4^+ and C_5^+; $Q^{2,2}_{n,n+1}$ contains C_3^-, C_4^- and C_5^+; $Q^{2,3}_{n,n+1}$ contains C_3^-, C_4^+, C_5^-; and $Q^{2,4}_{n,n+1}$ contains C_3^+, C_4^- and C_5^-. There are three switching classes on the signings of $Q^{3,1}_{n,n+1}$. Let $Q^{3,1}_{n,n+1}$, $Q^{3,2}_{n,n+1}$ and $Q^{3,3}_{n,n+1}$, respectively be the representative for these three switching classes, where $Q^{3,2}_{n,n+1}$ is the signed graphs obtained from $Q^{3,1}_{n,n+1}$, by making both triangles negative in $Q^{3,1}_{n,n+1}$. Also, $Q^{3,3}_{n,n+1}$ is the signed graph obtained from $Q^{3,1}_{n,n+1}$ by making one triangle negative and other triangle positive in $Q^{3,1}_{n,n+1}$. There are three switching classes on the signings of $H^1_{n,n+1}$. We use $H^r_{n,n+1}$ for $r = 1,2,3$, as the representative for these switching classes. $H^1_{n,n+1}$ is balanced, $H^2_{n,n+1}$ has both triangles negative and $H^3_{n,n+1}$ has one positive triangle and one negative triangle. With these notations, we have the following observation.

Lemma 9 (i) *For all* $n \geq 10$, $\mathcal{E}(B^{1,1}_{n,n+1}) < \mathcal{E}(Q^{1,1}_{n,n+1}) < \mathcal{E}(S^{2,1}_{n,n+1})$.
(ii) *For all* $n \geq 10$, $\mathcal{E}(Q^{1,2}_{n,n+1}) > \mathcal{E}(B^{2,2}_{n,n+1})$.
(iii) *For all* $n \geq 10$, $\mathcal{E}(S^{2,3}_{n,n+1}) < \mathcal{E}(Q^{2,1}_{n,n+1}) = \mathcal{E}(Q^{2,3}_{n,n+1}) < \mathcal{E}(B^{1,2}_{n,n+1})$.
(iv) *For all* $n \geq 10$, $\mathcal{E}(Q^{2,2}_{n,n+1}) = \mathcal{E}(Q^{2,4}_{n,n+1}) > \mathcal{E}(B^{2,2}_{n,n+1})$.
(v) *For all* $n \geq 10$, $\mathcal{E}(B^{1,2}_{n,n+1}) < \mathcal{E}(Q^{3,1}_{n,n+1}) = \mathcal{E}(Q^{3,2}_{n,n+1}) < \mathcal{E}(Q^{3,3}_{n,n+1}) < \mathcal{E}(H^3_{n,n+1})$.

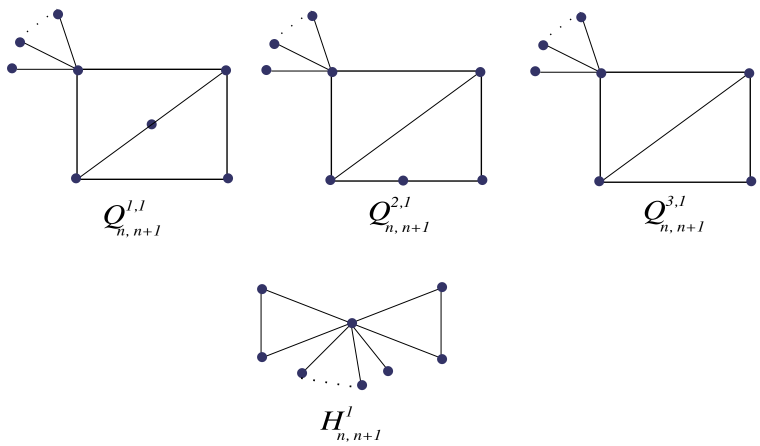

Figure 4: Signed graphs $Q^{r,1}_{n,n+1}$, $r = 1, 2, 3$ and $H^1_{n,n+1}$

(vi) *For all* $n \geq 10$, $\mathcal{E}(H^3_{n,n+1}) < \mathcal{E}(H^1_{n,n+1}) = \mathcal{E}(H^2_{n,n+1})$.
(vii) *For all* $n \geq 30$, $\mathcal{E}(H^1_{n,n+1}) = \mathcal{E}(H^2_{n,n+1}) < \mathcal{E}(B^{2,1}_{n,n+1})$.

Proof. (i). We have

$$\psi(B^{1,1}_{n,n+1}, x) = x^{n-4}\{x^4 - (n+1)x^2 + (4n-21)\},$$

$$\psi(Q^{1,1}_{n,n+1}, x) = x^{n-4}\{x^4 - (n+1)x^2 + (4n-20)\},$$

$$\psi(S^{2,1}_{n,n+1}, x) = x^{n-4}\{x^4 - (n+1)x^2 - 4x + (5n-20)\}.$$

It is easy to see that even and odd coefficients of signed graphs $B^{1,1}_{n,n+1}$, $Q^{1,1}_{n,n+1}$ and $S^{2,1}_{n,n+1}$ alternate in sign. Clearly, $B^{1,1}_{n,n+1} \prec Q^{1,1}_{n,n+1}$ and $Q^{1,1}_{n,n+1} \prec S^{2,1}_{n,n+1}$ for all $n \geq 10$. Therefore, $\mathcal{E}(B^{1,1}_{n,n+1}) < \mathcal{E}(Q^{1,1}_{n,n+1}) < \mathcal{E}(S^{2,1}_{n,n+1})$ for all $n \geq 10$.
(ii). We have

$$\psi(B^{2,2}_{n,n+1}, x) = x^{n-4}\{x^4 - (n+1)x^2 + (5n-21)\},$$

$$\psi(Q^{1,2}_{n,n+1}, x) = x^{n-6}\{x^6 - (n+1)x^4 + (4n-12)x^2 - (4n-20)\}.$$

The signed graphs $B^{2,2}_{n,n+1}$ and $Q^{1,2}_{n,n+1}$ are not quasi-order comparable. Therefore, consider the functions $\alpha_2(x) = x^6-(n+1)x^4+(4n-12)x^2-(4n-20)$ and $\beta_2(x) = x^4-(n+1)x^2+(5n-21)$. It is easy to see that $\beta_2(2) > 0$, $\beta_2(\sqrt{5}) < 0$, $\beta_2(\sqrt{n-4}) < 0$ and $\beta_2(\sqrt{n-3}) > 0$ for all $n \geq 10$. Also, $\alpha_2(\sqrt{2}) = 0$, $\alpha_2(1) < 0$, $\alpha_2(\frac{7071}{5000}) > 0$, $\alpha_2(\sqrt{n-3}) < 0$ and $\alpha_2(\sqrt{n-2}) > 0$ for all $n \geq 10$.

We observe that $\alpha_2(x) = \alpha_2(-x)$ and $\beta_2(x) = \beta_2(-x)$. Therefore $\alpha_2(x)$ has three positive and three negative zeros and $\beta_2(x)$ has two positive and two negative zeros. Recall that the energy of signed graph is twice the sum of its positive eigenvalues. Therefore, we have

$$\mathcal{E}(Q^{1,2}_{n,n+1}) > 2(\sqrt{2}+1+\sqrt{n-3}) > 2(\sqrt{5}+\sqrt{n-3}) > \mathcal{E}(B^{2,2}_{n,n+1})$$

for all $n \geq 10$, which proves part (ii).

(iii). We have

$$\psi(S^{2,3}_{n,n+1}, x) = x^{n-4}\{x^4 - (n+1)x^2 + (4n-16)\},$$

$$\psi(Q^{2,1}_{n,n+1}, x) = x^{n-4}\{x^4 - (n+1)x^2 - 2x + (4n-16)\},$$

$$\psi(Q^{2,3}_{n,n+1}, x) = x^{n-4}\{x^4 - (n+1)x^2 + 2x + (4n-16)\},$$

$$\psi(B^{1,2}_{n,n+1}, x) = x^{n-4}\{x^4 - (n+1)x^2 + (4n-13)\}.$$

Clearly, $S^{2,3}_{n,n+1} \prec Q^{2,1}_{n,n+1}$ and $Q^{2,1}_{n,n+1} \sim Q^{2,3}_{n,n+1}$ for all $n \geq 10$. Therefore, $\mathcal{E}(S^{2,3}_{n,n+1}) < \mathcal{E}(Q^{2,1}_{n,n+1}) = \mathcal{E}(Q^{2,3}_{n,n+1})$ for all $n \geq 10$. Thus, to prove the result, it is enough to show that $\mathcal{E}(Q^{2,1}_{n,n+1}) < \mathcal{E}(B^{1,2}_{n,n+1})$ for all $n \geq 10$. We see that even and odd coefficients of $Q^{2,1}_{n,n+1}$ and $B^{1,2}_{n,n+1}$ alternate in sign but the coefficients are not quasi-order comparable. We will compare the energy using Coulson's integral formula by directly solving the integrals. We have

$$\mathcal{E}(B^{1,2}_{n,n+1}) - \mathcal{E}(Q^{2,1}_{n,n+1}) = \frac{1}{\pi}\int_0^\infty \ln \frac{\{1+(n+1)x^2+(4n-13)x^4\}^2}{\{1+(n+1)x^2+(4n-16)x^4\}^2+4x^6}\,dx.$$

Put $\alpha_3(x) = \{1+(n+1)x^2+(4n-13)x^4\}^2$ and

$$\beta_3(x) = \{1+(n+1)x^2+(4n-16)x^4\}^2+4x^6,$$

we get $\alpha_3(x) - \beta_3(x) = 6x^4 + (6n+2)x^6 + (24n-87)x^8 > 0$ for $n \geq 10$ and $x > 0$. Thus, $\mathcal{E}(B^{1,2}_{n,n+1}) > \mathcal{E}(Q^{2,1}_{n,n+1})$.

(iv). We have

$$\psi(Q^{2,2}_{n,n+1}, x) = x^{n-6}\{x^6 - (n+1)x^4 + 2x^3 + (4n-12)x^2 - 4x - (4n-20)\},$$

$$\psi(Q^{2,4}_{n,n+1}, x) = x^{n-6}\{x^6 - (n+1)x^4 - 2x^3 + (4n-12)x^2 + 4x - (4n-20)\}.$$

Clearly, $Q^{2,2}_{n,n+1} \sim Q^{2,4}_{n,n+1}$ for all $n \geq 10$ and therefore $\mathcal{E}(Q^{2,2}_{n,n+1}) = \mathcal{E}(Q^{2,4}_{n,n+1})$ for all $n \geq 10$. Thus, to prove the result, it is enough to show that $\mathcal{E}(Q^{2,2}_{n,n+1}) > \mathcal{E}(B^{2,2}_{n,n+1})$ for all $n \geq 10$. The signed graphs $B^{2,2}_{n,n+1}$ and $Q^{2,2}_{n,n+1}$ are not quasi-order comparable. Consider the function $\alpha_4(x) = x^6 - (n+1)x^4 + 2x^3 + (4n-12)x^2 - 4x - (4n-20)$. It is easy to see that $\alpha_4(\sqrt{2}) = 0$, $\alpha_4(1) < 0$, $\alpha_4(\frac{7071}{5000}) > 0$, $\alpha_4(\sqrt{n-4}) < 0$ and $\alpha_4(\sqrt{n}) > 0$ for all $n \geq 10$. By Descartes' rule of signs, $\alpha_4(x)$ has three positive and three negative zeros. Therefore,

$$\mathcal{E}(Q^{2,2}_{n,n+1}) > 2(\sqrt{2} + 1 + \sqrt{n-4}) > 2(\sqrt{5} + \sqrt{n-3}) > \mathcal{E}(B^{2,2}_{n,n+1})$$

for all $n \geq 12$. We verified the result directly for $n = 10, 11$. This proves part (iv).

