Acta Univ. Sapientiae Informatica **15,** 1 (2023) 38–45

DOI: 10.2478/ausi-2023-0004

On the spread of the distance signless Laplacian matrix of a graph

S. PIRZADA
Department of Mathematics,
University of Kashmir, Srinagar,
India
email:
pirzadasd@kashmiruniversity.ac.in

Mohd Abrar Ul HAQ
Department of Mathematics,
University of Kashmir, Srinagar,
India
email: abrar14789@gmail.com

Abstract. Let G be a connected graph with n vertices, m edges. The distance signless Laplacian matrix $D^Q(G)$ is defined as $D^Q(G) = \text{Diag}(Tr(G)) + D(G)$, where $\text{Diag}(Tr(G))$ is the diagonal matrix of vertex transmissions and $D(G)$ is the distance matrix of G. The distance signless Laplacian eigenvalues of G are the eigenvalues of $D^Q(G)$ and are denoted by $\partial_1^Q(G), \partial_2^Q(G), \ldots, \partial_n^Q(G)$. ∂_1^Q is called the distance signless Laplacian spectral radius of $D^Q(G)$. In this paper, we obtain upper and lower bounds for $S_{D^Q}(G)$ in terms of the Wiener index, the transmission degree and the order of the graph.

Key words and phrases: distance matrix; distance signless Laplacian matrix; distance signless Laplacian eigenvalues; spread; Wiener index; transmission degree

1 Introduction

Let G be a connected simple graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\})$ and edges set $E(G)$. In G, the distance $d(v_i, v_j)$ between the vertices v_i and v_j is the length of (number of edges) the shortest path that connects v_i and v_j. The diameter of G is the maximum distance between any two vertices of G. The distance matrix of G is an $n \times n$ matrix in which the $(i, j)^{th}$-entry is equal to the distance between vertices v_i and v_j, that is, $D_{i,j}(G) = d_{i,j} = d(v_i, v_j)$. For more definitions and notations, we refer to [10].

In G, the distance degree of a vertex v, denoted by $Tr_G(v)$, is defined to be the sum of the distances from v to all other vertices in G, that is, $Tr_G(v) = \sum_{u \in V(G)} d(u, v)$. We can also write $Tr_G(v_i)$ as Tr_i. A graph G is said to be k-transmission regular if $Tr_i = k$, for each $i = 1, 2, \ldots, n$. The transmission degree sequence is given by $\{Tr_1, Tr_2, Tr_3, \ldots, Tr_n\}$. The second transmission degree of v_i, denoted by T_i, is given by $T_i = \sum_{j=1}^{n} d_{ij} Tr_j$. The Wiener index of graph G, denoted by $W(G)$, is the sum of the distances between all unordered pairs of vertices in G, that is,

$$W(G) = \frac{1}{2} \sum_{u,v \in V(G)} d(u, v) = \frac{1}{2} \sum_{u \in V(G)} Tr_G(v).$$

Let $Tr_G = diag(Tr_1, Tr_2, \ldots, Tr_n)$ be the diagonal matrix of vertex transmissions of G. Aouchiche and Hansen [5] introduced the Laplacian and the signless Laplacian for the distance matrix of a connected graph G. The matrix $DQ(G) = Tr(G) + D(G)$ (or simply D^Q) is called the distance signless Laplacian matrix of G. Since $DQ(G)$ is symmetric (positive semi-definite), its eigenvalues can be arranged as: $\partial_1^Q(G) \geq \partial_2^Q(G) \geq \cdots \geq \partial_n^Q(G)$, where $\partial_1^Q(G)$ is called the distance signless Laplacian spectral radius of G. If $\partial_i^Q(G)$ is repeated p times, then we say that the multiplicity of $\partial_i^Q(G)$ is p and we write $m(\partial_i^Q(G)) = p$. As $D^Q(G)$ is nonnegative and irreducible, by the Perron-Frobenius theorem, $\partial_1^Q(G)$ is positive, simple and there is a unique positive unit eigenvector X corresponding to $\partial_1^Q(G)$, which is called the distance signless Laplacian Perron vector of G. The distance signless Laplacian spread of a graph G, denoted by $S_{DQ}(G)$, is defined as $S_{DQ}(G) = \partial_1^Q(G) - \partial_n^Q(G)$, where $\partial_1^Q(G)$ and $\partial_n^Q(G)$ are respectively the largest and the smallest distance signless Laplacian eigenvalues of G. Some recent work on distance signless Laplacian eigenvalues can be seen in [1, 4, 8, 11, 12, 13, 14].

