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Annihilator graphs of a commutative semigroup whose Zero-divisor graphs are a complete graph with an end vertex

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Abstract. Suppose that the zero-divisor graph of a commutative semigroup S, be a complete graph with an end vertex. In this paper, we determine the structure of the annihilator graph S and we show that if $Z(S) = S$, then the annihilator graph S is a disconnected graph.

1 Introduction

In this paper S is a commutative semigroup with zero whose operation is written multiplicatively and $Z(S)$ is the set of all zero-divisors of S also $Z(S)^* = Z(S) \setminus \{0\}$.

The zero-divisor graph of a commutative semigroup S with zero, is denoted by $\Gamma(S)$, is an undirected graph with vertex set $Z(S)^*$ and two distinct vertices x and y are adjacent if and only if $xy = 0$. $\Gamma(S)$ is a connected graph and the

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diameter of $\Gamma(S)$ is less than or equal to three. For other results on zero divisor graphs one can see [5, 6, 7, 8, 9, 10].

In [1], we introduced and studied the annihilator graph for a commutative semigroup S, and showed it with $AG(S)$. The graph $AG(S)$ is an undirected graph with vertex set $Z(S)^*$ and two distinct vertices x and y are adjacent if and only if $\text{ann}_S(xy) \neq \text{ann}_S(x) \cup \text{ann}_S(y)$, where $\text{ann}_S(x) = \{s \in S \mid xs = 0\}$. We proved that if $Z(S) \neq S$, then $\Gamma(S)$ is a subgraph of $AG(S)$, and so $AG(S)$ is connected. Also if $Z(S) = S$, then $AG(S)$ may be connected or disconnected and if there exists $x \in S^* = S \setminus \{0\}$ such that x is adjacent to all vertices in $\Gamma(S)$, then x is an isolated vertex in $AG(S)$.

In [1, section 4] and in [2], we characterized all annihilator graphs with three and four vertices. Also in [3], we studied the structure of the annihilator graph of a commutative semigroup S whose $\Gamma(S)$ is a refinement of a star graph.

A complete graph and a complete graph with an end vertex are one of the graphs can be zero-divisor graph of a commutative semigroup.

In this paper, we study the annihilator graph associated with a commutative semigroup with zero using the zero-divisor graph $\Gamma(S)$, where $\Gamma(S)$ is a complete graph K_n with an end vertex $u \notin V(K_n)$ and u is only adjacent to $z \in V(K_n)$. Let m be the number of edges between u and $V(K_n)$ in $AG(S)$. We show that the following four statements hold.

(i) Let $u^2 = 0$. If $Z(S) \neq S$, then $m \in \{1, 2, 3, ..., n\}$ and if $Z(S) = S$, then $m \in \{0, 1, 2, 3, ..., n-1\}$.

(ii) Let $u^2 = z$. If $Z(S) \neq S$, then $m = n$ and so u is adjacent to all vertices of $V(K_n)$ in $AG(S)$ and if $Z(S) = S$, then $m = n-1$ and u is not adjacent to z in $AG(S)$.

(iii) Let $u^2 = u$. If $Z(S) \neq S$, then $m \in \{1, 2, 3, ..., n-1\}$ and if $Z(S) = S$, then $m \in \{0, 1, 2, 3, ..., n-2\}$ and so there is at least one vertex of $V(K_n)$ that u is not adjacent to it in $AG(S)$.

(iv) Let $u^2 = b \notin \{0, z, u\}$. If $Z(S) \neq S$, then $m \in \{n-1, n\}$ and so there is at most one vertex ($u^2 = b$) of $V(K_n)$ that u is not adjacent to it in $AG(S)$. Also if $Z(S) = S$, then $m \in \{n-1, n-2\}$ and u is not adjacent to z in $AG(S)$.

2 Preliminaries

In this section, we recall some definitions and notations of graphs and we use the standard terminology of graphs is contained in [4]. Here, G is a graph with vertex set $V(G)$ and edge set $E(G)$. If a is adjacent to b in G, then the edge between a and b will denote by $\{ab\}$ and we write $a \sim b$.

The *distance* between two distinct vertices x and y is the length of the shortest path connecting x and y and will denote by $d(x, y)$, if such a path exists; otherwise, we use $d(x, y) := \infty$. Also $\text{diam}(G) = \sup\{d(x, y) : x$ and y are distinct vertices of $G\}$ is the *diameter* of the graph G.

The *girth* of G, denoted by $\text{gr}(G)$, is the length of the shortest cycle in G. If there exists a path between any two distinct vertices of G, we say that graph G is a *connected* graph, and if for each two vertices x and y of $V(G)$ we have x is adjacent to y, we say that G is a complete graph and K_n is the complete graph with n vertices. If no two vertices of G are adjacent, we say that G is a *totally disconnected* graph and nK_1 is the totally disconnected graph with n vertices.

We say that u is an end vertex in G, If u is adjacent to only one vertex of G and if for each vertex $x \in V(G)$ we have u is not adjacent to x, then we say that u is an *isolated* vertex in G.

Suppose that H and G are two graphs. We use the notation $G \leq H$ to denote that G is a subgraph of H and if H is isomorphic to G, we write $H \cong G$. Let G be a graph. $G \setminus \{\{x_1y_1\}, \{x_2y_2\}, \{x_3y_3\}, ..., \{x_ny_n\}\}$ is a graph such that edges $\{x_1y_1\}, \{x_2y_2\}, \{x_3y_3\}, ..., \{x_ny_n\}$ are deleted.
P_n is the path of length n and C_n is the cycle of length n.
mK_n is a graph with m components such that each component is isomorphic to K_n. $G \cup H$, the union of the graphs G and H, is a graph with vertex set $V(G) \cup V(H)$ and edge set $E(H) \cup E(G)$.

Now, we recall some results which we are used in the next section.

Theorem 1 [1] *If* $Z(S) \neq S$, *then we have* $\Gamma(S) \leq AG(S)$.

Theorem 2 [1] *Let* $Z(S) = S$ *and there exists* $x \in S^*$ *such that, for each non zero element* $y \neq x$ *of* S, *we have* $xy = 0$. *Then* x *is an isolated vertex in* $AG(S)$.

Lemma 3 [2] *If* $Z(S) \neq S$ *and* $\Gamma(S) \cong P_3$, *then* $AG(S) \cong C_4$.

Lemma 4 [2] *Let* $Z(S) \neq S$. *Then* $AG(S) \cong C_4$ *if and only if we either have* $\Gamma(S) \cong P_3$ *or* $\Gamma(S) \cong C_4$.

Lemma 5 [2] *Let* $Z(S) = S$. *Then* $AG(S) \cong P_3$ *with* $x \sim w \sim z \sim y$ *if and only if* $\Gamma(S) \cong P_3$ *with* $w \sim x \sim y \sim z$.

Lemma 6 [2] *Let* S *be a commutative semigroup with* $Z(S) \neq S$, *and let* $\Gamma(S) \cong K_3 + \{wx\}$ *with* $wx = xy = yz = zx = 0$. *Then* $AG(S) \cong K_4 \setminus \{wy\}$ *if and only if the relations between the zero-divisors of* S *satisfies in one of the following four conditions.*

(i) $wy = y$, $wz = x$, $w^2 = y^2 = y$, $x^2 = 0$, *and* $z^2 \in \{0, x\}$.