(v). We have

$$\psi(B^{1,2}_{n,n+1}, x) = x^{n-4}\{x^4 - (n+1)x^2 + (4n-13)\},$$

$$\psi(Q^{3,1}_{n,n+1}, x) = x^{n-5}\{x^5 - (n+1)x^3 - 4x^2 + (3n-12)x + 2(n-4)\},$$

$$\psi(Q^{3,2}_{n,n+1}, x) = x^{n-5}\{x^5 - (n+1)x^3 + 4x^2 + (3n-12)x - 2(n-4)\},$$

$$\psi(Q^{3,3}_{n,n+1}, x) = x^{n-5}\{x^5 - (n+1)x^3 + (3n-8)x - 2(n-4)\},$$

$$\psi(H^{3}_{n,n+1}, x) = x^{n-6}\{x^6 - (n+1)x^4 + (2n-5)x^2 - (n-5)\}.$$

First, we will show that $\mathcal{E}(Q^{3,1}_{n,n+1}) < \mathcal{E}(Q^{3,3}_{n,n+1})$. We see that even and odd coefficients of $Q^{3,1}_{n,n+1}$ and $Q^{3,3}_{n,n+1}$ alternate in sign but the coefficients are not quasi-order comparable. We will compare energy using Coulson's integral formula by directly solving the integrals, and we have $\mathcal{E}(Q^{3,3}_{n,n+1}) - \mathcal{E}(Q^{3,1}_{n,n+1})$

$$= \frac{1}{\pi}\int_0^\infty \ln \frac{\{1+(n+1)x^2+(3n-8)x^4\}^2 + \{(2n-8)x^5\}^2}{\{1+(n+1)x^2+(3n-12)x^4\}^2 + \{4x^3+(2n-8)x^5\}^2} dx.$$

Put

$$\alpha_5(x) = \{1+(n+1)x^2+(3n-8)x^4\}^2 + \{(2n-8)x^5\}^2$$

and

$$\beta_5(x) = \{1+(n+1)x^2+(3n-12)x^4\}^2 + \{4x^3+(2n-8)x^5\}^2,$$

we get $\alpha_5(x) - \beta_5(x) = 8x^4 + 8(n-1)x^6 + 8(n-2)x^8 > 0$ for $n \geq 10$ and $x > 0$. Thus, $\mathcal{E}(Q^{3,3}_{n,n+1}) > \mathcal{E}(Q^{3,1}_{n,n+1})$.

Next we will show that, $\mathcal{E}(Q^{3,3}_{n,n+1}) < \mathcal{E}(H^{3}_{n,n+1})$. The signed graphs $Q^{3,3}_{n,n+1}$ and $H^{3}_{n,n+1}$ are not quasi-order comparable. Therefore, consider the functions $\alpha_6(x) = x^5 - (n+1)x^3 + (3n-8)x - 2(n-4)$ and $\beta_6(x) = x^6 - (n+1)x^4 + (2n-5)x^2 - (n-5)$. It is easy to see that $\alpha_6(-2) = 0$, $\alpha_6(-\sqrt{n-3}) > 0$, $\alpha_6(-\sqrt{n-\frac{5}{2}}) < 0$ and $\beta_6(\frac{n-3}{n}) < 0$, $\beta_6(\frac{n-1}{n}) > 0$, $\beta_6(1) = 0$, $\beta_6(\sqrt{n-1}) < 0$ and $\beta_6(\sqrt{n}) > 0$. By Descartes' rule of signs, $\alpha_6(x)$ has three positive and two negative zeros and $\beta_6(x)$ has three positive and three negative zeros. As the energy of signed graph is twice the sum of its positive eigenvalues or -2 times the sum of negative eigenvalues, therefore,

$$\mathcal{E}(H^{3}_{n,n+1}) > 2(\frac{n-3}{n} + 1 + \sqrt{n-1}) > 2(2 + \sqrt{n-\frac{5}{2}}) > \mathcal{E}(Q^{3,3}_{n,n+1})$$

for all $n \geq 14$. We verified the result directly for $n = 10, 11, 13$.
Clearly, $Q^{3,1}_{n,n+1} \sim Q^{3,2}_{n,n+1}$ for all $n \geq 10$ and therefore $\mathcal{E}(Q^{3,1}_{n,n+1}) = \mathcal{E}(Q^{3,2}_{n,n+1})$ for all $n \geq 10$. Thus, to prove the result, it is enough to show that $\mathcal{E}(Q^{3,1}_{n,n+1}) > \mathcal{E}(B^{1,2}_{n,n+1})$ for all $n \geq 10$. The signed graphs $B^{1,2}_{n,n+1}$ and $Q^{3,1}_{n,n+1}$ are not quasi-order comparable, therefore consider the functions $\alpha_7(x) = x^5 - (n+1)x^3 - 4x^2 + (3n-12)x + 2(n-4)$ and $\beta_7(x) = x^4 - (n+1)x^2 + (4n-13)$ and proceeding similarly as above, we can prove that $\mathcal{E}(Q^{3,1}_{n,n+1}) > \mathcal{E}(B^{1,2}_{n,n+1})$ for all $n \geq 10$. This proves part (v).
(vi). We have

$$\psi(H^{1}_{n,n+1}, x) = x^{n-6}\{x^6 - (n+1)x^4 - 4x^3 + (2n-5)x^2 + 4x - (n-5)\},$$

$$\psi(H^{2}_{n,n+1}, x) = x^{n-6}\{x^6 - (n+1)x^4 + 4x^3 + (2n-5)x^2 - 4x - (n-5)\},$$

$$\psi(H^{3}_{n,n+1}, x) = x^{n-6}\{x^6 - (n+1)x^4 + (2n-5)x^2 - (n-5)\}.$$

It is clear that $H^{1}_{n,n+1} \sim H^{2}_{n,n+1}$ and $H^{1}_{n,n+1}, H^{2}_{n,n+1} \succ H^{3}_{n,n+1}$ for all $n \geq 10$. Therefore, $\mathcal{E}(H^{1}_{n,n+1}) = \mathcal{E}(H^{2}_{n,n+1}) > \mathcal{E}(H^{3}_{n,n+1})$ for all $n \geq 10$.
(vii). We have

$$\psi(B^{2,1}_{n,n+1}, x) = x^{n-4}\{x^4 - (n+1)x^2 + (5n-29)\}.$$

The signed graphs $H^{r}_{n,n+1}$ for $r = 1, 2$ and $B^{2,1}_{n,n+1}$ are not quasi-order comparable. Therefore consider the functions $\alpha_8(x) = x^6 - (n+1)x^4 - 4x^3 + (2n-5)x^2 + 4x - (n-5)$ and $\beta_8(x) = x^4 - (n+1)x^2 + (5n-29)$. Again, it is easy to see that $\alpha_8(-\sqrt{n}) > 0$ and $\alpha_8(-\sqrt{n-2}) < 0$ for all $n \geq 12$. Also, -1 is a zero of $\alpha_8(x)$ with multiplicity 2 and $\beta_8(\sqrt{\frac{9}{2}}) > 0$, $\beta_8(\sqrt{5}) < 0$, $\beta_8(\sqrt{n-3}) > 0$ and $\beta_8(\sqrt{n-4}) < 0$ for all $n \geq 22$. By Descartes' rule of signs, $\alpha_8(x)$ has three negative and three positive zeros and $\beta_8(x)$ has two positive and two negative zeros. Therefore,

$$\mathcal{E}(B^{2,1}_{n,n+1}) > 2(\sqrt{\frac{9}{2}} + \sqrt{n-4}) > 2(2+\sqrt{n}) > \mathcal{E}(H^{1}_{n,n+1})$$

for all $n \geq 275$. We can directly verify the result from $n = 30$ to 274. As $\mathcal{E}(H^1_{n,n+1}) = \mathcal{E}(H^2_{n,n+1})$, therefore $\mathcal{E}(H^1_{n,n+1}) = \mathcal{E}(H^2_{n,n+1}) < \mathcal{E}(B^{2,1}_{n,n+1})$ for all ≥ 30. □

Combining Lemmas 8 and 9, we have the following result.

Corollary 10 (i) *For all* $n \geq 30$*, we have*
$\mathcal{E}(S^{0,1}_{n,n+1}) = \mathcal{E}(S^{0,2}_{n,n+1}) < \mathcal{E}(S^{0,3}_{n,n+1}) < \mathcal{E}(B^{0,1}_{n,n+1}) < \mathcal{E}(S^{1,1}_{n,n+1}) = \mathcal{E}(S^{1,2}_{n,n+1}) < \mathcal{E}(S^{1,3}_{n,n+1})$
$< \mathcal{E}(B^{0,2}_{n,n+1}) < \mathcal{E}(B^{1,1}_{n,n+1}) < \mathcal{E}(Q^{1,1}_{n,n+1}) < \mathcal{E}(S^{2,1}_{n,n+1}) = \mathcal{E}(S^{2,2}_{n,n+1}) < \mathcal{E}(S^{2,3}_{n,n+1})$
$< \mathcal{E}(Q^{2,1}_{n,n+1}) = \mathcal{E}(Q^{2,3}_{n,n+1}) < \mathcal{E}(B^{1,2}_{n,n+1}) < \mathcal{E}(Q^{3,1}_{n,n+1}) = \mathcal{E}(Q^{3,2}_{n,n+1}) < \mathcal{E}(Q^{3,3}_{n,n+1})$
$< \mathcal{E}(H^{3}_{n,n+1}) < \mathcal{E}(H^{1}_{n,n+1}) = \mathcal{E}(H^{2}_{n,n+1}) < \mathcal{E}(B^{2,1}_{n,n+1}) < \mathcal{E}(S^{3,1}_{n,n+1}) = \mathcal{E}(S^{3,2}_{n,n+1})$
$< \mathcal{E}(S^{3,3}_{n,n+1}) < \mathcal{E}(B^{2,2}_{n,n+1})$.

(ii) *For all* $17 \leq n \leq 29$*, we have*
$\mathcal{E}(S^{0,1}_{n,n+1}) = \mathcal{E}(S^{0,2}_{n,n+1}) < \mathcal{E}(S^{0,3}_{n,n+1}) < \mathcal{E}(B^{0,1}_{n,n+1}) < \mathcal{E}(S^{1,1}_{n,n+1}) = \mathcal{E}(S^{1,2}_{n,n+1}) < \mathcal{E}(S^{1,3}_{n,n+1})$
$< \mathcal{E}(B^{0,2}_{n,n+1}) < \mathcal{E}(B^{1,1}_{n,n+1}) < \mathcal{E}(Q^{1,1}_{n,n+1}) < \mathcal{E}(S^{2,1}_{n,n+1}) = \mathcal{E}(S^{2,2}_{n,n+1}) < \mathcal{E}(S^{2,3}_{n,n+1}) < \mathcal{E}(Q^{2,1}_{n,n+1})$
$= \mathcal{E}(Q^{2,3}_{n,n+1}) < \mathcal{E}(B^{1,2}_{n,n+1}) < \mathcal{E}(B^{2,1}_{n,n+1}) < \mathcal{E}(S^{3,1}_{n,n+1}) = \mathcal{E}(S^{3,2}_{n,n+1}) < \mathcal{E}(S^{3,3}_{n,n+1}) < \mathcal{E}(B^{2,2}_{n,n+1})$.

The following lemma [4] will be useful in the sequel.

Lemma 11 *Let* S' *and* S'' *be two unicyclic signed graphs of order* $m_1, m_2 \geq 6$ *and let* $n = m_1 + m_2$*. Then, for* $t = 1, 2$,

$$\mathcal{E}(S' \cup S'') \geq \mathcal{E}(S^{t}_{m_1,m_1} \cup S^{t}_{m_2,m_2}) \geq \mathcal{E}(S^{t}_{n-6,n-6} \cup S^{t}_{6,6})$$

with equality if and only if $m_1, m_2 \in \{6, n-6\}$.

Now, we have the following theorem.

Theorem 12 *If* $S \in CC[n, p, q]$*, with* $n \geq 12$ *and* $p, q \geq 3$*, then* $\mathcal{E}(S) > \mathcal{E}(B^{2,2}_{n,n+1})$.

Proof. As $S \in CC[n, p, q]$, with $n \geq 12$ and $p, q \geq 3$, therefore S has a cut-edge say e, such that $S - \{e\}$ is disconnected with two components, which are unicyclic signed graphs, say S' and S''. Let m_1 and m_2 respectively be the number of vertices in S' and S''. Without loss of generality, we assume that $m_1 \geq m_2$. The following cases arise. (i) $m_1, m_2 \geq 6$, (ii) $m_1 \geq 7$ and $m_2 \geq 5$,(iii) $m_1 \geq 8$ and $m_2 \geq 4$,(iv) $m_1 \geq 9$ and $m_2 \geq 3$.
Case (i). $m_1, m_2 \geq 6$. By Lemmas 5, 6 and 11, we have

$$\begin{aligned}\mathcal{E}(S) > \mathcal{E}(S-e) &= \mathcal{E}(S' \cup S'') = \mathcal{E}(S') + \mathcal{E}(S'')\\ &\geq \mathcal{E}(S^t_{m_1,m_1}) + \mathcal{E}(S^t_{m_2,m_2}) = \mathcal{E}(S^t_{m_1,m_1} \cup S^t_{m_2,m_2})\\ &\geq \mathcal{E}(S^t_{n-6,n-6} \cup S^t_{6,6}) = \mathcal{E}(S^1_{n-6,n-6} \cup S^1_{6,6}).\end{aligned}$$