The rest of the paper is organized as follows. In Section 2, we obtain lower and upper bounds for $S_{DQ}(G)$ in terms of the Wiener index $W(G)$, the transmission Tr and the order n of G.

2 Bounds for spread of distance signless Laplacian matrix

For a graph G with n vertices, let $Tr_{max}(G) = max\{Tr(v) : v \in V(G)\}$ and $Tr_{min}(G) = min\{Tr(v) : v \in V(G)\}$. Whenever the graph G is understood, we will write Tr_{max} and Tr_{min} in place of $Tr_{max}(G)$ and $Tr_{min}(G)$, respectively. From the definitions, we have $2W(G) = \partial_1^Q + \partial_2^Q + \cdots + \partial_n^Q$. Also, $Tr_{max} \geq \frac{2W(G)}{n}$ and $Tr_{min} \leq \frac{2W(G)}{n}$, where $\frac{2W(G)}{n}$ is the average transmission degree. First we note the following observations.

Lemma 1 [2] *Let* G *be a simple, connected graph. Then*

$$\frac{Tr_{min} + \sqrt{Tr_{min}^2 + 8T_{min}}}{2} \leq \partial_1^Q(G) \leq \frac{Tr_{max} + \sqrt{Tr_{max}^2 + 8T_{max}}}{2},$$

equality hold if and only if the graph is transmission regular.

Lemma 2 [6] *Let* G *be a connected graph with minimum and maximum transmissions* Tr_{min} *and* Tr_{max}*. Then* $2Tr_{min} \leq \partial_1^Q(G) \leq 2Tr_{max}$*, and the equality hold if and only if* G *is transmission regular.*

Now, we obtain bounds for the distance signless Laplacian spread $S_{DQ}(G)$ of a graph G in terms of the Wiener index $W(G)$, the order n, the maximum transmission degree $Tr_{max}(G)$ and the minimum transmission degree Tr_{min} of G.

Theorem 3 *Let G be a connected graph with n vertices having Wiener index W(G). Then*

$$\frac{n(Tr_{min} + \sqrt{Tr_{min}^2 + 8T_{min}}) - 4W(G)}{2(n-1)} \leq S_{DQ}(G)$$
$$< \frac{n(Tr_{max} + \sqrt{Tr_{max}^2 + 8T_{min}}) - 4W(G)}{2}.$$

Equality holds in the left if and only if $G \cong K_n$.

Proof. Let $\partial_1^Q(G), \partial_2^Q(G), \ldots, \partial_n^Q(G)$ be $D^Q(G)$-eigenvalues. Then we have

$$2W(G) = \partial_1^Q(G) + \partial_2^Q(G) + \cdots + \partial_n^Q(G) \geq \partial_1^Q(G) + (n-1)\partial_n^Q(G),$$

which implies that $\partial_n^Q(G) \leq \frac{2W(G)-\partial_1^Q(G)}{n-1}$, with equality if and only if $\partial_2^Q(G) = \partial_3^Q(G) = \cdots = \partial_n^Q(G)$. For equality, consider the following two cases.

Case 1. Clearly, $\partial_1^Q(G) = \partial_2^Q(G) = \partial_3^Q(G) = \cdots = \partial_n^Q(G)$, is not possible, since the spectral radius of D^Q is always simple.

Case 2. $\partial_1^Q(G) > \partial_2^Q(G)$ and $\partial_2^Q(G) = \partial_3^Q(G) = \cdots = \partial_n^Q(G)$. Then $G \cong K_n$, as K_n is the unique graph having only two distinct distance signless Laplacian eigenvalues. Therefore,

$$\begin{aligned} S_{D^Q}(G) = \partial_1^Q(G) - \partial_n^Q(G) &\geq \partial_1^Q(G) - \frac{2W(G) - \partial_1^Q(G)}{n-1} \\ &= \frac{(n-1)\partial_1^Q(G) - 2W(G) - \partial_1^Q(G)}{n-1} \\ &= \frac{n\partial_1^Q(G) - 2W(G)}{n-1}. \end{aligned}$$

Now, using Lemma 1, we get

$$\begin{aligned} S_{D^Q}(G) &\geq \frac{n(\frac{Tr_{min}+\sqrt{Tr_{min}^2+8T_{min}})}{2} - 2W(G)}{n-1} \\ &= \frac{n(Tr_{min} + \sqrt{Tr_{min}^2 + 8T_{min}}) - 4W(G)}{2(n-1)}, \end{aligned}$$

with equality if and only if $G \cong K_n$. Also, we have $2W(G) = \partial_1^Q(G) + \partial_2^Q(G) + \cdots + \partial_n^Q(G) \leq (n-1)\partial_1^Q(G) + \partial_n^Q(G)$. We observe that the above inequality is strict as the distance signless Laplacian spectral radius is always simple, that is, $\partial_n^Q(G) \geq 2W(G) - (n-1)\partial_1^Q(G)$. Therefore,

$$S_{D^Q}(G) = \partial_1^Q(G) - \partial_n^Q(G) < \partial_1^Q(G) - 2W(G) + (n-1)\partial_1^Q(G).$$