(ii) $wy = wz = x$, $w^2 = y^2 = x^2 = 0$, *and* $z^2 = x$.

(iii) $wy = y = wz$, $w^2 = w$, $z^2 = x$, *and* $y^2 = x^2 = 0$.

Lemma 7 [2] *Let* S *be a commutative semigroup with* $Z(S) \neq S$, *and let* $\Gamma(S) \cong K_3 + \{wx\}$ *with* $wx = xy = yz = zx = 0$. *Then* $AG(S) \cong K_4$ *if and only if the relations between the zero-divisors of* S *satisfies in one of the following eleven conditions.*

(i) $wx = xy = yz = zx = 0$, $wy = x$, $wz = y$, $y^2 = x^2 = 0$, $w^2 = z$ *and* $z^2 = x$.

(ii) $wx = xy = yz = zx = 0$, $wy = z$, $wz = x$, $w^2 = y$, $y^2 = x$ *and* $z^2 = x^2 = 0$.

(iii) $wx = xy = yz = zx = 0$, $wy = wz = x$, $x^2 = 0$ *and one of the following nine cases holds.*

- *(1)* $w^2 = 0$, $y^2 = x$ *and* $z^2 = x$.
- *(2)* $w^2 = y$, $y^2 = 0$ *and* $z^2 \in \{0, x\}$.
- *(3)* $w^2 = z$, $z^2 = 0$ *and* $y^2 \in \{0, x\}$.
- *(4)* $w^2 = x$, $y^2 = 0$ *and* $z^2 \in \{0, x\}$.
- *(5)* $w^2 = x$, $y^2 = x$ *and* $z^2 \in \{0, x\}$.

Lemma 8 [2] *Let* S *be a commutative semigroup with* $Z(S) \neq S$, *and let* $\Gamma(S) \cong K_3 + \{wx\}$ *with* $wx = xy = yz = zx = 0$. *Then* $AG(S) \cong K_3 + \{wx\}$ *with* $w \sim x \sim y \sim z \sim x$ *if and only if the relations between the zero-divisors of* S *satisfies in one of the following nineteen conditions.*

(i) $wx = xy = yz = zx = 0$, $wy = y = wz$, $z^2 = y^2 = 0$, $w^2 = w$ *and* $x^2 \in \{0, x\}$.

(ii) $wx = xy = yz = zx = 0$, $wz = wy = x$ *and* $w^2 = y^2 = z^2 = x^2 = 0$.

(iii) $wx = xy = yz = zx = 0$, $wz = z = wy$, $w^2 = w$, $y^2 = z^2 = 0$ *and* $x^2 \in \{0, x\}$.

(iv) $wx = xy = yz = zx = 0$, $wz = y$, $wy = z$, $w^2 = w$, $y^2 = z^2 = 0$ *and* $x^2 \in \{0, x\}$.

(v) $wx = xy = yz = zx = 0$, $wy = y$, $wz = z$, $w^2 = w$ *and we have the following twelve situations.*

(1) $y^2 = 0$, $z^2 = 0$ *and* $x^2 \in \{0, x\}$.
(2) $y^2 = 0$, $z^2 = z$ *and* $x^2 \in \{0, x\}$.
(3) $y^2 = 0$, $z^2 = y$ *and* $x^2 \in \{0, x\}$.
(4) $y^2 = y$, $z^2 = 0$ *and* $x^2 \in \{0, x\}$.
(5) $y^2 = y$, $z^2 = z$ *and* $x^2 \in \{0, x\}$.
(6) $y^2 = z$, $z^2 = 0$ *and* $x^2 \in \{0, x\}$.

Lemma 9 [2] *Let* S *be a commutative semigroup with* $Z(S) = S$*, and let* $\Gamma(S) \cong K_3 + \{wx\}$ *with* $wx = xy = yz = zx = 0$*. Then* $AG(S) \cong 2K_1 \cup K_2$*, where* x *and* w *are isolated vertices and* z *is adjacent to* y*, if and only if the semigroup* S *satisfies in one of the nineteen conditions of Lemma* (8)*.*

Lemma 10 [2] *Let* S *be a commutative semigroup with* $Z(S) = S$*. Then* $AG(S) \cong K_{1,2} \cup K_1$*, where* x *is an isolated vertex and the vertices* y*,* z*,* w *form a star graph with center* z*, if and only if* $\Gamma(S) \cong K_3 + \{wx\}$ *with* $wx = xy = yz = zx = 0$*, and the semigroup* S *satisfies in one of the four conditions of Lemma* (6)*.*

Lemma 11 [2] *Let* S *be a commutative semigroup with* $Z(S) = S$*, and let* $\Gamma(S) \cong K_3 + \{wx\}$ *with* $wx = xy = yz = zx = 0$*. Then* $AG(S) \cong K_3 \cup K_1$*, where* x *is an isolated vertex and the vertices* w*,* z*,* y *form a triangle if and only if the semigroup* S *satisfies in one of the eleven conditions of Lemma* (7)*.*

Suppose that G is a complete graph K_n with an end vertex u that u is adjacent to $z \in V(K_n)$ and $n = 1$. Then $\Gamma(S) \cong K_2$. Now if $Z(S) = S$, then clearly $AG(S) \cong 2K_1$, and if $Z(S) \neq S$, then $AG(S) \cong \Gamma(S) \cong K_2$.

Let $n = 2$. We have $\Gamma(S) \cong K_{1,2} = P_2$ with $u \sim z \sim x$. In [1], we show that, if $Z(S) = S$, then $AG(S) \cong 3K_1$ or $AG(S) \cong K_1 \cup K_2$, and if $Z(S) \neq S$, then $AG(S) \cong K_{1,2}$ or $AG(S) \cong K_3$.

Morover assume that complete graph K_2 has two end vertices u_1 and u_2 adjacent to z_1 and z_2. Then $\Gamma(S) \cong P_3$ with $u_1 \sim z_1 \sim z_2 \sim u_2$. Now by lemma 5, if $Z(S) = S$, then $AG(S) \cong P_3$ with $z_1 \sim u_1 \sim u_2 \sim z_2$ such that z_1 and z_2 are two end vertices in $AG(S)$, and by lemma 3, if $Z(S) \neq S$, then $AG(S) \cong C_4$.

3 Properties of $AG(S)$

In this section, we assume that $|Z(S)^*| \geq 4$ and K_n is a complete graph with at least three vertices and $z \in V(\ K_n)$ and $u \notin V(K_n)$. we add to K_n an end vertex u, which is adjacent to a unique vertex z of $V(K_n)$ and denote it by $\Gamma(S) \cong K_n + \{uz\}$ and so $\Gamma(S) \cong K_n + \{uz\}$ is the graph of a commutative semigroup such that $Z(S) = V(K_n) \cup \{0\} \cup \{u\}$. Thus for each two distinct vertices x and y in$V(K_n)$, we have $xy = zu = 0$ and $xu \neq 0$ and Since z is a cut vertex in $\Gamma(S)$, thus $\{0, z\}$ is an ideal of S and so $z^2 = 0$ or $z^2 = z$.