We see that $\mathcal{E}(S^1_{6,6}) > 6$. Consider the functions, $\alpha_9(x) = x^4 - (n-6)x^2 - 2x + (n-9)$ and $\beta_9(x) = x^4 - (n+1)x^2 + (5n-21)$. It is easy to see that $\alpha_9(\frac{1}{2}) > 0$, $\alpha_9(1) < 0$, $\alpha_9(\sqrt{n-7}) < 0$ and $\alpha_9(\sqrt{n-4}) > 0$. Similarly, $\beta_9(2) > 0$, $\beta_9(\sqrt{5}) < 0$, $\beta_9(\sqrt{n-4}) < 0$ and $\beta_9(\sqrt{n-3}) > 0$. By Descartes' rule of signs, both $\alpha_9(x)$ and $\beta_9(x)$ have two positive and two negative zeros. Let the positive zeros of $\alpha_9(x)$ and $\beta_9(x)$ be x_1, x_2 and y_1, y_2, respectively. Therefore, we have $\mathcal{E}(S^1_{n-6,n-6} \cup S^1_{6,6}.) > 2(x_1 + x_2) + 6 > 2(\frac{1}{2} + \sqrt{n-7}) + 6 = 7 + 2\sqrt{n-7} > 2(\sqrt{5} + \sqrt{n-3}) > 2(y_1 + y_2) = \mathcal{E}(B^{2,2}_{n,n+1})$ for all $n \geq 12$. This completes the proof of case(i).
Case (ii). Proceeding similarly as in case (i), we can prove that $\mathcal{E}(S) > \mathcal{E}(S^1_{n-5,n-5}) + \mathcal{E}(S)$, where S is the signed graph shown in Figure 5. Note that $\mathcal{E}(S) > 5.5$. Therefore, we have
$\mathcal{E}(S) > \mathcal{E}(S^1_{n-5,n-5}) + \mathcal{E}(S) > 2(\frac{1}{2} + \sqrt{n-6}) + 5.5 = 6.5 + 2\sqrt{n-6} > 2(\sqrt{5} + \sqrt{n-3}) > \mathcal{E}(B^{2,2}_{n,n+1})$ for all ≥ 12.
Case (iii). Again, for $n = 12$, we proved the result directly. For $n \geq 13$, we have

$$\begin{aligned}\mathcal{E}(S) &> \mathcal{E}(S^1_{n-4,n-4}) + \mathcal{E}(C^+_4) > 2(\frac{1}{2} + \sqrt{n-5}) + 4 = 5 + 2\sqrt{n-5}\\ &> 2(\sqrt{5} + \sqrt{n-3}) > \mathcal{E}(B^{2,2}_{n,n+1}).\end{aligned}$$

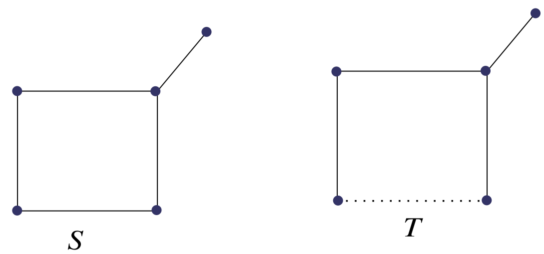

Figure 5: Signed graphs S and T

Case (iv). Finally, we have

$$\mathcal{E}(S) > \mathcal{E}(S^1_{n-3,n-3}) + \mathcal{E}(C_3) > 2(\frac{1}{2} + \sqrt{n-4}) + 4 = 5 + 2\sqrt{n-4}$$
$$> 2(\sqrt{5} + \sqrt{n-3}) > \mathcal{E}(B^{2,2}_{n,n+1}),$$

for all $n \geq 12$. This completes the proof. □

Recall that $\infty(n, p, q)$ is the class of bicyclic signed graphs on n vertices, which have exactly two edge-disjoint cycles sharing a common vertex, which we call the meet vertex. According to the sign and order of cycles in $\infty(n, p, q)$, we divide the class $\infty(n, p, q)$ into three main subclasses.

Subclass 1. According to the sign of C_p(p is even) and C_q(q is odd).This class is further divided into four subclasses:
(1.1) $\infty^{11}(n, p, q) : \sigma(C_p) = \sigma(C_q) = +$; (1.2) $\infty^{12}(n, p, q) : \sigma(C_p) = +$ and $\sigma(C_q) = -$;
(1.3) $\infty^{13}(n, p, q) : \sigma(C_p) = -$ and $\sigma(C_q) = +$; (1.4) $\infty^{14}(n, p, q) : \sigma(C_p) = \sigma(C_q) = -$;

Subclass 2. According to the sign of C_p(p is odd) and C_q(q is odd). Also, this class is further divided into four subclasses:
(2.1) $\infty^{21}(n, p, q) : \sigma(C_p) = \sigma(C_q) = +$; (2.2) $\infty^{22}(n, p, q) : \sigma(C_p) = +$ and $\sigma(C_q) = -$;
(2.3) $\infty^{23}(n, p, q) : \sigma(C_p) = -$ and $\sigma(C_q) = +$; (2.4) $\infty^{24}(n, p, q) : \sigma(C_p) = \sigma(C_q) = -$;

Subclass 3. According to the sign of C_p(p is even) and C_q(q is even). This class is further divided into four subclasses:
(3.1) $\infty^{31}(n, p, q) : \sigma(C_p) = \sigma(C_q) = +$; (3.2) $\infty^{32}(n, p, q) : \sigma(C_p) = +$ and $\sigma(C_q) = -$;

(3.3) $\infty^{33}(n,p,q) : \sigma(C_p) = -$ and $\sigma(C_q) = +$; (3.4) $\infty^{34}(n,p,q) : \sigma(C_p) = \sigma(C_q) = -$;

For an underlying graph G and the corresponding signed graph $S_{ij} = (G, \sigma_{ij}) \in \infty^{ij}(n,p,q)$ $(i = 1,2,3$ and $j = 1,2,3,4)$, we can easily obtain $\lambda_k(A_{\sigma_{1.1}}) = -\lambda_k(A_{\sigma_{1.2}})$, $\lambda_k(A_{\sigma_{1.3}}) = -\lambda_k(A_{\sigma_{1.4}})$, $\lambda_k(A_{\sigma_{2.1}}) = -\lambda_k(A_{\sigma_{2.4}})$ and $\lambda_k(A_{\sigma_{2.2}}) = -\lambda_k(A_{\sigma_{2.3}})$ for $k = 1, 2, \ldots, n$. So $\mathcal{E}(G, \sigma_{1.1}) = \mathcal{E}(G, \sigma_{1.2})$, $\mathcal{E}(G, \sigma_{1.3}) = \mathcal{E}(G, \sigma_{1.4})$, $\mathcal{E}(G, \sigma_{2.1}) = \mathcal{E}(G, \sigma_{2.4})$ and $\mathcal{E}(G, \sigma_{2.2}) = \mathcal{E}(G, \sigma_{2.3})$. Thus we can regard (1.1) and (1.2) as identical, (1.3) and (1.4) as identical, (2.1) and (2.4) as identical and (2.2) and (2.3) as identical. Let $\infty_*(n,p,q)$ denote the collection of signed graphs in $\infty(n,p,q)$ having all $(n-p-q+1)$ pendent vertices adjacent to the meet vertex in $\infty_*(n,p,q)$. Let $\infty_*^{ij}(n,p,q)$ $(i = 1,2,3$ and $j = 1,2,3,4)$ be the corresponding switching class, as shown in Figure 6, in $\infty_*(n,p,q)$, where p and q are not equal to 3 simultaneously, that is according to sign and order of cycles as defined above in subclasses. Also let $\infty_2^*(n,3,3), \infty_2^{**}(n,3,3) \in \infty(n,3,3)$ be signed graphs having both cycles of length 3, $(n-6)$ pendent vertices are adjacent to meet vertex, remaining one pendent vertex is adjacent to any vertex of either cycle other than the meet vertex in $\infty_2^*(n,3,3)$ and $(n-5)$ pendent vertices are adjacent to a single vertex in either of the cycles other than the meet vertex in $\infty_2^{**}(n,3,3)$, respectively. We use $\infty_{2r}^*(n,3,3)$,$\infty_{2r}^{**}(n,3,3)$ $r = 1,2,3,4$ as the representative of these switching classes, as shown in Figure 7, in $\infty_2^*(n,3,3)$ and $\infty_2^{**}(n,3,3)$, respectively corresponding to subclass 2. We know that the necessary condition to use quasi-order method is that the coefficients of the characteristic polynomials of signed graphs must have uniform sign. We next have the following result.

Lemma 13 *(i) If* $S \in \infty^{1j}(n,p,q)$ $(j = 1,3)$*, contains an even cycle* C_p *and an odd cycle* C_q*,* $q = 2t+1$*, then, for all* $i \geq 0$*, we have*
(a) $(-1)^i a_{2i}(S) \geq 0$, (b) $(-1)^i a_{2i+1}(S) \geq 0$ (resp. ≤ 0) *if* t *is odd (resp,even)*
(ii) *If* $S \in \infty^{3j}(n,p,q)$ $(j = 1,2,3,4)$*, containing both even cycles, then for all* $i \geq 0$*, we have*
(a) $(-1)^i a_{2i}(S) \geq 0$, (b) $(-1)^i a_{2i+1}(S) = 0$
(iii) (a) *If* $S \in \infty^{2j}(n,p,q)$ $(j = 1,2,3,4)$*, containing both odd cycles,then for all* $i \geq 0$*, we have* $(-1)^i a_{2i}(S) \geq 0$
(b) *If* $S \in \infty^{2j}(n,p,p)$ $(j = 1,2)$*, containing both odd cycles of equal length* $p = 2t+1$*,then for all* $i \geq 0$*, we have* $(-1)^i a_{2i+1}(S) \geq 0$ (or ≤ 0).

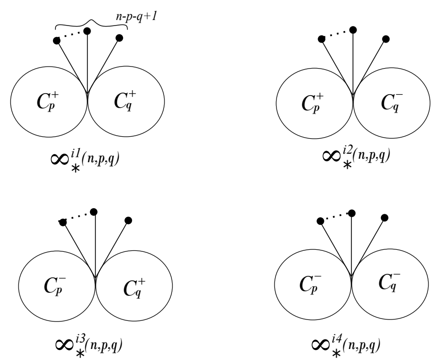

Figure 6: Switching classes corresponding to $\infty_*(n, p, q)$

Proof. (i). If $S \in \infty^{11}(n, p, q)$, then the proof follows from Lemma 1.8 in [8] and if $S \in \infty^{13}(n, p, q)$, then the proof follows from Lemma 4.3 in [12] .
(ii). If $S \in \infty^{3j}(n, p, q)$ $(j = 1, 2, 3, 4)$,then the proof follows from Theorem 2.1 in [3]
(iii). Let $\mathscr{L}_{2i}$, $\mathscr{L}^{(1)}_{2i+1}$ and $\mathscr{L}^{(2)}_{2i+1}$ denote the basic figures of $S \in \infty^{2j}(n, p, q)$ $(j = 1, 2, 3, 4)$ containing only edges, an odd cycle C_p and an odd cycle C_q, respectively. Then

(a). Since $S \in \infty^{2j}(n, p, q)$ $(j = 1, 2, 3, 4)$, therefore the odd cycles share a common vertex in S and hence the basic figure on even vertices does not contain any odd cycle. Therefore, from Theorem 1, we have

$$\begin{aligned}(-1)^i a_{2i}(S) &= (-1)^i \Bigg(\sum_{L \in \mathscr{L}_{2i}} (-1)^{P(L)} 2^{|c(L)|} \prod_{X \in c(L)} \sigma(X) \Bigg) \\ &= (-1)^i \Bigg(\sum_{L \in \mathscr{L}_{2i}} (-1)^i \Bigg) = m(S, i).\end{aligned}$$

Thus, $(-1)^i a_{2i}(S) = m(S, i) \geq 0$ for all i. This proves part (a).
(b). There are two cases to be executed as follows.
Case 1. If $S \in \infty^{21}(n, p, p)$, that is, containing both positive cycles of equal odd lengths $p = 2t + 1$. If $2i + 1 < p = 2t + 1$, then $(-1)^i a_{2i+1}(S) = 0$ and if

$2i+1 \geq p = 2t+1$, then

$$(-1)^i a_{2i+1}(S) = (-1)^i \left(2 \sum_{L \in \mathscr{L}^{(1)}_{2i+1}} (-1)^{\frac{2i+1-(2t+1)}{2}+1} + 2 \sum_{L \in \mathscr{L}^{(2)}_{2i+1}} (-1)^{\frac{2i+1-(2t+1)}{2}+1} \right)$$
$$= \left(2 \sum_{L \in \mathscr{L}^{(1)}_{2i+1}} (-1)^{-t+1} + 2 \sum_{L \in \mathscr{L}^{(2)}_{2i+1}} (-1)^{-t+1} \right)$$

Thus $(-1)^i a_{2i+1}(S) \geq 0$ if $t = 2k+1$, and $(-1)^i a_{2i+1}(S) \leq 0$ if $t = 2k$ for all i.
Case 2. $S \in \infty^{22}(n, p, p)$, that is, containing one positive cycle and one negative cycle of equal odd lengths $p = 2t+1$, respectively. If $2i+1 < p = 2t+1$, then $(-1)^i a_{2i+1}(S) = 0$ and if $2i+1 \geq p = 2t+1$, then

$$(-1)^i a_{2i+1}(S) = (-1)^i \left(2 \sum_{L \in \mathscr{L}^{(1)}_{2i+1}} (-1)^{\frac{2i+1-(2t+1)}{2}+1} - 2 \sum_{L \in \mathscr{L}^{(2)}_{2i+1}} (-1)^{\frac{2i+1-(2t+1)}{2}+1} \right)$$
$$= \left(2 \sum_{L \in \mathscr{L}^{(1)}_{2i+1}} (-1)^{-t+1} - 2 \sum_{L \in \mathscr{L}^{(2)}_{2i+1}} (-1)^{-t+1} \right).$$

Thus, $(-1)^i a_{2i+1}(S) \geq 0$ if $t = 2k+1$ and $|\mathscr{L}^{(1)}_{2i+1}| \geq |\mathscr{L}^{(2)}_{2i+1}|$; $(-1)^i a_{2i+1}(S) \leq 0$ if $t = 2k+1$ and $|\mathscr{L}^{(1)}_{2i+1}| \leq |\mathscr{L}^{(2)}_{2i+1}|$; $(-1)^i a_{2i+1}(S) \geq 0$ if $t = 2k$ and $|\mathscr{L}^{(1)}_{2i+1}| \leq |\mathscr{L}^{(2)}_{2i+1}|$; and $(-1)^i a_{2i+1}(S) \leq 0$ if $t = 2k$ and $|\mathscr{L}^{(1)}_{2i+1}| \geq |\mathscr{L}^{(2)}_{2i+1}|$ for all i, where $|Z|$ denotes the cardinality of a set Z. This completes the proof. □

The following two lemmas can be easily established.