By using Lemma 1, we get

$$\begin{aligned} S_{DQ}(G) &\leq \frac{n(Tr_{max} + \sqrt{Tr_{max}^2 + 8T_{min}})}{2} - 2W(G) \\ &= \frac{n(Tr_{max} + \sqrt{Tr_{max}^2 + 8T_{max}}) - 4W(G)}{2} \end{aligned}$$

and we get the desired result. □

The following lemma will be used in the next theorem.

Lemma 4 [15] *Let* G *be a connected graph on n vertices. Then* $\partial_1^Q(G) \geq \frac{4W(G)}{n}$ *with equality holding if and only if* G *is transmission regular.*

Theorem 5 *Let* G *be a connected graph of order* n*. Then* $S_{DQ}(G) \geq \frac{2W(G)}{n-1}$*, and equality holds if and only if* $G \cong K_n$*.*

Proof. If $\partial_1^Q(G), \partial_2^Q(G), \ldots, \partial_n^Q(G)$ are $D^Q(G)$-eigenvalues, then we have

$$2W(G) = \partial_1^Q(G) + \partial_2^Q(G) + \cdots + \partial_n^Q(G) \geq \partial_1^Q(G) + (n-1)\partial_n^Q(G),$$

which implies that $\partial_n^Q \leq \frac{2W(G) - \partial_1^Q(G)}{n-1}$, with equality if and only if $G \cong K_n$. Therefore,

$$\begin{aligned} S_{DQ}(G) = \partial_1^Q(G) - \partial_n^Q(G) &\geq \partial_1^Q(G) - \frac{2W(G) - \partial_1^Q(G)}{n-1} \\ &= \frac{(n-1)\partial_1^Q(G) - 2W(G) + \partial_1^Q(G)}{n-1} \\ &= \frac{n\partial_1^Q(G) - 2W(G)}{n-1} \end{aligned}$$

Using Lemma 4, we get $S_{DQ}(G) = \partial_1^Q(G) - \partial_n^Q(G) \geq \frac{2W(G)}{n-1}$, equality holds if and only if $G \cong K_n$. □

Since $D^Q(G)$ is nonnegative and irreducible, by the Perron-Frobenius theorem, ∂_1^Q is positive, simple and there is a unique positive unit eigenvector X corresponding to ∂_1^Q. Using Lemma 4 and the fact that $\partial_1^Q(G) \geq$

$\frac{2\sqrt{\sum_{i=1}^{n} Tr_i^2}}{n}$, equality hold if and only if G is transmission degree regular graph [9], we get

$$S_{D^Q}(G) \geq \frac{2(n-1)\sqrt{\sum_{i=1}^{n} Tr_i^2} - 2W(G)}{n-1},$$

and equality holds if and only if G is transmission degree regular graph.

Lemma 6 [3] *If the transmission degree sequence of* G *is* $\{Tr_1, Tr_2 \ldots, Tr_n\}$, *then*

$$\sum_{i=1}^{n} \partial_i^Q(G)^2 = 2\sum_{1\leq i<j\leq n} (d_{ij})^2 + \sum_{i=1}^{n} {Tr_i}^2.$$

Theorem 7 *Let G be a connected graph with n vertices. Then*

$$S_{D^Q}(G) \geq 2Tr_{min} - \sqrt{\frac{R_1 - 4Tr_{min}^2}{n-1}},$$

and equality holds if and only if $G \cong K_n$.

Proof. From Lemma 6, we have $\sum_{i=1}^{n} \partial_i^Q(G)^2 = 2\sum_{1\leq i<j\leq n}(d_{ij})^2 + \sum_{i=1}^{n} {Tr_i}^2 = R_1$. Clearly, $R_1 = \sum_{i=1}^{n} \partial_i^Q(G)^2 \geq \partial_1^Q(G)^2 + (n-1)\partial_n^Q(G)^2$, which implies that $\partial_n^Q(G) \leq \sqrt{\frac{R_1 - \partial_1^Q(G)^2}{n-1}}$, with equality if and only if $G \cong K_n$. By using this inequality for $\partial_n^Q(G)$, we have

$$S_{D^Q}(G) = \partial_1^Q(G) - \partial_n^Q(G) \geq \partial_1^Q(G) - \sqrt{\frac{R_1 - \partial_1^Q(G)^2}{n-1}}$$