In following, we distinguish the structure of the annihilator graph a commutative semigroup whose $\Gamma(S) \cong K_n + \{uz\}$, for cases $u^2 = 0$ or $u^2 = z$ or $u^2 = u$ or $u^2 \neq 0, z, u$.

The following lemma show that if $\Gamma(S)$ is a complete graph K_n with an end vertex u, then for all $x, y \in V(K_n) \setminus \{z\}$ always, x is adjacent to y in $AG(S)$.

Lemma 12 *Suppose that $\Gamma(S)$ is a complete graph K_n with an end vertex u. Then for all $x, y \in V(K_n) \setminus \{z\}$, we have x is adjacent to y in $AG(S)$.*

Proof. Since $\Gamma(S) \cong K_n + \{uz\}$ and $x, y \in V(K_n) \setminus \{z\}$, we have $xy = 0$ and so $\mathrm{ann}_S(xy) = S$. since u is an end vertex adjacent to only z in $\Gamma(S)$ thus $ux \neq 0$ and $uy \neq 0$ so $u \notin \mathrm{ann}_S(x) \cup \mathrm{ann}_S(y)$ which follows that $\mathrm{ann}_S(x) \cup \mathrm{ann}_S(y) \neq \mathrm{ann}_S(xy)$. Therefore x is adjacent to y in $AG(S)$. □

Lemma 13 *Suppose that $\Gamma(S)$ is a complete graph K_n with an end vertex u. Then the following statements hold.*

(i) If $Z(S) \neq S$, then $AG(S)$ is a connected graph and u is adjacent to z in $AG(S)$.

(ii) If $Z(S) = S$, then $AG(S)$ is a disconnected graph and z is an isolated vertex in $AG(S)$.

Proof. (i) Since $Z(S) \neq S$ by theorem 1, we have $\Gamma(S) \leq AG(S)$. Since $\Gamma(S)$ is a connected graph and z is adjacent to u in $\Gamma(S)$, we have $AG(S)$ is a connected graph and u is adjacent to z in $AG(S)$.

(ii) Since z is adjacent to all vertices in $\Gamma(S)$ and $Z(S) = S$ by theorem 2, z is an isolated vertex in $AG(S)$ and so $AG(S)$ is a disconnected graph. □

Let $\Gamma(S) \cong K_n + \{uz\}$. By lemma 12 and lemma 13, to study the graph $AG(S)$, it is sufficient to examine the edges between u and x, for all $x \in V(K_n) \setminus \{z\}$.

Proposition 14 *Suppose that* $\Gamma(S)$ *is a complete graph* K_n *with an end vertex* u. *Also assume that* $u^2 = 0$ *and* $x, y \in V(K_n) \setminus \{z\}$. *Then* $ux = z$, $z^2 = 0$ *and* $x^2 = 0$ *or* $x^2 = z$.

Proof. Since u is not adjacent to x in $\Gamma(S)$, we have $ux \neq 0$. If $ux = u$, then $uy = (ux)y = u(xy) = 0$, which is impossible and so $ux \neq u$. Now let $ux = y$. We have $uy = u(ux) = u^2x = 0$ which is again impossible. Since $Z(S) = V(K_n) \cup \{0\} \cup \{u\}$, we have $ux = z$ and so $z^2 = (ux)z = u(xz) = 0$. Finally, since $ux = z$, we have $ux^2 = (ux)x = zx = 0$ and so $x^2 \in \mathrm{ann}_S(u) = \{0, u, z\}$. If $x^2 = u$, then $uy = x^2y = x(xy) = 0$, which is impossible. Therefore $x^2 = 0$ or $x^2 = z$. □

Let $u^2 = 0$. The following lemma states which vertices of $V(K_n) \setminus \{z\}$ are connected to the end vertex u in $AG(S)$

Lemma 15 *Suppose that* $\Gamma(S)$ *is a complete graph* K_n *with an end vertex* u. *Also assume that* $u^2 = 0$ *and* $x, y \in V(K_n) \setminus \{z\}$. *Then the following statements hold.*

(i) u *is adjacent to* x *in* $AG(S)$ *if and only if* $x^2 = z$.

(ii) u *is not adjacent to* x *in* $AG(S)$ *if and only if* $x^2 = 0$.

Proof. (i) By proposition 14, we have $u^2 = z^2 = uz = 0$, $ux = z$ and $x^2 = 0$ or $x^2 = z$.

First suppose that $x^2 = z$. Then $x \notin \mathrm{ann}_S(x)$. Since $ux = z$ so $x \notin \mathrm{ann}_S(u)$ and since $zx = 0$, we have $x \in \mathrm{ann}_S(z) = \mathrm{ann}_S(ux)$. Thus $\mathrm{ann}_S(x) \cup \mathrm{ann}_S(u) \neq \mathrm{ann}_S(ux)$. Therefore x is adjacent to u in $AG(S)$.

Conversely, assume that u is adjacent to x in $AG(S)$ and $x^2 = 0$. Then $\mathrm{ann}_S(x) = V(K_n)$. Also $\mathrm{ann}_S(u) = \{0, u, z\}$ hence $\mathrm{ann}_S(x) \cup \mathrm{ann}_S(u) = Z(S) = \mathrm{ann}_S(z) = \mathrm{ann}_S(ux)$. Thus u is not adjacent to x in $AG(S)$ which is impossible. Therefore $x^2 \neq 0$ and by proposition 14, $x^2 = z$.

(ii) It is clear. □

By the above lemma, we have the following theorem.

Theorem 16 *Suppose that* $\Gamma(S)$ *is a complete graph* K_n *with an end vertex* u *and* $u^2 = 0$. *Also assume that* $Z(S) \neq S$ *and* $V(K_n) = \{x_1, x_2, x_3, ., ., ., x_{n-1}, z\}$.

Then $AG(S) \cong K_{n+1} \setminus \{\{ux_1\}, \{ux_2\}, \{ux_3\}, ., ., ., \{ux_m\}\}$ *if and only if for all* $0 \leq i \leq m$ *and* $m+1 \leq j \leq n-1$*, we have* ${x_i}^2 = 0$ *and* ${x_j}^2 = z$*.*

Proof. First suppose that $AG(S) \cong K_{n+1} \setminus \{\{ux_1\}, \{ux_2\}, \{ux_3\}, ., ., ., \{ux_m\}\}$. Then for all $0 \leq i \leq m$, we have u is not adjacent to x_i in $AG(S)$ and for all $m+1 \leq j \leq n-1$, we have u is adjacent to x_j in $AG(S)$. By lemma 15, for all $0 \leq i \leq m$ and $m+1 \leq j \leq n-1$, we have ${x_i}^2 = 0$ and ${x_j}^2 = z$.

Conversely, Since $Z(S) \neq S$ by theorem 1, we have $\Gamma(S) \leq AG(S)$ and by lemma 15, for all $0 \leq i \leq m$ and $m+1 \leq j \leq n-1$, we have u is not adjacent to x_i in $AG(S)$ and u is adjacent to x_j in $AG(S)$. Therefore $AG(S) \cong K_{n+1} \setminus \{\{ux_1\}, \{ux_2\}, \{ux_3\}, ., ., ., \{ux_m\}\}$. □

If $m = 0$ or $m = 1$ or $m = n-1$, we have the following corollary.