Lemma 14 *For positive integers* $m_1, m_2 \geq 5$, $m_1 + m_2 = n \geq 17$ *and* $t = 1, 2$,

$$\mathcal{E}(S_{m_1} \cup S^t_{m_2,m_2}) \geq \mathcal{E}(S_5 \cup S^t_{n-5,n-5}),$$

with equality if and only if $m_1, m_2 \in \{5, n-5\}$.

Lemma 15 (i) $\mathcal{E}(S_{n-4} \cup S^1_{4,4}) > \mathcal{E}(B^{2,2}_{n,n+1})$ *for all* $n \geq 12$.
(ii) $\mathcal{E}(S_{n-4} \cup C^-_4) > \mathcal{E}(B^{2,2}_{n,n+1})$ *for all* $n \geq 12$.
(iii) $\mathcal{E}(S_5 \cup S^1_{n-5,n-5}) > \mathcal{E}(B^{2,2}_{n,n+1})$ *for all* $n \geq 12$.
(iv) $\mathcal{E}(S_{n-5} \cup S) > \mathcal{E}(B^{2,2}_{n,n+1})$ *for all* $n \geq 12$, *where* S *is the signed graph shown in Figure* 5.
(v) $\mathcal{E}(S_{n-5} \cup T) > \mathcal{E}(B^{2,2}_{n,n+1})$ *for all* $n \geq 12$, *where* T *is the signed graph shown in Figure* 5.

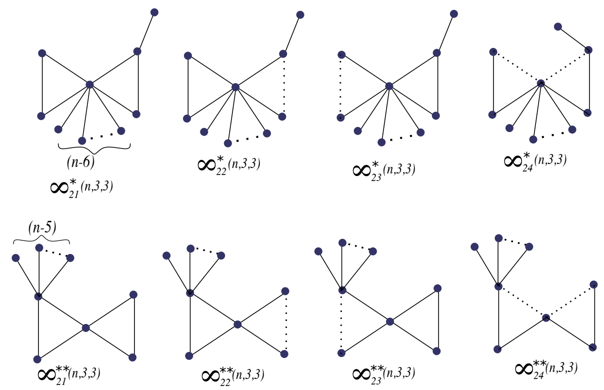

Figure 7: Switching classes $\infty^*_{2r}(n,3,3)$ and $\infty^{**}_{2r}(n,3,3)$, $r = 1,2,3,4$

Now, as the proof of the following result is similar as in Lemma 9. So we skip the proof here.

Lemma 16 (i) *For all* $n \geq 12$, $\mathcal{E}[\infty^*_{21}(n,3,3)] > \mathcal{E}[\infty^*_{22}(n,3,3)] > \mathcal{E}(B^{2,2}_{n,n+1})$.

(ii) *For all* $n \geq 12$, $\mathcal{E}[\infty^{**}_{21}(n,3,3)] > \mathcal{E}[\infty^{**}_{22}(n,3,3)] > \mathcal{E}(B^{2,2}_{n,n+1})$.

(iii) *For all* $n \geq 12$, $\mathcal{E}[\infty^{13}_*(n,4,3)] > \mathcal{E}[\infty^{11}_*(n,4,3)] > \mathcal{E}(B^{2,2}_{n,n+1})$.

(iv) *For all* $n \geq 12$, $\mathcal{E}[\infty^{34}_*(n,4,4)] > \mathcal{E}[\infty^{32}_*(n,4,4)] > \mathcal{E}[\infty^{31}_*(n,4,4)] > \mathcal{E}(B^{2,2}_{n,n+1})$.

(v) *For all* $n \geq 12$, $\mathcal{E}[\infty^{22}_*(n,5,3)] > \mathcal{E}[\infty^{21}_*(n,5,3)] > \mathcal{E}[\infty^*_{22}(n,3,3)]$.

Now, we proceed to prove the following theorem.

Theorem 17 *If* $S \in \infty(n,p,q)$, $n \geq 17$, $p,q \geq 3$ *and* $S \neq H^r_{n,n+1}$ $(r = 1,2,3)$, *then* $\mathcal{E}(S) > \mathcal{E}(B^{2,2}_{n,n+1})$.

Proof. Let v be the meet vertex of the two cycles C_p and C_q. The following cases arise.
Case 1. Let $S \in \infty(n,3,3)$. We have the following claim.
Claim. If $S \in \infty^{2r}(n,3,3)$ $(r = 1,2)$ and $S \neq H^s_{n,n+1}(s = 1,3)$, $\infty^{**}_{2t}(n,3,3)(t = 1,2)$, $\infty^*_{2r}(n,3,3)(r = 1,2)$, then for all $n \geq 7$, $S \succ \infty^*_{22}(n,3,3)$.

We shall prove the claim by induction on n. Assume that the claim holds for smaller values of n. Let v_1 be a pendent vertex which is adjacent to the meet vertex v in $\infty^*_{22}(n,3,3)$. For $n \geq 7$, $S \in \infty^{2r}(n,3,3)$, $r = 1,2$, has at least one pendent vertex say v_2 (such that $S - v_2$ is again different from signed graphs forbidden in this claim), which is adjacent to u (say) in S. Then from Lemmas 4 and 13, we obtain

$$b_i(S) = b_i(S - v_2) + b_{i-2}(S - v_2 - u)$$

and

$$b_i(\infty^*_{22}(n,3,3)) = b_i(\infty^*_{22}(n,3,3) - v_1) + b_{i-2}(P_3 \cup P_2).$$

By induction assumption, $S - v_2 \succ \infty^*_{22}(n,3,3) - v_1$.
Since $S \neq H^s_{n,n+1}(s = 1,3), \infty^{**}_{2t}(n,3,3)(t = 1,2)$, $\infty^*_{2r}(n,3,3)$ $(r = 1,2)$, therefore $S - v_2 - u$ has $P_3 \cup P_2$ as a subgraph and hence we have $S - v_2 - u \succeq P_3 \cup P_2$. This proves the claim.
By Lemma 13, the even and odd coefficients of S alternate in sign. Thus, by the above claim, for $S \neq H^s_{n,n+1}(s = 1,3), \infty^{**}_{2t}(n,3,3)(t = 1,2)$, $\infty^*_{2r}(n,3,3)$ $(r = 1,2)$, we have $\mathcal{E}(S) > \mathcal{E}[\infty^*_{22}(n,3,3)]$. Also for the same underlined graph , the energy of signed graphs $S \in \infty^{21}(n,3,3)$ and $T \in \infty^{24}(n,3,3)$ is same, $S \in \infty^{22}(n,3,3)$ and $T \in \infty^{23}(n,3,3)$ is same and therefore the result follows by Lemma 16 in this case.
Case 2. Let $S \in \infty(n,4,3)$. We have the following claim.
Claim. If $S \in \infty^{1r}(n,4,3)$ $(r = 1,3)$ and $S \neq \infty^{1r}_*(n,4,3)$ $(r = 1,3)$, then for all $n \geq 7$, $S \succ \infty^{11}_*(n,4,3)$.
We shall prove the claim by induction on n. Assume that the claim holds for smaller values of n. Let v_1 be a pendent vertex which is adjacent to the meet vertex v in $\infty^{11}_*(n,4,3)$. For $n \geq 7$, $S \in \infty^{1r}(n,4,3)$, $r = 1,2$, has at least one pendent vertex say v_2, which is adjacent to u (say) in S. Then from Lemmas 4 and 13, we obtain

$$b_i(S) = b_i(S - v_2) + b_{i-2}(S - v_2 - u).$$

and

$$b_i(\infty^{11}_*(n,4,3)) = b_i(\infty^{11}_*(n,4,3) - v_1) + b_{i-2}(P_3 \cup P_2).$$

By induction assumption, $S - v_2 \succ \infty^{11}_*(n,4,3) - v_1$. Since $S \neq \infty^{1r}_*(n,4,3)$ $(r = 1,3)$ and therefore $S - v_2 - u$ has $P_3 \cup P_2$ as a subgraph and thus $S - v_2 - u \succeq$

$P_3 \cup P_2$. This proves the claim.
By Lemma 13, the even and odd coefficients of S alternate in sign. Thus, by the above claim, we have $\mathcal{E}(S) > \mathcal{E}[\infty_*^{11}(n,3,3)]$. Also, for the same underlined graph , the energy of signed graphs $S \in \infty^{11}(n,4,3)$ and $T \in \infty^{12}(n,4,3)$ is same, $S \in \infty^{13}(n,4,3)$ and $T \in \infty^{14}(n,4,3)$ is same and therefore the result follows by Lemma 16 in this case.
Case 3. If $S \in \infty(n,4,4)$ and $S \neq \infty_*^{3r}(n,4,4)(r = 1,2,4)$, then proceeding similarly as in case 2, one can easily prove that, $\mathcal{E}(S) > \mathcal{E}[\infty_*^{31}(n,4,4)]$ and hence the result follows by Lemma 16.
Case 4. If $S \in \infty(n,5,3), \infty(n,5,4), \infty(n,5,5)$ and $S \neq \infty_*^{2r}(n,5,3), r = 1,2,3,4$ then it easy to see that $b_4(S) > b_4(B^{2,2}_{n,n+1}) = 5n - 21$. Since by Lemma 13 even coefficients S alternate in sign and therefore $\mathcal{E}(S) > \mathcal{E}(B^{2,2}_{n,n+1})$, and so the result follows in this case. If $S = \infty_*^{2r}(n,5,3), r = 1,2,3,4$, then the result follows by Lemma 16, since for the same underlined graph, the energy of signed graphs $S \in \infty^{21}(n,5,3)$ and $T \in \infty^{24}(n,5,3)$ is same, $S \in \infty^{22}(n,5,3)$ and $T \in \infty^{23}(n,5,3)$ is same.
Case 5. Let $S \in \infty(n,p,q)$ and at least one of p and q is greater or equal to 6. Without loss of generality, we assume C_p is such a cycle. Then the following four subcases arise.
Subcase 5.1. Let $C_q \neq C_3^+, C_3^-, C_4^+, C_4^-$, if there be at most a single noncyclic signed edge incident to any vertex of C_3^+ or C_3^- and no noncyclic signed edge incident to the vertices of C_4^+, C_4^-. Then choose the cut set $Z = \{e_1, e_2\}$, where e_1 and e_2 are the edges on the cycle C_p adjacent to v. Then $S - Z$ has two components, say S' and S'', where S' is a signed tree on $m_1 \geq 5$ vertices and S'' is unicyclic signed graph with $m_2 \geq 5$ vertices such that $m_1 + m_2 = n \geq 17$. Then the result follows by Lemmas 14 and 15.
Subcase 5.2. If $C_q = C_4^-, C_3^+, C_3^-$ such that there is exactly single noncyclic signed edge incident to any vertex of C_3^+, C_3^- and there is no noncyclic signed edge incident to any vertex of C_4^-, then choose the cut set $Z = \{e_1, e_2\}$, where e_1 and e_2 are the edges on the cycle C_p adjacent to v. Then $S - Z$ has two components, say S' and S'', where S' is a signed tree on $n-4$ vertices and S'' is unicyclic signed graph with 4 vertices such that $m_1 + m_2 = n \geq 17$. Since S'' is either $S^t_{4,4}, t = 1,2$ or C_4^- and $\mathcal{E}(S^1_{4,4}) = \mathcal{E}(S^2_{4,4})$, then the result follows by Lemma 15.
Subcase 5.3. Let $C_q = C_4^+$ and there be no noncyclic signed edge incident to

any vertex of C_4^+. Let $\{e_1, e_2, \ldots, e_{p-1}, e_p\}$ be the edges of cycle C_p. Without loss of generality, suppose that the edges e_1 and e_p are incident to meet vertex v such that the edges e_r and e_s are adjacent in C_p if $|r-s| = 1$ and e_1 , e_p are incident to meet vertex v. Then choose the cut set either $\{e_1, e_{p-1}\}$ or $\{e_2, e_p\}$ such that $S - \{e_1, e_{p-1}\}$ or $S - \{e_2, e_p\}$ has two components, say S' and S'', where S' is a signed tree on $m_1 \geq 5$ vertices and S'' is unicyclic signed graph with $m_2 \geq 5$ vertices such that $m_1 + m_2 = n \geq 17$. Then the result follows by Lemmas 14 and 15.