Now, using Lemma 2, we get

$$S_{D^Q}(G) \geq 2Tr_{min} - \sqrt{\frac{R_1 - 4Tr_{min}^2}{n-1}},$$

which is the required inequality. Clearly, the equality holds if and only if $G \cong K_n$. □

Remarks. If G is a connected graph of order n, then $\partial_n^Q(G) \leq Tr_{min}$, where Tr_{min} is the smallest transmission [7]. From Theorem 7, we have $S_{DQ}(G) \geq 2Tr_{min} - \partial_n^Q(G)$. Combining, we get $\partial_1^Q(G) - \partial_n^Q(G) \geq Tr_{min}$.

If G is a connected graph of order $n > 2$, then $\partial_1^Q(G) \geq 2(n-1)$ [9]. Using the inequality $\partial_n^Q(G) \leq \frac{2W(G)}{n}$., we get $S_{DQ}(G) = \partial_1^Q(G) - \partial_n^Q(G) \geq 2(n-1) - \frac{2W(G))}{n} = \frac{2(n(n-1)-W(G))}{n}$.

References

[1] A. Alhevaz, M. Baghipur, S. Pirzada, Y. Shang, Some inequalities involving the distance signless Laplacian eigenvalues of graphs, *Transactions Combinatorics* **10(1)** (2021) 9–29. ⇒40

[2] A. Alhevaz, M. Baghipur, E. Hashemi, H. Ramane, On the distance signless Laplacian spectrum of graphs, *Bull. Malays. Math. Sci. Soc.* **42** (2019) 2603–2621. ⇒40

[3] A. Alhevaz, M. Baghipur, S. Paul, On the distance signless Laplacian spectral radius and the distance signless Laplacian energy of graphs, *Discrete Math. Algorithms and Appl.* **10(3)** (2018) 1850035 (19 pages). ⇒43

[4] A. Alhevaz, M. Baghipur, H. Ahmad, S. Pirzada, Brouwer type conjecture for the eigenvalues of distance signless Laplacian matrix of a graph, *Linear Multilinear Algebra* **69** (2019) 1–18. ⇒40

[5] M. Aouchiche, P. Hansen, Two Laplacians for the distance matrix of a graph, *Linear Algebra Appl.* **439** (2013) 21–33. ⇒39

[6] M. Aouchiche, P. Hansen, A signless Laplacian for the distance matrix of a graph, Cahiers du GERAD G-2011-78. ⇒40

[7] K. C. Das, H. Lin, J. Guo, Distance signless Laplacian eigenvalues of graphs, *Front. Math. China* **14** (2019) 693–713. ⇒44

[8] S. Khan, S. Pirzada, Distance signless Laplacian eigenvalues, diameter, and clique number, *Discrete Mathematics Letters* **10** (2022) 28–31. ⇒40

[9] L. Medina, H. Nina, M. Trigo, On distance signless Laplacian spectral radius and distance signless Laplacian energy, *Mathematics* **8** (2020) 792. ⇒ 43, 44

[10] S. Pirzada, *An Introduction to Graph Theory*, Universities Press, Hyderabad, India, 2012. ⇒ 39

[11] S. Pirzada, B. A. Rather, M. Aijaz, T. A. Chishti, On distance signless Laplacian spectrum of graphs and spectrum of zero divisor graphs of Z_n, *Linear Multilinear Algebra* **70** (2022) 3354–3369. ⇒ 40

[12] S. Pirzada, B. A. Rather, R. U. Shaban, M. I. Bhat, On distance Laplacian (signless) eigenvalues of commuting graphs of dihedral and dicyclic groups, *Springer Proceedings on Algebra and Related Topics with Applications*, ICARTA 2019 (2022) 413–425. ⇒ 40

[13] B. A. Rather, S. Pirzada, T. A. Naikoo, On distance signless Laplacian spectra of power graphs of the integer modulo group, *Art Discrete Appl. Math.* **5** (2022) P2.09. ⇒ 40

[14] R. U. Shaban, B. A. Rather, S. Pirzada, A. Somasundaram, On distance signless Laplacian spectral radius of power graphs of cyclic and dihedral groups, *Annales Mathematicae et Informaticae* **55** (2022) 172–183. ⇒ 40

[15] R. Xing, B. Zhou, J. Li, On the distance signless Laplacian spectral radius of graphs, *Linear Multilinear Algebra* **62** (2014) 1377–1387. ⇒ 42

Received: April 19, 2023 • Revised: May 15, 2023