Corollary 17 *Suppose that* $\Gamma(S)$ *is a complete graph* K_n *with an end vertex* u *and* $u^2 = 0$*. Also assume that* $Z(S) \neq S$ *and* $V(K_n) = \{x_1, x_2, x_3, ., ., ., x_{n-1}, z\}$*. Then the following statements hold.*

(i) $AG(S) \cong K_{n+1}$ *if and only if for all* $1 \leq i \leq n-1$*, we have* ${x_i}^2 = z$*.*

(ii) $AG(S) \cong K_{n+1} \setminus \{\{ux_1\}\}$ *if and only if* ${x_1}^2 = 0$ *and for all* $2 \leq i \leq n-1$*, we have* ${x_i}^2 = z$*.*

(iii) $AG(S) \cong K_n + \{uz\}$ *if and only if for all* $1 \leq i \leq n-1$*, we have* ${x_i}^2 = 0$*.*

The next corollary follows from theorem 16.

Corollary 18 *Suppose that* $\Gamma(S)$ *is a complete graph* K_n *with an end vertex* u *and* $u^2 = 0$*. Also assume that* $V(K_n) = \{x_1, x_2, x_3, ., ., ., x_{n-1}, z\}$*. Then the following statements hold.*

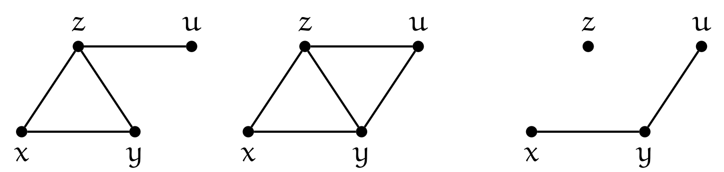
(i) If $Z(S) \neq S$*, then* $AG(S)$ *can be one of the graphs:* K_{n+1} *or* $K_{n+1} \setminus \{\{ux_1\}\}$ *or* $K_{n+1} \setminus \{\{ux_1\}, \{ux_2\}\}$ *or* $K_{n+1} \setminus \{\{ux_1\}, \{ux_2\}, \{ux_3\}\}$ *or ..... or* $K_{n+1} \setminus \{\{ux_1\}, \{ux_2\}, \{ux_3\}, ., ., ., \{ux_{n-1}\}\} = K_n + \{uz\}$

(ii) If $Z(S) = S$*, then* $AG(S)$ *can be one of the graphs:* $K_1 \cup K_n$ *or* $K_1 \cup K_n \setminus \{\{ux_1\}\}$ *or* $K_1 \cup K_n \setminus \{\{ux_1\}, \{ux_2\}\}$ *or* $K_1 \cup K_n \setminus \{\{ux_1\}, \{ux_2\}, \{ux_3\}\}$ *or ....or* $K_1 \cup K_n \setminus \{\{ux_1\}, \{ux_2\}, \{ux_3\}, ., ., ., \{ux_{n-1}\}\} = 2K_1 \cup K_{n-1}$ *with* u *and* z *are two isolated vertices.*

Proof. If $Z(S) \neq S$, by theorem 1, then $\Gamma(S) \leq AG(S)$ and if $Z(S) = S$, by theorem 2, then z is an isolated vertex in $AG(S)$. Now by theorem 16, the results hold. □

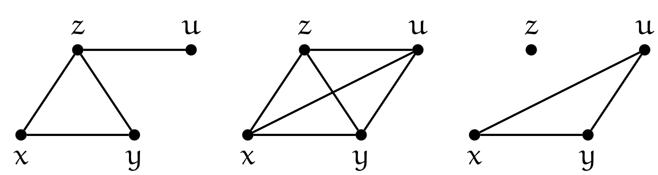
Example 19 *Suppose that $\Gamma(S)$ is a complete graph K_3 with an end vertex u and $u^2 = 0$. Also assume that $V(K_3) = \{x, y, z\}$. Then $xy = xz = yz = uz = 0$ and we have $ux = uy = z$ and $z^2 = 0$. Moreover we have one of the following three statements.*

(i) $x^2 = 0$ *and* $y^2 = z$ *or* $x^2 = z$ *and* $y^2 = 0$*. In this case if* $Z(S) \neq S$*, by lemma 6, we have* $AG(S) \cong K_4 \setminus \{\{ux\}\}$ *or* $AG(S) \cong K_4 \setminus \{\{uy\}\}$ *and if* $Z(S) = S$*, by lemma 10, we have* $AG(S) \cong K_1 \cup K_3 \setminus \{\{ux\}\}$ *or* $AG(S) \cong K_1 \cup K_3 \setminus \{\{uy\}\}$.



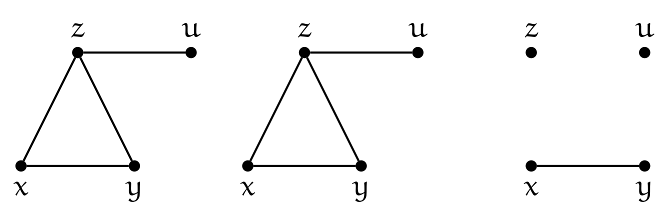
$\Gamma(S)$ $AG(S)$, $Z(S) \neq S$ $AG(Z(S))$

(ii) $x^2 = z$ *and* $y^2 = z$*. In this case if* $Z(S) \neq S$*, by lemma 7, we have* $AG(S) \cong K_4$ *and if* $Z(S) = S$*, by lemma 11, we have* $AG(S) \cong K_1 \cup K_3$.



$\Gamma(S)$ $AG(S)$, $Z(S) \neq S$ $AG(Z(S))$

(iii) $x^2 = y^2 = 0$*. In this case if* $Z(S) \neq S$*, by lemma 8, we have* $AG(S) \cong K_4 \setminus \{\{ux\}, \{uy\}\} = K_3 + \{uz\}$ *and if* $Z(S) = S$*, by lemma 9, we have* $AG(S) \cong 2K_1 \cup K_2$ *such that* u *and* z *are two isolated vertices in* $AG(S)$.



$\Gamma(S)$ $AG(S)$, $Z(S) \neq S$ $AG(Z(S))$

In the following we study the case of $u^2 = z$ and we show that u is adjacent to x, for all $x \in V(K_n) \setminus \{z\}$.

Proposition 20 *Suppose that* $\Gamma(S)$ *is a complete graph* K_n *with an end vertex* u*. Also assume that* $u^2 = z$ *and* $x, y \in V(K_n) \setminus \{z\}$*. Then* $ux = z$*,* $z^2 = 0$ *and* $x^2 = 0$ *or* $x^2 = z$*.*

Proof. Since $u^2 = z$, we have $z^2 = u^2 z = u(uz) = u0 = 0$ and since u is not adjacent to x in $\Gamma(S)$, we have $ux \neq 0$. If $ux = u$, then $uy = (ux)y = u(xy) = 0$ and if for all $y \in V(K_n) \setminus \{z\}$, we have $ux = y$, then $uy = u(ux) = u^2 x = zx = 0$ which are impossible. Thus $ux \notin \{0, u\} \cup V(K_n) \setminus \{z\}$. Therefore $ux = z$. Since $ux = z$, we have $ux^2 = (ux)x = zx = 0$ and so $x^2 \in \mathrm{ann}_S(u) = \{0, z\}$.