Subcase 5.4. Let $C_q = C_3^+, C_3^-$ and there is no noncyclic signed edge incident to any vertex of C_3^+, C_3^-. Then there exists a cut set Z consisting of the two edges of C_p such that $S - Z$ has two components, say S' and S'', where S' is a signed tree on $m_1 \geq 5$ vertices and S'' is unicyclic signed graph with $m_2 \geq 5$ vertices such that $m_1 + m_2 = n \geq 17$. Then the result follows by Lemmas 14 and 15. This completes the proof. □

Let L_q^- and L_q^+ respectively denote the number of negative and positive 4-cycles in a signed graph. Then, by Theorem 1, we have

$$b_4(S) = m(S, 2) - 2(L_q^+ - L_q^-). \tag{1}$$

Where $m(S, k)$ denote the number of matchings of size k. Let $\mathscr{L}_k$ denote the set of basic figures of order k of a signed graph S and let $\mathscr{L}_k^1$ denote the set of basic figures which do not contain any cycle and $\mathscr{L}_k^2 = \mathscr{L}_k - \mathscr{L}_k^1$. It is clear that for a signed graph $S \in \theta(n, p, q, r)$, $\left|\mathscr{L}_{2k}^1\right| \geq 2\left|\mathscr{L}_{2k}^2\right|$. From this, we can easily see that if $S \in \theta(n, p, q, r)$, then $b_{2k}(S) \geq 0$ for all $k \geq 0$. Also, if $b_4(S) > 5n - 21$, then it is easy to see that $\mathcal{E}(S) > \mathcal{E}(B_{n,n+1}^{2,2})$. Let $Q_k^{1,1}$, $k = 1, 2, 3, ..., 52$ be the graphs as shown in Figure 8. Then it is easy to see that there are three switching classes on the signings of $Q_k^{1,1}$ $k = 1, 2, 3, \ldots, 52$. Let $Q_k^{1,1}$, $Q_k^{1,2}$ and $Q_k^{1,3}$ $k = 1, 2, 3, \ldots, 52$, respectively be the representative for these three switching classes, where $Q_k^{1,2}$ are the signed graphs obtained from $Q_k^{1,1}$, by making both triangles negative in $Q_k^{1,1}$ and $Q_k^{1,3}$ be the signed graphs obtained from $Q_k^{1,1}$ by making one triangle negative(left sided triangle) and other triangle positive(right sided triangle) in $Q_k^{1,1}$. It is easy to see that $Q_k^{1,2}$ is switching equivalent to $-Q_k^{1,1}$ and therefore $\mathcal{E}(Q_k^{1,1}) = \mathcal{E}(Q_k^{1,2})$ for all $k = 1, 2, 3, \ldots, 52$. Thus, we can regard $Q_k^{1,1}$ and $Q_k^{1,2}$ as identical. Also, $b_4(Q_k^{1,3}) > 5n - 21$ for all $k = 15, 16, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 31, 32, \ldots, 52$ and therefore we omit these signed graphs here, as these signed graphs will be considered later (Theorem 22). With these notations, we have the following observation.

Lemma 18 *For all* $n \geq 10$*, we have*
(i) $\mathcal{E}(B^{2,2}_{n,n+1}) < \mathcal{E}(Q^{1,1}_1) < \mathcal{E}(Q^{1,1}_k)$ *for all* $k = 2, 3, \ldots, 52$.
(ii) $\mathcal{E}(B^{2,2}_{n,n+1}) < \mathcal{E}(Q^{1,3}_1) < \mathcal{E}(Q^{1,3}_k)$ *for all* $k = 3, 4, 6, 8, 13, 14, 17, 39, 40, \ldots, 44$.
(iii) $\mathcal{E}(B^{2,2}_{n,n+1}) < \mathcal{E}(Q^{1,3}_2) < \mathcal{E}(Q^{1,3}_k)$ *for all* $k = 5, 9, 10, 11, 12, 18, 30$.
(iv) $\mathcal{E}(B^{2,2}_{n,n+1}) < \mathcal{E}(Q^{1,3}_7)$.

Proof. (i). We have

$$\psi(Q^{1,1}_1, x) = x^{n-6}\{x^6 - (n+1)x^4 - 4x^3 + (3n-12)x^2 + 2x - (n-5)\},$$

$$\psi(Q^{1,1}_2, x) = x^{n-6}\{x^6 - (n+1)x^4 - 4x^3 + (3n-11)x^2 + 4x - 2(n-6)\},$$

$$\psi(Q^{1,1}_3, x) = x^{n-6}\{x^6 - (n+1)x^4 - 4x^3 + (4n-18)x^2 + 4x - 2(n-6)\},$$

$$\psi(Q^{1,1}_4, x) = x^{n-6}\{x^6 - (n+1)x^4 - 4x^3 + (4n-18)x^2 + 2x - (2n-11)\},$$

$$\psi(Q^{1,1}_5, x) = x^{n-6}\{x^6 - (n+1)x^4 - 4x^3 + (4n-17)x^2 + 4x - (3n-17)\},$$

$$\psi(Q^{1,1}_6, x) = x^{n-8}\{x^8 - (n+1)x^6 - 4x^5 + (4n-16)x^4 + 6x^3 - (4n-24)x^2 - 2x + (n-7)\},$$

$$\psi(Q^{1,1}_7, x) = x^{n-6}\{x^6 - (n+1)x^4 - 4x^3 + (4n-17)x^2 + 4x - (2n-14)\},$$

$$\psi(Q^{1,1}_8, x) = x^{n-6}\{x^6 - (n+1)x^4 - 4x^3 + (4n-16)x^2 + 6x - 3(n-6)\},$$

$$\psi(Q^{1,1}_9, x) = x^{n-6}\{x^6 - (n+1)x^4 - 4x^3 + (4n-21)x^2 + 4x - (3n-22)\},$$

$$\psi(Q^{1,1}_{10}, x) = x^{n-6}\{x^6 - (n+1)x^4 - 4x^3 + (4n-19)x^2 + 8x - (4n-30)\},$$

$$\psi(Q^{1,1}_{11}, x) = x^{n-8}\{x^8 - (n+1)x^6 - 4x^5 + (4n-15)x^4 + 8x^3 - (5n-31)x^2 - 4x + 2(n-8)\},$$

$$\psi(Q^{1,1}_{12}, x) = x^{n-6}\{x^6 - (n+1)x^4 - 4x^3 + (4n-16)x^2 + 8x - 4(n-7)\},$$

$$\psi(Q^{1,1}_{13}, x) = x^{n-6}\{x^6 - (n+1)x^4 - 4x^3 + (5n-26)x^2 + 6x - 3(n-7)\},$$

$$\psi(Q^{1,1}_{14}, x) = x^{n-6}\{x^6 - (n+1)x^4 - 4x^3 + (5n-25)x^2 + 4x - (4n-26)\},$$

$$\psi(Q^{1,1}_{15}, x) = x^{n-6}\{x^6 - (n+1)x^4 - 4x^3 + (5n-24)x^2 + 6x - (5n-33)\},$$

$$\psi(Q^{1,1}_{16}, x) = x^{n-8}\{x^8 - (n+1)x^6 - 4x^5 + (5n-23)x^4 + 8x^3 - (6n-40)x^2 - 4x + 2(n-8)\},$$

$$\psi(Q^{1,1}_{17}, x) = x^{n-6}\{x^6 - (n+1)x^4 - 4x^3 + (5n-26)x^2 + 2x - (3n-19)\},$$

$$\psi(Q^{1,1}_{18}, x) = x^{n-6}\{x^6 - (n+1)x^4 - 4x^3 + (5n-25)x^2 + 4x - (4n-28)\},$$

$$\psi(Q_{19}^{1,1},x) = x^{n-8}\{x^8-(n+1)x^6-4x^5+(5n-24)x^4+4x^3-4(n-7)x^2+(n-7)\},$$

$$\psi(Q_{20}^{1,1},x)=x^{n-8}\{x^8-(n+1)x^6-4x^5+(5n-23)x^4+6x^3-(6n-39)x^2-2x+(2n-15)\},$$

$$\psi(Q_{21}^{1,1},x)=x^{n-8}\{x^8-(n+1)x^6-4x^5+(5n-22)x^4+8x^3-(7n-45)x^2-4x+(3n-23)\},$$

$$\begin{aligned}\psi(Q_{22}^{1,1},x) = x^{n-10}\{&x^{10}-(n+1)x^8-4x^7+(5n-21)x^6+10x^5-(8n-52)x^4\\ &-8x^3+(5n-40)x^2+2x-(n-9)\},\end{aligned}$$

$$\psi(Q_{23}^{1,1},x)=x^{n-8}\{x^8-(n+1)x^6-4x^5+(5n-22)x^4+8x^3-(7n-47)x^2-4x+(3n-25)\},$$

$$\psi(Q_{24}^{1,1},x)=x^{n-8}\{x^8-(n+1)x^6-4x^5+(5n-21)x^4+10x^3-(6n-39)x^2-2x+(n-7)\},$$

$$\psi(Q_{25}^{1,1},x)=x^{n-8}\{x^8-(n+1)x^6-4x^5+(5n-23)x^4+8x^3-(5n-32)x^2-2x+(n-7)\},$$

$$\psi(Q_{26}^{1,1},x) = x^{n-6}\{x^6-(n+1)x^4-4x^3+(5n-23)x^2+6x-(5n-32)\},$$

$$\psi(Q_{27}^{1,1},x)=x^{n-8}\{x^8-(n+1)x^6-4x^5+(5n-22)x^4+8x^3-(5n-28)x^2-2x+(n-7)\},$$

$$\psi(Q_{28}^{1,1},x)=x^{n-8}\{x^8-(n+1)x^6-4x^5+(5n-21)x^4+10x^3-(7n-43)x^2-6x+3(n-8)\},$$

$$\psi(Q_{29}^{1,1},x) = x^{n-6}\{x^6-(n+1)x^4-4x^3+(5n-24)x^2+8x-(5n-33)\},$$

$$\psi(Q_{30}^{1,1},x) = x^{n-6}\{x^6-(n+1)x^4-4x^3+(5n-25)x^2+4x-(3n-19)\},$$

$$\psi(Q_{31}^{1,1},x)=x^{n-8}\{x^8-(n+1)x^6-4x^5+(5n-23)x^4+6x^3-(5n-31)x^2-2x+(n-7)\},$$

$$\psi(Q_{32}^{1,1},x)=x^{n-8}\{x^8-(n+1)x^6-4x^5+(5n-22)x^4+8x^3-(6n-39)x^2-4x+2(n-8)\},$$

$$\psi(Q_{33}^{1,1},x) = x^{n-6}\{x^6-(n+1)x^4-4x^3+(5n-22)x^2+10x-5(n-7)\},$$

$$\psi(Q_{34}^{1,1},x) = x^{n-6}\{x^6-(n+1)x^4-4x^3+(5n-23)x^2+8x-4(n-7)\},$$

$$\psi(Q_{35}^{1,1},x)=x^{n-8}\{x^8-(n+1)x^6-4x^5+(5n-21)x^4+10x^3-(7n-46)x^2-4x+(2n-15)\},$$

$$\psi(Q_{36}^{1,1},x) = x^{n-6}\{x^6-(n+1)x^4-4x^3+(5n-22)x^2+8x-(6n-41)\},$$

$$\psi(Q_{37}^{1,1},x) = x^{n-6}\{x^6-(n+1)x^4-4x^3+(5n-21)x^2+12x-(6n-44)\},$$

$$\psi(Q_{38}^{1,1},x) = x^{n-6}\{x^6-(n+1)x^4-4x^3+(5n-23)x^2+12x-6(n-8)\},$$

$$\psi(Q_{39}^{1,1},x) = x^{n-6}\{x^6-(n+1)x^4-4x^3+(4n-18)x^2+2(n-5)x-(n-5)\},$$

$$\psi(Q_{40}^{1,1},x) = x^{n-6}\{x^6-(n+1)x^4-4x^3+(4n-17)x^2+(2n-8)x-(n-5)\},$$

$$\psi(Q_{41}^{1,1},x)=x^{n-7}\{x^7-(n+1)x^5-4x^4+(4n-16)x^3+(2n-6)x^2-3(n-6)x-2(n-6)\},$$

$$\psi(Q_{42}^{1,1},x)=x^{n-6}\{x^6-(n+1)x^4-4x^3+(5n-26)x^2+2(n-6)x-2(n-6)\},$$

$$\psi(Q_{43}^{1,1},x) = x^{n-6}\{x^6-(n+1)x^4-4x^3+(5n-26)x^2+(2n-10)x-3(n-6)\},$$

$$\psi(Q_{44}^{1,1},x) = x^{n-6}\{x^6-(n+1)x^4-4x^3+(5n-25)x^2+2(n-6)x-3(n-6)\},$$

$$\psi(Q_{45}^{1,1},x) = x^{n-8}\{x^8-(n+1)x^6-4x^5+(5n-23)x^4+(2n-8)x^3 - (5n-32)x^2-2(n-7)x+(n-7)\},$$

$$\psi(Q_{46}^{1,1},x) = x^{n-6}\{x^6-(n+1)x^4-4x^3+(5n-24)x^2+(2n-8)x-2(n-6)\},$$

$$\psi(Q_{47}^{1,1},x) = x^{n-8}\{x^8-(n+1)x^6-4x^5+(5n-22)x^4+(2n-6)x^3 - (5n-31)x^2-(2n-12)x+(n-7)\},$$

$$\psi(Q_{48}^{1,1},x) = x^{n-7}\{x^7-(n+1)x^5-4x^4+(5n-21)x^3+(2n-4)x^2 - (6n-39)x-(4n-26)\},$$

$$\psi(Q_{49}^{1,1},x)=x^{n-7}\{x^7-(n+1)x^5-4x^4+(5n-23)x^3+(2n-8)x^2-4(n-6)x-2(n-6)\},$$

$$\psi(Q_{50}^{1,1},x)=x^{n-7}\{x^7-(n+1)x^5-4x^4+(5n-22)x^3+(2n-6)x^2-4(n-6)x-2(n-6)\},$$

$$\psi(Q_{51}^{1,1},x) = x^{n-9}\{x^9-(n+1)x^7-4x^6+(5n-21)x^5+(2n-4)x^4 - (7n-45)x^3-(4n-24)x^2+3(n-8)+2(n-8)\},$$

$$\psi(Q_{52}^{1,1},x)=x^{n-7}\{x^7-(n+1)x^5-4x^4+(5n-22)x^3+(2n-4)x^2-6(n-7)x-4(n-7)\}.$$

First we will show that $\mathcal{E}(B_{n,n+1}^{2,2}) < \mathcal{E}(Q_1^{1,1})$. Now, consider the function

$$\alpha_{10}(x) = x^6-(n+1)x^4-4x^3+(3n-12)x^2+2x-(n-5)$$

and proceeding similarly as in part (ii), Lemma 9, we can prove that $\mathcal{E}(Q_1^{1,1}) > E(B_{n,n+1}^{2,2})$ for all $n \geq 10$. Also, it is easy to see that even and odd coefficients of $Q_k^{1,1}$, $k = 1,2,3,\ldots,52$, alternate in sign and clearly $Q_1^{1,1} \prec Q_k^{1,1}$ for all $k = 2,3,\ldots,52$. Therefore, $\mathcal{E}(Q_1^{1,1}) < \mathcal{E}(Q_k^{1,1})$ for all $k = 2,3,\ldots,52$. This completes the proof.

(ii). We have

$$\psi(Q_1^{1,3},x) = x^{n-6}\{x^6-(n+1)x^4+(3n-8)x^2+2x-(n-5)\},$$

$$\psi(Q_3^{1,3},x) = x^{n-6}\{x^6-(n+1)x^4+(4n-14)x^2-4x-2(n-6)\},$$

$$\psi(Q_4^{1,3},x) = x^{n-6}\{x^6-(n+1)x^4+(4n-14)x^2-2x-(2n-11)\},$$

$$\psi(Q_6^{1,3},x) = x^{n-8}\{x^8-(n+1)x^6+(4n-12)x^4+2x^3-(4n-20)x^2-2x+(n-7)\},$$

$$\psi(Q_8^{1,3},x) = x^{n-6}\{x^6-(n+1)x^4+(4n-12)x^2+2x-(3n-14)\},$$

$$\psi(Q_{13}^{1,3},x) = x^{n-6}\{x^6-(n+1)x^4+(5n-22)x^2+6x-3(n-7)\},$$

$$\psi(Q_{14}^{1,3},x) = x^{n-6}\{x^6-(n+1)x^4+(5n-21)x^2+4x-(4n-26)\},$$

$$\psi(Q_{17}^{1,3},x) = x^{n-6}\{x^6-(n+1)x^4+(5n-22)x^2-2x-(3n-19)\},$$

$$\psi(Q_{39}^{1,3},x) = x^{n-6}\{x^6-(n+1)x^4+(4n-14)x^2-2(n-5)x-(n-5)\}$$

$$\psi(Q_{40}^{1,3},x) = x^{n-6}\{x^6-(n+1)x^4+(4n-13)x^2-(2n-8)x-(n-5)\},$$

$$\psi(Q_{41}^{1,3},x) = x^{n-7}\{x^7-(n+1)x^5+(4n-12)x^3-(2n-10)x^2-(3n-14)x+2(n-6)\},$$

$$\psi(Q_{42}^{1,3},x) = x^{n-6}\{x^6-(n+1)x^4+(5n-22)x^2-2(n-6)x-2(n-6)\},$$

$$\psi(Q_{43}^{1,3},x) = x^{n-6}\{x^6-(n+1)x^4+(5n-22)x^2-(2n-14)x-3(n-6)\},$$

$$\psi(Q_{44}^{1,3},x) = x^{n-6}\{x^6-(n+1)x^4+(5n-21)x^2-2(n-6)x-3(n-6)\}.$$

First we will show that $\mathcal{E}(B_{n,n+1}^{2,2}) < \mathcal{E}(Q_1^{1,3})$. Consider the function

$$\alpha_{11}(x) = x^6-(n+1)x^4+(3n-8)x^2+2x-(n-5)$$

and proceeding similarly as in part (ii), Lemma 9, we can prove that $\mathcal{E}Q_1^{1,3}) > \mathcal{E}(B_{n,n+1}^{2,2})$ for all $n \geq 10$. Also, the even and odd coefficients of $Q_k^{1,3}$,

$$k = 2,3,4,6,8,13,14,17,39,40,41,42,43,44,$$

alternate in sign and clearly $Q_1^{1,3} \prec Q_k^{1,1}$, for all

$$k = 3,4,6,8,13,14,17,39,40,41,42,43,44.$$

Therefore, $\mathcal{E}(Q_1^{1,1}) < \mathcal{E}(Q_k^{1,1})$ for all

$$k = 3,4,6,8,13,14,17,39,40,41,42,43,44.$$

This completes the proof.

(iii, iv). The proof is similar to (1). □

Let $Q_k^{2,1}$, $k = 1,2,3,\dots,34$ be the graphs as shown in Figure 9. Then it is easy to see that there are four switching classes on the signings of $Q_k^{2,1}$, $k = 1,2,3,\dots,34$. Let $Q_k^{2,1}$, $Q_k^{2,2}$, $Q_k^{2,3}$, $Q_k^{2,4}$, $k = 1,2,3,\dots,34$, respectively be the representative for these four switching classes, where $Q_k^{2,1}$ contains C_3^+, C_4^+ and C_5^+;$Q_k^{2,2}$ contains C_3^-, C_4^- and C_5^+; $Q_k^{2,3}$ contains C_3^-, C_4^+ and C_5^-; and $Q_k^{2,4}$ contains C_3^+, C_4^- and C_5^-. It is easy to see that $Q_k^{2,1}$ are switching equivalent to $-Q_k^{2,3}$ and $Q_k^{2,2}$ are switching equivalent to $-Q_k^{2,4}$, therefore $\mathcal{E}(Q_k^{2,1}) = \mathcal{E}(Q_k^{2,3})$ and $\mathcal{E}(Q_k^{2,2}) = \mathcal{E}(Q_k^{2,4})$ for all $k = 1, 2, 3, \ldots, 34$. Thus, we can regard $Q_k^{2,1}$ and $Q_k^{2,3}$ as identical, $Q_k^{2,2}$ and $Q_k^{2,4}$ as identical, respectively. Also, $b_4(Q_k^{2,r}) > 5n - 21$ for all $k = 7, 8, \ldots, 34$, $k \neq 26$ and $r = 2, 4$, therefore we omit these signed graphs here, as these signed graphs will be considered later. With these notations, we have the following result, whose proof is similar as in Lemma 18. So we skip the proof here.

Lemma 19 *For all* $n \geq 10$*, we have*
(i) $\mathcal{E}(B_{n,n+1}^{2,2}) < \mathcal{E}(Q_1^{2,1}) < \mathcal{E}(Q_k^{2,1})$ *for all* $k = 2, 3, \ldots, 34$.
(ii) $\mathcal{E}(Q_1^{2,1}) < \mathcal{E}(Q_1^{2,2}) < \mathcal{E}(Q_k^{2,2})$ *for all* $k = 2, 3, 4, 5, 6, 26$.

Let $Q_1^{3,1}$ be the graph as shown in Figure 10. It is easy to see that there are four switching classes on the signings of $Q_1^{3,1}$. Let $Q_1^{3,1}$, $Q_1^{3,2}$, $Q_1^{3,3}$, $Q_1^{3,3}$ be the representative for these four switching classes, where $Q_1^{3,1}$ contains C_3^+, C_5^+ and C_6^+; $Q_1^{3,2}$ contains C_3^-, C_5^- and C_6^+; $Q_1^{3,3}$ contains C_3^-, C_5^+ and C_6^-; and $Q_1^{3,4}$ contains C_3^+, C_5^- and C_6^-. It is easy to see that $Q_1^{3,1}$ is switching equivalent to $-Q_1^{3,2}$ and $Q_1^{3,3}$ is switching equivalent to $-Q_1^{3,4}$, therefore $\mathcal{E}(Q_1^{3,1}) = \mathcal{E}(Q_1^{3,2})$ and $\mathcal{E}(Q_1^{3,3}) = \mathcal{E}(Q_1^{3,4})$. Thus, we can regard $Q_1^{3,1}$ and $Q_1^{3,2}$ as identical, $Q_1^{3,3}$ and $Q_1^{3,4}$ as identical. Also, let $Q_k^{4,1}$, $k = 1, 2, 3, 4, 5, 6$ be the graphs as shown in Figure 10. Then it is easy to see that there are three switching classes on the signings of $Q_k^{4,1}$, $k = 1, 2, 3, 4, 5$. Let $Q_k^{4,1}$, $Q_k^{4,2}$, $Q_k^{4,3}$, $k = 1, 2, 3, 4, 5, 6$, respectively be the representative for these three switching classes, where $Q_k^{4,1}$ contains C_4^+, C_4^+ and C_6^+; $Q_k^{4,2}$ contains C_4^-, C_4^+ and C_6^-; $Q_k^{4,3}$ contains C_4^-, C_4^- and C_6^+. It is easy to see that $b_4(Q_k^{4,r}) > 5n-21$ for all $k = 2, 3, 4, 5, 6$, $r = 2, 3$ and therefore we omit these signed graphs here, as these signed graphs will be considered later. With these notations, we have the following result, whose proof is similar as in Lemma 18. So we skip the proof here.

Lemma 20 *For all* $n \geq 10$*, we have* (i) $\mathcal{E}(B_{n,n+1}^{2,2}) < \mathcal{E}(Q_1^{3,1})$.
(ii) $\mathcal{E}(B_{n,n+1}^{2,2}) < \mathcal{E}(Q_1^{3,3})$.
(iii) $\mathcal{E}(B_{n,n+1}^{2,2}) < \mathcal{E}(Q_1^{4,1}) < \mathcal{E}(Q_k^{4,t})$ *for all* $k = 1$ *when* $t = 2, 3$ *and* $k = 2, 3, 4, 5$ *when* $t = 1$.
(iv) $\mathcal{E}(B_{n,n+1}^{2,2}) < \mathcal{E}(Q_6^{4,1})$.

Let $Q_k^{5,1}$, $k = 1, 2, 3, \ldots, 15$, be the graphs as shown in Figure 11. Clearly, there are two switching classes on the signings of $Q_k^{5,1}$ $k = 1, 2, 3, \ldots, 15$. Let $Q_k^{5,1}$ and $Q_k^{5,2}$, $k = 1, 2, 3, \ldots, 15$, be the representative for these two switching classes, where $Q_k^{5,1}$ contains C_4^+, C_4^+, C_4^+; and $Q_k^{5,2}$ contains C_4^-, C_4^- and C_4^+ respectively. Also, let $Q_k^{6,1}$, $k = 1, 2, \ldots, 7$, be the graphs as shown in Figure 11. Then it is easy to see that there are three switching classes on the signings of $Q_k^{6,1}$, $k = 1, 2, \ldots, 7$. Let $Q_k^{6,1}$, $Q_k^{6,2}$, $Q_k^{6,3}$,$k = 1, 2, \ldots, 7$ respectively, be the representative for these three switching classes, where $Q_k^{6,1}$ contains C_4^+, C_5^+ and C_5^+; $Q_k^{3,2}$ contains C_4^+, C_5^- and C_5^-; $Q_k^{6,3}$ contains C_4^-, C_5^+ and C_5^-. It is easy to see that $Q_k^{6,1}$ is switching equivalent to $-Q_k^{6,2}$. Therefore, $\mathcal{E}(Q_k^{6,1}) = \mathcal{E}(Q_k^{6,2})$ for all $k = 1, 2, \ldots, 7$, thus we can regard $Q_k^{6,1}$ and $Q_k^{6,2}$ as identical. Also, $b_4(Q_k^{5,2}) > 5n - 21$ for all $k = 3, 4, \ldots, 15$ and $b_4(Q_k^{6,3}) > 5n - 21$ for all $k = 2, 3, \ldots, 7$. Therefore, we omit these signed graphs here, as these signed graphs will be considered later. With these notations, we have the following result, whose proof is similar as in Lemma 18. So we skip the proof here.