□

Lemma 21 *Suppose that* $\Gamma(S)$ *is a complete graph* K_n *with an end vertex* u*. Also assume that* $u^2 = z$ *and* $x \in V(K_n) \setminus \{z\}$*. Then* u *is adjacent to* x *in* $AG(S)$*.*

Proof. By proposition 20, we have $z^2 = uz = 0$, $ux = z$ and $x^2 = 0$ or $x^2 = z$. Since $z^2 = uz = 0$, we have $\mathrm{ann}_S(z) = Z(S)$. On the other hand, since $u^2 = z$ and $ux = z$, so $u \notin \mathrm{ann}_S(x) \cup \mathrm{ann}_S(u)$ which follows that $\mathrm{ann}_S(x) \cup \mathrm{ann}_S(u) \neq Z(S) = \mathrm{ann}_S(z) = \mathrm{ann}_S(ux)$. Therefore x is adjacent to u in $AG(S)$. □

By the above lemma, we have the following theorem.

Theorem 22 *Suppose that* $\Gamma(S)$ *is a complete graph* K_n *with an end vertex* u *and* $u^2 = z$*. Then the following two statements hold.*

(i) If $Z(S) \neq S$*, then* $AG(S) \cong K_{n+1}$*.*

(ii) If $Z(S) = S$*, then* $AG(S) \cong K_1 \cup K_n$*.*

Proof. (i) Since $Z(S) \neq S$ by theorem 1, we have $\Gamma(S) \leq AG(S)$. By lemma 21, for all $x \in V(K_n) \setminus \{z\}$, we have u is adjacent to x in $AG(S)$. Also by lemmas 12 and 13, for all $x, y \in V(K_n)$, we have x is adjacent to y in $AG(S)$. Therefore $AG(S) \cong K_{n+1}$.

(ii) Sincs $Z(S) = S$ by theorem 2, we have z is an isolated vertex in $AG(S)$. Now by lemmas 12, 13, 21, we have $AG(S) \cong K_1 \cup K_n$. □

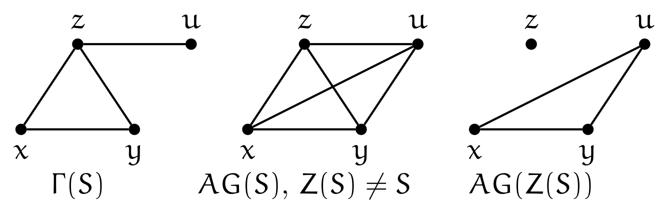
Example 23 *Suppose that* $\Gamma(S)$ *is a complete graph* K_3 *with an end vertex* u *and* $u^2 = z$*. Also assume that* $V(K_3) = \{x, y, z\}$*. Then* $xy = xz = yz = uz = 0$ *and we have* $ux = uy = z$ *and* $z^2 = 0$*. Moreover we have one of the following three statements.*

(i) $x^2 = 0$ *and* $y^2 = z$ *or* $x^2 = z$ *and* $y^2 = 0$*.*

(ii) $x^2 = y^2 = z$.

(iii) $x^2 = y^2 = 0$.

In three cases if $Z(S) \neq S$, *by lemma 7, we have* $AG(S) \cong K_4$ *and if* $Z(S) = S$, *by ,lemma 11, we have* $AG(S) \cong K_1 \cup K_3$



In the following we study the case of $u^2 = u$ and we show that there is at least one vertex $y \in V(K_n)$ such that u is not adjacent to y in $AG(S)$ and so in this case $AG(S)$ is not a complete graph.

Proposition 24 *Suppose that* $\Gamma(S)$ *is a complete graph* K_n *with an end vertex* u. *Also assume that* $u^2 = u$ *and* x, y *are two distincet vertices in* $V(K_n) \setminus \{z\}$. *Then* $z^2 = 0$ *or* $z^2 = z$. *and* $ux \in Z(S) \setminus \{0, z, u\} = V(K_n) \setminus \{z\}$. *Also we have the following two statements.*

(i) If $ux = x$, *then* $x^2 = 0$ *or* $x^2 = x$ *or* $x^2 = y$ *and* $uy = y$.

(ii) If $ux = y$, *then* $uy = y$ *and* $y^2 = 0$ *and also* $x^2 = 0$ *or* $x^2 = z$.

Proof. Since u is not adjacent to x in $\Gamma(S)$, we have $ux \neq 0$. If $ux = z$, then $z = ux = u^2x = u(ux) = uz = 0$ this is impossible and if $ux = u$, then $uy = (ux)y = u(xy) = 0$ which is again impossible. So $ux \notin \{0, z, u\}$ and thus $ux \in Z(S) \setminus \{0, z, u\} = V(K_n) \setminus \{z\}$.

(i) Also suppose that $ux = x$. Then $ux^2 = x^2$. If $x^2 = z$, then $z = uz = 0$ and if $x^2 = u$, then $uy = x^2y = x(xy) = 0$ which are impossible. So $x^2 \notin \{z, u\}$ and thus $x^2 \in Z(S) \setminus \{z, u\} = V(K_n) \setminus \{z\}$. Therefore $x^2 = 0$ or $x^2 = x$ or $x^2 = y$. Also if $x^2 = y$, then $uy = ux^2 = (ux)x = x^2 = y$.

(ii) Now assume that $ux = y$. Then $y^2 = (ux)y = u(xy) = 0$ and $uy = u(ux) = u^2x = ux = y$. Since $ux^2 = (ux)x = yx = 0$, we have $x^2 \in \text{ann}_S(u) = \{0, z\}$ and thus $x^2 = 0$ or $x^2 = z$. □

Let $u^2 = u$. The following lemma states which vertices of $V(K_n) \setminus \{z\}$ are connected to the end vertex u in $AG(S)$.

Lemma 25 *Suppose that* $\Gamma(S)$ *is a complete graph* K_n *with an end vertex* u. *Also assume that* $u^2 = u$ *and* $x, y \in V(K_n) \setminus \{z\}$. *Then the following two statements hold.*

(i) u *is adjacent to* x *in* $AG(S)$ *if and only if* $ux = y$ *and* $x^2 = z$.

(ii) u *is not adjacent to* x *in* $AG(S)$ *if and only if* $ux = x$ *or* $ux = y$ *and* $x^2 = 0$. *Moreover if* $ux = y$, *then in both cases* $x^2 = z$, *and* $x^2 = 0$ *we have* u *is not adjacent to* y *in* $AG(S)$.

Proof.
(i) First suppose that u is adjacent to x in $AG(S)$. Then $ux \neq x$ and by proposition 24, $ux = y$ and $y^2 = 0$ and $x^2 = 0$ or $x^2 = z$. If $x^2 = 0$, then $ann_S(x) \cup ann_S(u) = V(K_n) \cup \{0, z\} = V(K_n) \cup \{0\} = ann_S(y) = ann_S(ux)$ and so u is not adjacent to x in $AG(S)$ this is impossible. Therefore $x^2 \neq 0$ and so $x^2 = z$.

Conversely, assum that $ux = y$ and $x^2 = z$. Then $x \notin ann_S(x) \cup ann_S(u)$ and $x \in ann_S(y)$ and so u is adjacent to x in $AG(S)$.

(ii) First suppose that u is not adjacent to x in $\Gamma(S)$ and $ux \neq x$. Then by proposition 24, we have $ux = y$, $y^2 = 0$ and also $x^2 = 0$ or $x^2 = z$. If $x^2 = z$, then u is adjacent to x in $AG(S)$ this is impossible. Therefore $x^2 = 0$.

Conversely, if $ux = x$, then u is not adjacent to x in $AG(S)$. Now assume that $ux \neq x$. Then by proposition 24, we have $ux = y$, $y^2 = 0$ and since $x^2 = 0$, we have $ann_S(x) = ann_S(y) = ann_S(ux)$ and so u is not adjacent to x in $AG(S)$.

Moreover if $ux = y$, then $uy = u(ux) = u^2x = ux = y$ and so u is not adjacent to y in $AG(S)$. □

By proposition 24, for all $x \in V(K_n) \setminus \{z\}$, we have $ux \in Z(S) \setminus \{0, z, u\} = V(K_n) \setminus \{z\}$ and $ux = x$ or there is $y \in V(K_n) \setminus \{z, x\}$ that $ux = y$ and $uy = y$. So u is not adjacent to x in $AG(S)$ or u is not adjacent to y in $AG(S)$. Therefore there is at least one vertex $x \in V(K_n) \setminus \{z\}$ that is not adjacent to u in $AG(S)$ and thus $AG(S)$ is not a commplete graph.

Corollary 26 *Suppose that* $\Gamma(S)$ *is a complete graph* K_n *with an end vertex* u. *Also assume that* $u^2 = u$. *Then* $AG(S)$ *is not a complete graph.*

Theorem 27 *Suppose that* $\Gamma(S)$ *is a complete graph* K_n *with an end vertex* u *and* $u^2 = u$. *Also assume that* $Z(S) \neq S$ *and* $V(K_n) = \{x_1, x_2, x_3, ., ., ., x_{n-1}, z\}$. *Then* $AG(S) \cong K_{n+1} \setminus \{\{ux_1\}, \{ux_2\}, \{ux_3\}, ., ., ., \{ux_m\}\}$ *if and only if for all* $1 \leq i \leq m$ *and* $m + 1 \leq j \leq n - 1$, *we have the following two statements.*

(i) *either* $ux_i = x_i$ *or* $ux_i = x_t$ *and* $1 \le t \le m$ *also* ${x_i}^2 = 0$.

(ii) $ux_j = x_i$ *and* ${x_j}^2 = z$

Proof. (i) First suppose that $AG(S) \cong K_{n+1} \setminus \{\{ux_1\}, \{ux_2\}, \{ux_3\}, ., ., ., \{ux_m\}\}$. Then u is not adjacent to x_i in $AG(S)$ and by lemma 25 for all $1 \le i \le m$, we have $ux_i = x_i$ or $ux_i = x_t$ and ${x_i}^2 = 0$. Moreover if $ux_i = x_t$, then $ux_t = x_t$ and so u is not adjacent to x_t in $AG(S)$. Thus $1 \le t \le m$.

(ii)Since u is adjacent to x_j in $AG(S)$ by lemma 25 for all $m+1 \le j \le n-1$, we have $ux_j = x_t$ and ${x_j}^2 = z$. Also if $ux_j = x_t$, then $ux_t = x_t$ and so u is not adjacent to x_t in $AG(S)$. Thus $1 \le t \le m$. Therefore $ux_j = x_i$.

Conversely, by lemma 25, if statement (i) holds, then for all $1 \le i \le m$, we have u is not adjacent to x_i in $AG(S)$ and if statement (ii) holds, then for all $m + 1 \le j \le n - 1$, we have u is adjacent to x_j in $AG(S)$ and so $AG(S) \cong K_{n+1} \setminus \{\{ux_1\}, \{ux_2\}, \{ux_3\}, ., ., ., \{ux_m\}\}$. □

In the above theorem, since $AG(S)$ is not a commplete graph so $m \neq 0$. If $m = 1$ or $m = n - 1$, then we have the following corollary.

Corollary 28 *Suppose that* $\Gamma(S)$ *is a complete graph* K_n *with an end vertex* u *and* $u^2 = u$. *Also assume that* $Z(S) \neq S$ *and* $V(K_n) = \{x_1, x_2, x_3, ., ., ., x_{n-1}, z\}$. *Then the following statements hold.*

(i) $AG(S) \cong K_{n+1} \setminus \{\{ux_1\}\}$ *if and only if* $ux_1 = x_1$ *and for all* $2 \le i \le n-1$, *we have* $ux_i = x_1$ *and* ${x_1}^2 = 0$ *and* ${x_i}^2 = z$.

(ii) $AG(S) \cong K_n + \{uz\}$ *if and only if for all* $1 \le i, j \le n - 1$, *we have if* $ux_i \neq x_i$, *then* $ux_i = x_j$ *and* ${x_i}^2 = {x_j}^2 = 0$.

The next corollary follows from theorem 27.

Corollary 29 *Suppose that* $\Gamma(S)$ *is a complete graph* K_n *with an end vertex* u *and* $u^2 = u$. *Also assume that* $V(K_n) = \{x_1, x_2, x_3, ., ., ., x_{n-1}, z\}$. *Then the following statements hold.*

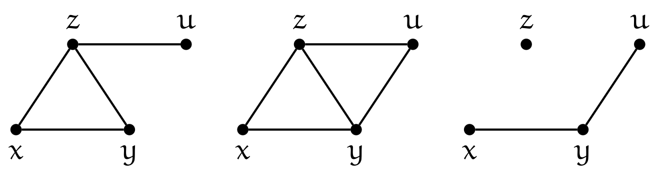
(i) *If* $Z(S) \neq S$, *then* $AG(S)$ *can be one of the graphs:* $K_{n+1} \setminus \{\{ux_1\}\}$ *or* $K_{n+1} \setminus \{\{ux_1\}, \{ux_2\}\}$ *or* $K_{n+1} \setminus \{\{ux_1\}, \{ux_2\}, \{ux_3\}\}$ *or,...... ,or* $K_{n+1} \setminus \{\{ux_1\}, \{ux_2\}, \{ux_3\}, ., ., ., \{ux_{n-1}\}\}$.