Lemma 21 *For all* $n \geq 10$, *we have* (i) $\mathcal{E}(B_{n,n+1}^{2,2}) < \mathcal{E}(Q_1^{5,1}) < \mathcal{E}(Q_k^{5,t})$ *for all* $k = 1, 2$ *when* $t = 2$ *and* $k = 2, 3, \ldots, 15$ *when* $t = 1$.
(ii) $\mathcal{E}(Q_1^{5,1}) < \mathcal{E}(Q_k^{6,t})$ *for all* $k = 1$ *when* $t = 3$ *and* $k = 1, 2, 3, \ldots, 7$ *when* $t = 1$.

Now, we have the following theorem.

Theorem 22 *Let* $S \in \theta(n, p, q, r)$, $S \neq S_{n,n+1}^{k,t}$ $(k = 0, 1, 2, 3$ *and* $t = 1, 2, 3)$, $B_{n,n+1}^{k,t}$ $(k = 0, 1, 2,$ *and* $t = 1, 2)$, $Q_{n,n+1}^{1,1}$, $Q_{n,n+1}^{2,t}$ $(t = 1, 3)$, $Q_{n,n+1}^{3,t}$ $(t = 1, 2, 3)$, *where* $n \geq 17$, $p \geq 3$, $q \geq 3$ *and* $r \geq 1$. *Then* $\mathcal{E}(S) > \mathcal{E}(B_{n,n+1}^{2,2})$.

Proof. As C_p and C_q have $r \geq 1$ common edges, we first assume that $r \geq 6$, and let $P_{r+1} = e_1 e_2 \ldots e_r$ be the path formed by these r edges. Choose the cut set as $Z = \{e_1, e_r\}$, so that $S - \{e_1, e_r\}$ has two components say S' and S'', where S' is a signed tree on $m_1 \geq 5$ vertices and S'' is a unicyclic signed graph on $m_2 \geq 5$ vertices. Therefore the result follows by Lemmas 14 and 15. This proves the result for $r \geq 6$ and $n \geq 17$. For $r \leq 5$, the above technique still applies but we need to consider several cases while choosing the cut set. For $r \leq 5$, we prove the result by induction on $n - p - q + r$. Clearly, $n - p - q + r \geq -1$. If $n - p - q + r = -1$, then S has no pendant edge. We consider the following cases, (i) $r = 1$, (ii) $r = 2$ and (iii) $3 \leq r \leq 5$.

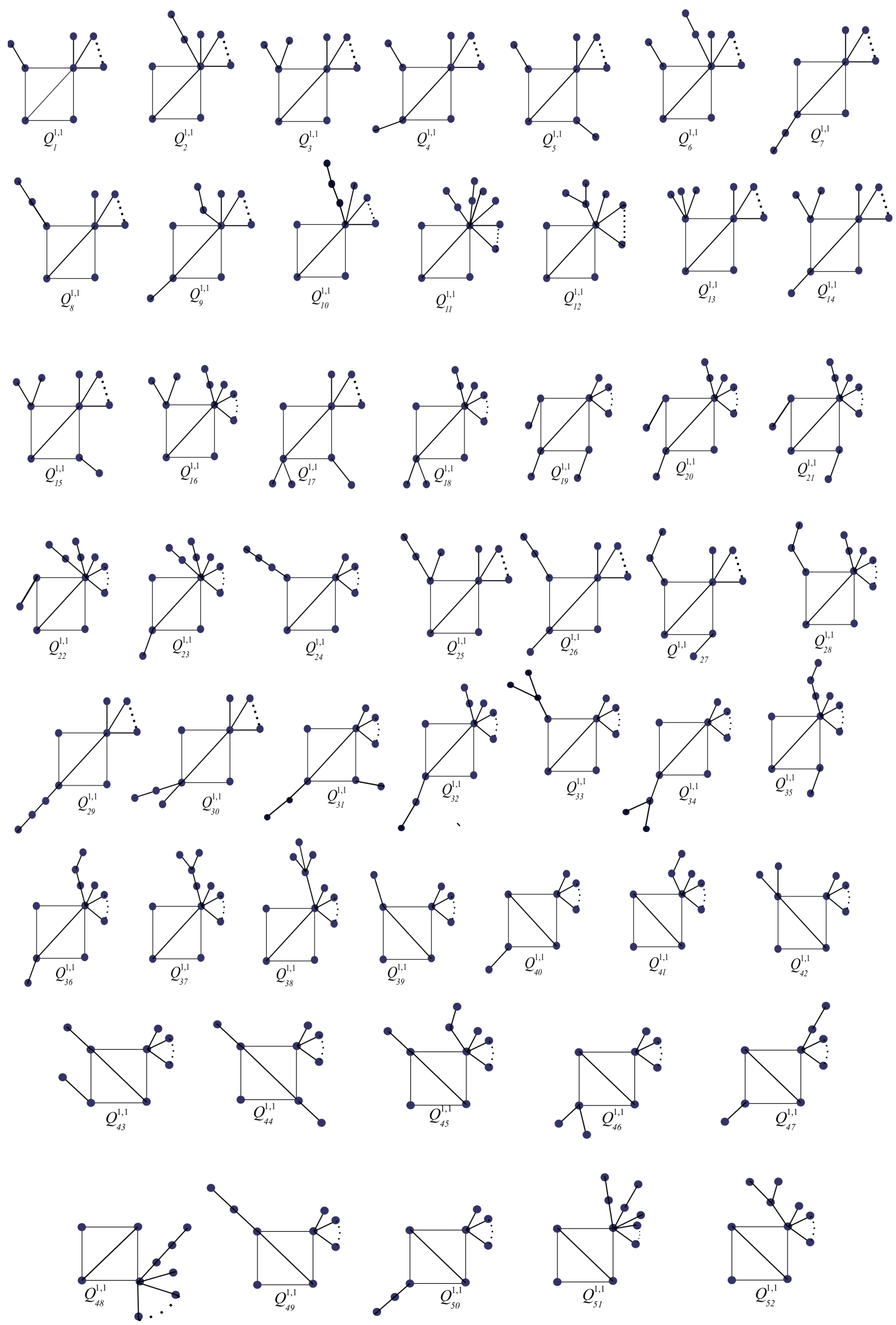

Figure 8: Signed graphs $Q_k^{1,1}$, $k = 1, 2, \ldots, 52$

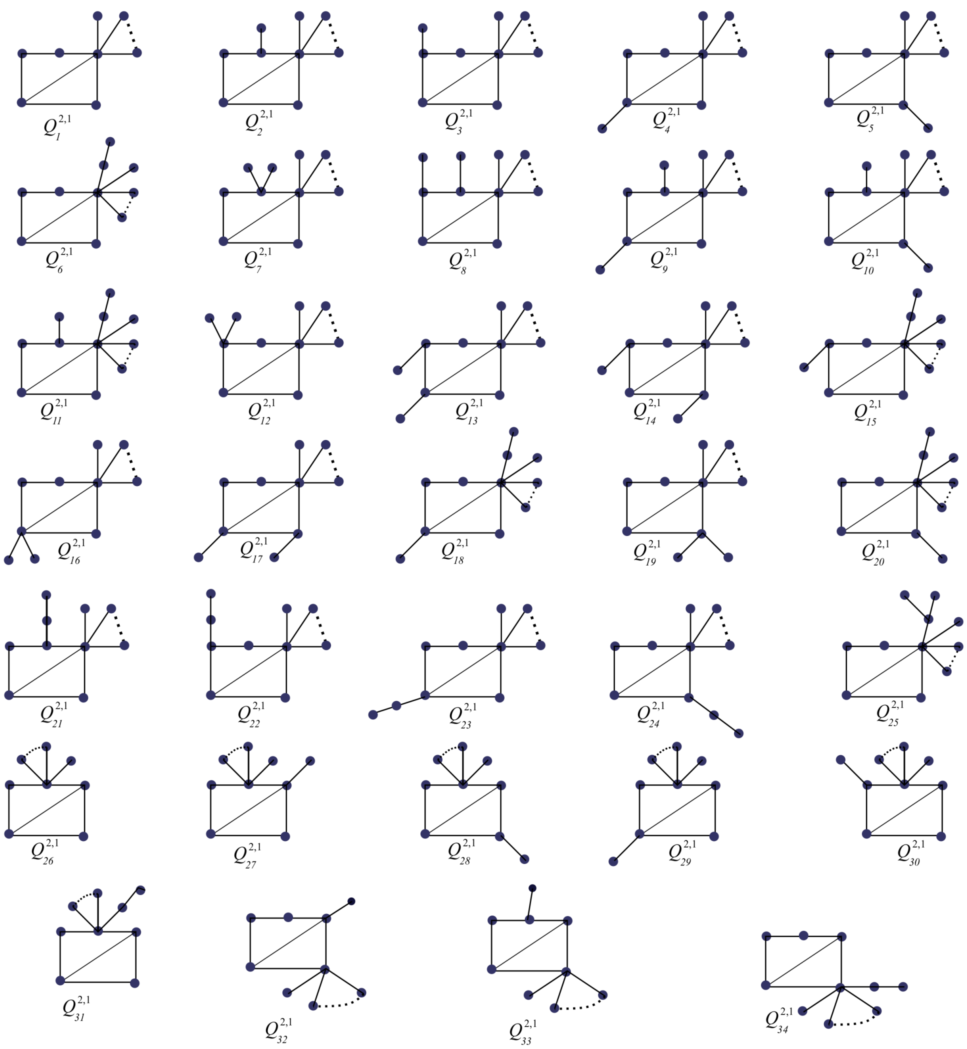

Figure 9: Signed graphs $Q_k^{2,1}$, $k = 1, 2, \ldots, 34$

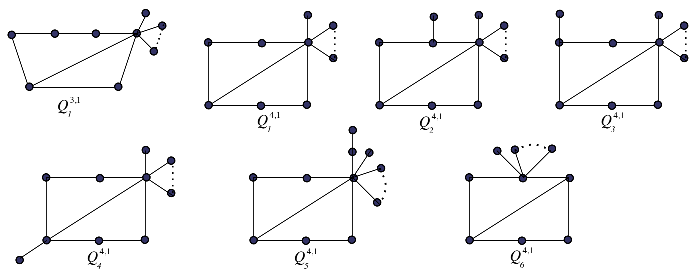

Figure 10: Signed graphs $Q_1^{3,1}$ and $Q_k^{4,1}$, $k = 1, 2, 3, 4, 5, 6$

Case (i). If $r = 1$, then $n - p - q = -2$ or $n = p + q - 2$. As $n \geq 17$, S can have at most one 4-cycle. If $p = 4$, then $n = q + 2$. Now, by (3.1), we have

$$b_4(S) = q - 2 + m(C_{q+2}, 2) \pm 2 = \begin{cases} \frac{1}{2}(q^2 + 3q - 10), & \text{if 4-cycle is positive} \\ \frac{1}{2}(q^2 + 3q - 2), & \text{if 4-cycle is negative.} \end{cases}$$

As $b_4(B^{2,2}_{n,n+1}) = 5n - 21$, so $b_4(B^{2,2}_{q+2,q+3}) = 5q - 11$ and therefore

$$b_4(S) - b_4(B^{2,2}_{q+2,q+3}) = b_4(S) - (5n - 11) = \begin{cases} \frac{1}{2}(q^2 - 7q + 12), & \text{if 4-cycle is positive} \\ \frac{1}{2}(q^2 - 7q + 20), & \text{if 4-cycle is negative.} \end{cases}$$

Since $n \geq 17$, so $q \geq 15$ and thus $b_4(S) - b_4(B^{2,2}_{q+2,q+3}) > 0$. If there is no 4-cycle in S, then since $n = p + 1 - 2$, we have $b_4(B^{2,2}_{n,n+1}) = 5n - 11 = 5(p + q) - 31$. Also,

$$b_4(S) = m(S, 2) = m(C_{p+q-2}, 2) + p + q - 6 = \frac{1}{2}[(p + q)^2 - 5(p + q) - 2]$$

and so

$$b_4(S) - b_4(B^{2,2}_{n,n+1}) = \frac{1}{2}[(p + q)^2 - 15(p + q) + 60] > 0$$

because $n \geq 17$ and so $p + q \geq 19$.