(ii) *If* $Z(S) = S$, *then* $AG(S)$ *can be one of the graphs:* $K_1 \cup K_n \setminus \{\{ux_1\}\}$ *or* $K_1 \cup K_n \setminus \{\{ux_1\}, \{ux_2\}\}$ *or* $K_1 \cup K_n \setminus \{\{ux_1\}, \{ux_2\}, \{ux_3\}\}$ *or,......,or* $K_1 \cup K_n \setminus \{\{ux_1\}, \{ux_2\}, \{ux_3\}, ., ., ., \{ux_{n-1}\}\} = 2K_1 \cup K_{n-1}$ *with* u *and* z *are two isolated vertices.*

Proof. By corollary 26, $AG(S)$ is not a complete graph. If $Z(S) \neq S$ by theorem 1, we have $\Gamma(S) \leq AG(S)$ and if $Z(S) = S$ by theorem 2, we have z is an isolated vertex in $AG(S)$. Now by theorem 27, the results hold. □

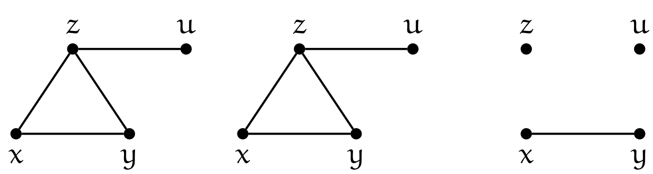
Example 30 *Suppose that $\Gamma(S)$ is a complete graph K_3 with an end vertex u and $u^2 = u$. Also assume that $V(K_3) = \{x, y, z\}$. Then $xy = xz = yz = uz = 0$ and $z^2 = 0$ or $z^2 = z$. Moreover we have one of the following three statements.*

(i) $ux = y = uy$, $x^2 = z$, $y^2 = z^2 = 0$ *or* $ux = x = uy$, $y^2 = z$ *and* $x^2 = z^2 = 0$,. *In this case if* $Z(S) \neq S$, *by lemma 6, we have* $AG(S) \cong K_4 \setminus \{\{uy\}\}$ *or* $AG(S) \cong K_4 \setminus \{\{ux\}\}$ *and if* $Z(S) = S$, *by lemma 10, we have* $AG(S) \cong K_1 \cup K_3 \setminus \{\{uy\}\}$ *or* $AG(S) \cong K_1 \cup K_3 \setminus \{\{ux\}\}$.



$\Gamma(S)$ $AG(S)$, $Z(S) \neq S$ $AG(Z(S))$

(ii) $z^2 \in \{0, z\}$ *and* $ux = y = uy$, $x^2 = y^2 = 0$, *or* $ux = x = uy$, $x^2 = y^2 = 0$ *or* $ux = y$, $uy = x$, $x^2 = y^2 = 0$. *In this case if* $Z(S) \neq S$, *by lemma 8, we have* $AG(S) \cong K_4 \setminus \{\{ux\}, \{uy\}\} = K_3 + \{uz\}$ *and if* $Z(S) = S$, *by lemma 9, we have* $AG(S) \cong 2K_1 \cup K_2$ *such that* u *and* z *are two isolated vertices in* $AG(S)$.



$\Gamma(S)$ $AG(S)$, $Z(S) \neq S$ $AG(Z(S))$

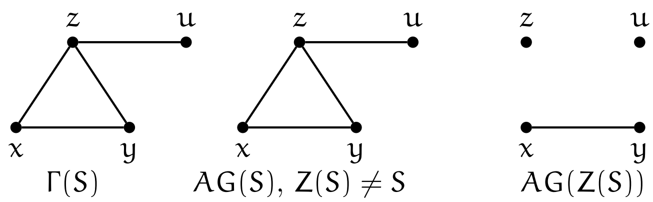
(iii) $z^2 \in \{0, z\}$ *and* $ux = x$, $uy = y$ *and we have the following six cases.*

(1) $y^2 = 0$, *and* $x^2 \in \{0, x, y\}$.

(2) $y^2 = y$,*and* $x^2 \in \{0, x\}$.

(3) $y^2 = x$, *and* $x^2 = 0$.

In this case if $Z(S) \neq S$*, by lemma 8, we have* $AG(S) \cong K_4 \setminus \{\{ux\}, \{uy\}\} = K_3 + \{uz\}$ *and if* $Z(S) = S$*, by lemma 9, we have* $AG(S) \cong 2K_1 \cup K_2$ *such that* u *and* z *are two isolated vertices in* $AG(S)$*.*



Finally, we study the case of $u^2 \notin \{0, z, u\}$ and so $u^2 = b \in V(K_n) \setminus \{z\}$. we show that u is adjacent to all vertices $y \in V(K_n) \setminus \{z, b\}$ in $AG(S)$ and u is adjacent to b in $AG(S)$ if and only if $ub \neq b$.

Proposition 31 *Suppose that* $\Gamma(S)$ *is a complete graph* K_n *with an end vertex* u*. Also assume that* $u^2 = b \notin \{0, z, u\}$*. Then* $u^2 = b \in V(K_n) \setminus \{z\}$ *and we have the following two statements.*

(i) For all $x \in Z(S) \setminus \{0, z, u, b\}$*, we have* $ux = z$*,* $z^2 = 0$ *and* $x^2 = 0$ *or* $x^2 = z$*.*

(ii) $ub = b$ *and* $b^2 = b$*, or* $ub = z$ *and* $b^2 = 0$ *or* $ub = y \in Z(S) \setminus \{0, z, u, b\}$ *and* $b^2 = z$*,* $y^2 = 0$*.*

Proof. (i) Suppose that $u^2 = b$. For all $x \in Z(S) \setminus \{0, z, u, b\}$, since u is not adjacent to x in $\Gamma(S)$, we have $ux \neq 0$. If $ux = u$, then $ub = (ux)b = u(xb) = 0$ which is impossible. For all $y \in V(K_n) \setminus \{z\}$, if $ux = y$, then $uy = u(ux) = u^2x = bx = 0$ which is again impossible and so $ux \notin \{0, u\} \cup V(K_n) \setminus \{z\}$. Therefore $ux = z$ and $z^2 = (ux)z = u(xz) = 0$.

Since $ux = z$, we have $ux^2 = (ux)x = zx = 0$ and so $x^2 \in \text{ann}_S(u) = \{0, u\}$. Therefore $x^2 = 0$ or $x^2 = z$.

(ii) Clearly $ub \neq 0$ and $ub \neq u$ so $ub \in V(K_n)$ and thus $ub = b$ or $ub = z$ or $ub = y \in V(K_n) \setminus \{z, b\}$. Since $u^2 = b$, if $ub = b$, then $u^3 = uu^2 = ub = b = u^2$ and so $u^4 = u^3 = u^2$. Thus $b^2 = u^4 = u^3 = u^2 = b$ and if $ub = z$ we have $b^2 = bb = u^2b = u(ub) = uz = 0$.

Now assume that $ub = y \in Z(S) \setminus \{0, z, u, b\}$. Then $y^2 = (ub)y = u(by) = 0$. Since $uy = z$, we have $b^2 = bb = u^2b = u(ub) = uy = z$. □

Lemma 32 *Suppose that* $\Gamma(S)$ *is a complete graph* K_n *with an end vertex* u*. Also assume that* $u^2 = b \in V(K_n) \setminus \{z\}$ *and* $x \in V(K_n) \setminus \{z, b\}$*. Then* u *is* *adjacent to* x *in* $AG(S)$*. Moreover* u *is adjacent to* b *in* $AG(S)$ *if and only if* $ub \neq b$*.*

Proof. By proposition 31, For all $x \in V(K_n) \setminus \{z, b\}$, we have $ux = z$. Since $u^2 = b$ so $u \notin ann_S(x) \cup ann_S(u)$ and $u \in ann_S(z) = ann_S(ux)$. Thus u is adjacent to x in $AG(S)$.

Moreover if u is adjacent to b in $AG(S)$, then $ub \neq b$.

Conversely assume that $ub \neq b$. By proposition 31, we have $ub = z$ and $b^2 = 0$ or $ub = y \in Z(S) \setminus \{0, z, u, b\}$ and $b^2 = z$, $y^2 = 0$.

If $ub = z$ and $b^2 = 0$, then $u \notin ann_S(b) \cup ann_S(u)$ and $u \in ann_S(z) = ann_S(ub)$. Thus u is adjacent to b in $AG(S)$. Also if $ub = y \in Z(S) \setminus \{0, z, u, b\}$ and $b^2 = z$, $y^2 = 0$, then $b \notin ann_S(b) \cup ann_S(u)$ and $b \in ann_S(y) = ann_S(ub)$. Therefore u is adjacent to b in $AG(S)$. □

Corollary 33 *Suppose that* $\Gamma(S)$ *is a complete graph* K_n *with an end vertex* u*. Also assume that* $u^2 = b \in V(K_n) \setminus \{z\}$ *and* $Z(S) \neq S$*. Then* $AG(S)$ *is not a commplete graph if and only if* $ub = b$*.*

Theorem 34 *Suppose that* $\Gamma(S)$ *is a complete graph* K_n *with an end vertex* u *and* $u^2 = b \in V(K_n) \setminus \{z\}$*. Then the following statements hold.*

(i) If $Z(S) \neq S$*, then we have two cases.*

  (1) $AG(S) \cong K_{n+1}$ *if and only if* $ub \neq b$*.*

  (2) $AG(S) \cong K_{n+1} \setminus \{\{ub\}\}$ *if and only if* $ub = b$*.*

(ii) If $Z(S) = S$*, then we have two cases.*

  (1) $AG(S) \cong K_1 \cup K_n$ *if and only if* $ub \neq b$*.*

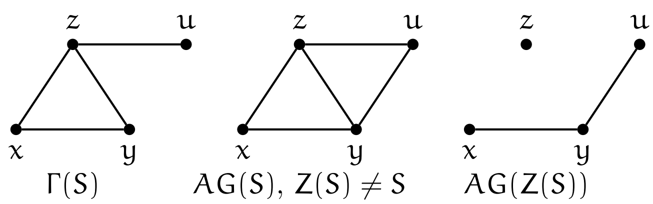
  (2) $AG(S) \cong K_1 \cup K_n \setminus \{\{ub\}\}$ *if and only if* $ub = b$*.*

Proof. If $Z(S) \neq S$, then $\Gamma(S) \leq AG(S)$. By lemma 32, for all $x \in V(K_n) \setminus \{z, b\}$, we have u is adjacent to x in $AG(S)$ and u is adjacent to b in $AG(S)$ if and only if $ub \neq b$. Thus the statement (i) holds.

(ii) Since $Z(S) = S$, we have z is an isolated vertex in $AG(S)$. Now by lemma 32, the results hold. □

Example 35 *Suppose that* $\Gamma(S)$ *is a complete graph* K_3 *with an end vertex* u *and* $u^2 = x$ *or* $u^2 = y$*. Also assume that* $V(K_3) = \{x, y, z\}$*. Then* $xy = xz = yz = uz = 0$ *and* $z^2 = 0$*. Moreover we have one of the following two statements.*

(i) $ux = z$, $uy = y$, $u^2 = y^2 = y$, *and* $x^2 = 0$ *or* $x^2 = z$ *or* $uy = z$, $ux = x$, $u^2 = x^2 = x$, *and* $y^2 = 0$ *or* $y^2 = z$. *In this case if* $Z(S) \neq S$, *by lemma 6, we have* $AG(S) \cong K_4 \setminus \{\{uy\}\}$ *or* $AG(S) \cong K_4 \setminus \{\{ux\}\}$ *and if* $Z(S) = S$, *by lemma 10, we have* $AG(S) \cong K_1 \cup K_3 \setminus \{\{uy\}\}$ *or* $AG(S) \cong K_1 \cup K_3 \setminus \{\{ux\}\}$.



(ii) *Also we have the following four cases.*

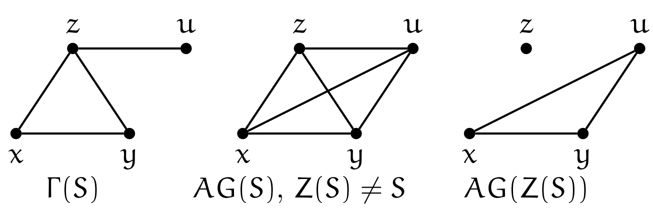
(1) $ux = y$, $uy = z$, $u^2 = x$, *and* $x^2 = y^2 = 0$.

(2) $ux = z$, $uy = x$, $u^2 = y$, *and* $x^2 = 0$, $y^2 = z$.

(3) $ux = uy = z$, $u^2 = y$, $y^2 = 0$ *and* $x^2 = 0$ *or* $x^2 = z$.

(4) $ux = uy = z$, $u^2 = x$, $x^2 = 0$ *and* $y^2 = 0$ *or* $y^2 = z$.

In this case if $Z(S) \neq S$, *by lemma 7, we have* $AG(S) \cong K_4$ *and if* $Z(S) = S$, *by lemma 11, we have* $AG(S) \cong K_1 \cup K_3$ *such that* u *is an isolated vertex in* $AG(S)$.



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References

[1] M. Afkhami, K. Khashyarmanesh, M. Sakhdari, On the annihilator graphs of semigroups, *J. Algebra and its Appl.* **1**4 (2015) 1550015 – 1550029. ⇒120, 121, 123

[2] M. Afkhami, K. Khashyarmanesh, M. Sakhdari, Annihilator graphs with four vertices, *Semigroup Forum.* **9**4 (2017) 139 – 166. ⇒120, 121, 122, 123

[3] M. Afkhami, K. Khashyarmanesh, M. Sakhdari, Annihilator graph of a commutative semigroup whose zero-divisor graph is a refinement of a star graph, *Quasigroups and Related Systems.* **2**9 (2021) 157–170. ⇒120

[4] J. A. Bondy and U. S. R. Murty, *Graph Theory with Applications,* American Elsevier, New York, 1976. ⇒121

[5] F. DeMeyer, L. DeMeyer, Zero-divisor graphs of semigroups, *J. Algebra.* **2**83 (2005) 190–198. ⇒120

[6] L. DeMeyer, M. Dsa, I. Epstein, A. Geiser and K. Smith, Semigroups and the zero-divisor graph, *Bull. Inst. Comb. Appl.* **57** (2009) 60–70. ⇒120

[7] F. DeMeyer, T. Mckenzie, K. Schneider, The zero-divisor graph of a commutative semigroup, *Semigroup Forum.* **6**5 (2002) 206–214. ⇒120

[8] T. Wu, D. Lu, Subsemigroups determined by the zero-divisor graph, *Discrete Math.* **3**08 (2008) 5122–5135. ⇒120

[9] T. Wu, F. Cheng, The structure of zero-divisor semigroups with graph $K_n \circ K_2$, *Semigroup Forum.* **7**6 (2008) 330–340. ⇒120

[10] T. Wu, Q. Liu, L. Chen, Zero-divisor semigroups and refinements of a star graph, *Discrete Math.* **3**09 (2009) 2510–2518. ⇒120

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