Case (ii). Let $r = 2$, so that in this case $n = p + q - 3$. Again, S can have at most one 4-cycle. Suppose S contains a 4-cycle and let $p = 4$. Then $n = q + 1$ and so $b_4(B^{2,2}_{n,n+1}) = b_4(B^{2,2}_{q+1,q+2}) = 5q - 16$. Also,

$$b_4(S) = 2(q - 2) + m(C_q, 2) \pm 2 = 2(q - 2) + \frac{q(q - 3)}{2} \pm 2.$$

Therefore,

$$b_4(S)-b_4(B^{2,2}_{q+1,q+2})=b_4(S)-(5q-16)=\begin{cases}\frac{1}{2}(q^2-9q+20), & \text{if 4 cycle is positive}\\ \frac{1}{2}(q^2-9q+28), & \text{if 4 cycle is negative.}\end{cases}$$

As $n \geq 17$, so $q \geq 16$ and therefore $b_4(S) - b_4(B^{2,2}_{q+1,q+2}) > 0$. Suppose S does not contain a 4-cycle. Then

$$\begin{aligned} b_4(S) &= m(S,2) = 2(p+q-6) + m(C_{p+q-4},2) \\ &= \frac{1}{2}\{4(p+q-6)(p+q-4)(p+q-7)\} \\ &= \frac{1}{2}\{(p+q)^2 - 7(p+q) + 4\}. \end{aligned}$$

Therefore

$$b_4(S) - b_4(B^{2,2}_{n,n+1}) = \frac{1}{2}\{(p+q)^2 - 17(p+q) + 76\} > 0,$$

since $p + q \geq 20$.

Case (iii). If $3 \leq r \leq 5$, then $n = p + q - r - 1$. So S does not contain a 4-cycle. Then proceeding similarly as above, we can prove that

$$b_4(S) - b_4(B^{2,2}_{n,n+1}) = \frac{1}{2}\{(p+q)^2 - 13(p+q) + r^2 + 13r - 2(p+q)r + 46\} > 0.$$

This proves that the result is true for $n - p - q + r = -1$. Assume the result to be true for $n - p - q + r < p'$, where $p' \geq 0$. Let $n - p - q + r = p'$. Then S has a pendent edge, say $e = uv$, with v as a pendant vertex. Apply Lemma 4, we have

$$b_4(S) = b_4(S - \{v\}) + b_2(S - \{u,v\})$$

and

$$b_4(B^{2,2}_{n,n+1}) = b_4(B^{2,2}_{n-1,n}) + b_2(S_5).$$

By induction, it is easy to see that $b_4(S-\{v\}) > b_4(B^{2,2}_{n-1,n})$ and $b_2(S-\{u,v\}) \geq 5 = b_2(S_5)$ if $S \neq S^{k,t}_{n,n+1}$ $(k = 0,1,2,3$ and $t = 1,2,3)$, $B^{k,t}_{n,n+1}$ $(k = 0,1,2,$ and $t = 1,2)$, $Q^{1,t}_{n,n+1}$ $(t = 1,2)$, $Q^{2,t}_{n,n+1}$ $(t = 1,2,3,4)$, $Q^{3,t}_{n,n+1}$ $(t = 1,2,3)$, $Q^{1,t}_k$ $(k = 1,2,\dots,52$ and $t = 1,2)$, $Q^{1,3}_k$ $(k = 1,2,\dots,14,17,18,30,39,40,\dots,44)$, $Q^{2,t}_k$ $(k = 1,2,3,\dots,34$ and $t = 1,3)$, $Q^{2,t}_k$ $(k = 1,2,3,4,5,6,26$ and $t = 2,4)$, $Q^{3,t}_1$ $(t = 1,2,3,4)$, $Q^{4,1}_k$ $(k = 1,2,\dots,6)$, $Q^{4,t}_1$ $(t = 2,3)$, $Q^{5,1}_k$ $(k =$

$1, 2, \ldots, 15)$, $Q_1^{5,2}$, $Q_2^{5,2}$, $Q_k^{6,t}$ $(k = 1, 2, \ldots, 7$ and $t = 1, 2)$ and $Q_1^{6,3}$. Thus $b_4(S) > b_4(B_{n,n+1}^{2,2})$. Further as $S \in \theta(n, p, q, r)$, so $b_{2j}(S) \geq 0$ for all $j \geq 3$. Also, $b_{2j}(B_{n,n+1}^{2,2}) = 0$ for all $j \geq 3$ and $b_{2j+1}(B_{n,n+1}^{2,2}) = 0$ for all $j \geq 0$. The second summand of logarithm in integral formula given in Theorem 2 is non negative for the signed graph $S \in \theta(n, p, q, r)$ (in fact for every signed graph). Hence, $S \succ B_{n,n+1}^{2,2}$. By integral formula given in Theorem 2, we see that $\mathcal{E}(S) > \mathcal{E}(B_{n,n+1}^{2,2})$. If $S =$ $Q_{n,n+1}^{1,2}$, $Q_{n,n+1}^{2,2}$, $Q_{n,n+1}^{2,4}$, $Q_k^{1,t}$ $(k = 1, 2, \ldots, 52$ and $t = 1, 2)$, $Q_k^{1,3}$ $(k = 1, 2, \ldots, 14, 17, 18, 30, 39, 40, \ldots, 44)$, $Q_k^{2,t}$ $(k = 1, 2, 3, \ldots, 34$ and $t = 1, 3)$, $Q_k^{2,t}$ $(k = 1, 2, 3, 4, 5, 6, 26$ and $t = 2, 4)$, $Q_1^{3,t}$ $(t = 1, 2, 3, 4)$, $Q_k^{4,1}$ $(k = 1, 2, \ldots, 6)$, $Q_1^{4,t}$ $(t = 2, 3)$, $Q_k^{5,1}$ $(k = 1, 2, \ldots, 15)$, $Q_1^{5,2}$, $Q_2^{5,2}$, $Q_k^{6,t}$ $(k = 1, 2, \ldots, 7$ and $t = 1, 2)$ and $Q_1^{6,3}$, then the result follows by Lemmas 9, 18, 19, 20 and 21. This completes the proof. □

Theorem 23 *Among all bicyclic signed graphs with* $n \geq 17$ *vertices,* $B_{n,n+1}^{2,2}$ *is the signed graph with* 20^{th} *minimal energy for all* $n \geq 30$ *and with* 16^{th} *minimal energy for all* $17 \leq n \leq 29$*. Also, we have ordering of energies in ascending order as follows.*

(i) *For all* $n \geq 30$*, we have*

$$\mathcal{E}(S_{n,n+1}^{0,1}) = \mathcal{E}(S_{n,n+1}^{0,2}) < \mathcal{E}(S_{n,n+1}^{0,3}) < \mathcal{E}(B_{n,n+1}^{0,1}) < \mathcal{E}(S_{n,n+1}^{1,1}) = \mathcal{E}(S_{n,n+1}^{1,2}) < \mathcal{E}(S_{n,n+1}^{1,3}) < \mathcal{E}(B_{n,n+1}^{0,2}) < \mathcal{E}(B_{n,n+1}^{1,1}) < \mathcal{E}(Q_{n,n+1}^{1,1}) < \mathcal{E}(S_{n,n+1}^{2,1}) = \mathcal{E}(S_{n,n+1}^{2,2}) < \mathcal{E}(S_{n,n+1}^{2,3}) < \mathcal{E}(Q_{n,n+1}^{2,1}) = \mathcal{E}(Q_{n,n+1}^{2,3}) < \mathcal{E}(B_{n,n+1}^{1,2}) < \mathcal{E}(Q_{n,n+1}^{3,1}) = \mathcal{E}(Q_{n,n+1}^{3,2}) < \mathcal{E}(Q_{n,n+1}^{3,3}) < \mathcal{E}(H_{n,n+1}^{3}) < \mathcal{E}(H_{n,n+1}^{1}) = \mathcal{E}(H_{n,n+1}^{2}) < \mathcal{E}(B_{n,n+1}^{2,1}) < \mathcal{E}(S_{n,n+1}^{3,1}) = \mathcal{E}(S_{n,n+1}^{3,2}) < \mathcal{E}(S_{n,n+1}^{3,3}) < \mathcal{E}(B_{n,n+1}^{2,2}).$$

(ii) *For all* $17 \leq n \leq 29$*, we have*

$$\mathcal{E}(S_{n,n+1}^{0,1}) = \mathcal{E}(S_{n,n+1}^{0,2}) < \mathcal{E}(S_{n,n+1}^{0,3}) < \mathcal{E}(B_{n,n+1}^{0,1}) < \mathcal{E}(S_{n,n+1}^{1,1}) = \mathcal{E}(S_{n,n+1}^{1,2}) < \mathcal{E}(S_{n,n+1}^{1,3}) < \mathcal{E}(B_{n,n+1}^{0,2}) < \mathcal{E}(B_{n,n+1}^{1,1}) < \mathcal{E}(Q_{n,n+1}^{1,1}) < \mathcal{E}(S_{n,n+1}^{2,1}) = \mathcal{E}(S_{n,n+1}^{2,2}) < \mathcal{E}(S_{n,n+1}^{2,3}) < \mathcal{E}(Q_{n,n+1}^{2,1}) = \mathcal{E}(Q_{n,n+1}^{2,3}) < \mathcal{E}(B_{n,n+1}^{1,2}) < \mathcal{E}(B_{n,n+1}^{2,1}) < \mathcal{E}(S_{n,n+1}^{3,1}) = \mathcal{E}(S_{n,n+1}^{3,2}) < \mathcal{E}(S_{n,n+1}^{3,3}) < \mathcal{E}(B_{n,n+1}^{2,2}).$$

Proof. This follows by Corollary 10 and Theorems 12, 17 and 22. □

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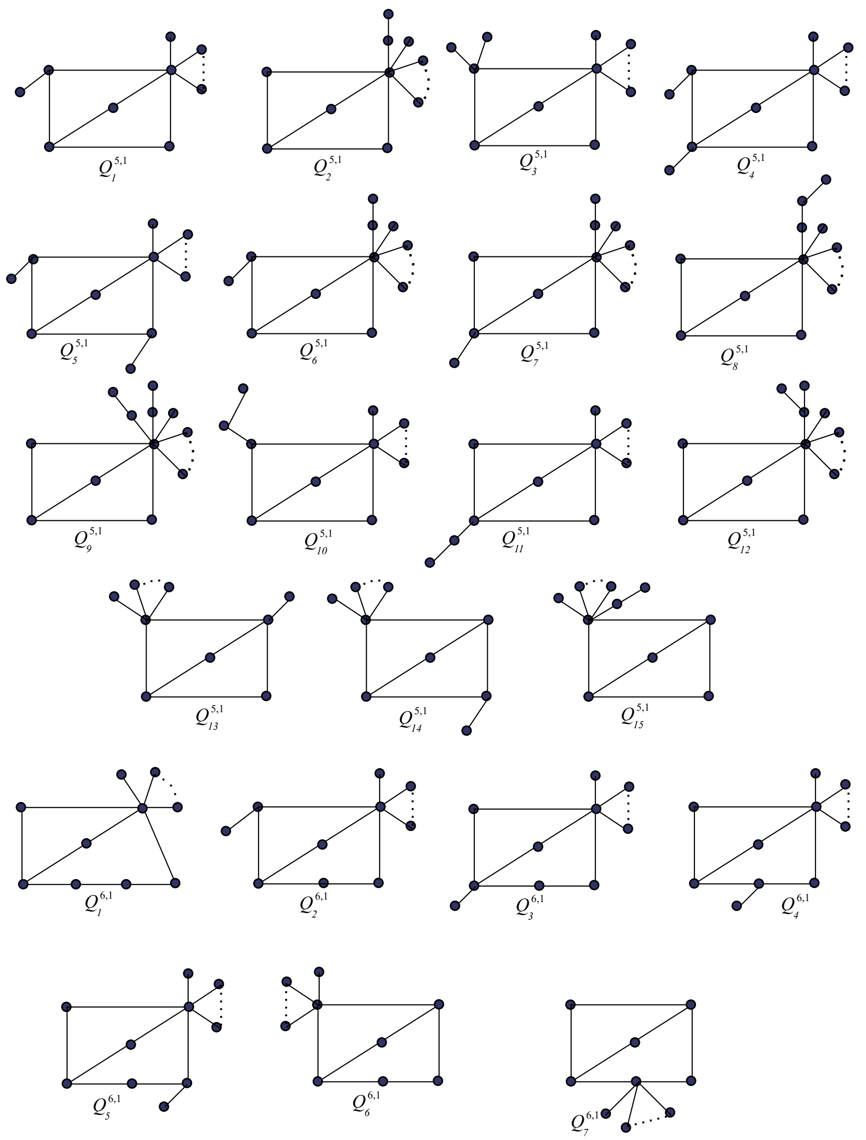

Figure 11: Signed raphs $Q_k^{5,1}$, $k = 1, 2, \dots 15$ and $Q_k^{6,1}$, $k = 1, 2, \dots, 7$

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