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Research Article

Mixed-type hypergeometric Bernoulli-Gegenbauer polynomials: some properties

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Abstract

We consider the novel family of the mixed-type hypergeometric Bernoulli-Gegenbauer polynomials. This family represents a fascinating fusion between two distinct categories of special functions: hypergeometric Bernoulli polynomials and Gegenbauer polynomials. We collect some recent results concerning algebraic and differential properties of this class of polynomials and use some them to deduce an ordinary differential equation satisfied by these polynomials. Some numerical illustrative examples about the behavior of the zeros of these polynomials are given.

Keywords: Gegenbauer polynomials; generalized Bernoulli polynomials; hypergeometric Bernoulli polynomials; operational methods

AMS subject classification: 33E20; 32A05; 11B83; 33C45.

1. Introduction

For a fixed integer $m \in \mathbb{N}$, the mixed-type hypergeometric Bernoulli-Gegenbauer polynomials $\mathscr{V}_n^{[m-1,\alpha]}(x)$ of order $\alpha \in (-1/2, \infty)$, where $n \geq 0$, are defined through the generating functions and series expansions as follows:

$$\left(\frac{z^m e^{xz}}{e^z - \sum_{l=0}^{m-1} \frac{z^l}{l!}}\right)\left(1 - \frac{xz}{\pi} + \frac{z^2}{4\pi^2}\right)^{-\alpha} = \sum_{n=0}^{\infty} \mathscr{V}_n^{[m-1,\alpha]}(x)\frac{z^n}{n!}, \tag{1}$$

where $|z| < 2\pi$, $|x| \leq 1$, and $\alpha \in (-1/2, \infty) \setminus \{0\}$.

$$\left(\frac{z^m e^{xz}}{e^z - \sum_{l=0}^{m-1} \frac{z^l}{l!}}\right)\left(\frac{2\pi - xz}{1 - \frac{xz}{\pi} + \frac{z^2}{4\pi^2}}\right) = \sum_{n=0}^{\infty} \mathscr{V}_n^{[m-1,0]}(x)\frac{z^n}{n!}, \qquad |z| < 2\pi, \quad |x| \leq 1. \tag{2}$$

The polynomials $\left\{\mathscr{V}_n^{[m-1,\alpha]}(x)\right\}_{n\geq 0}$ represent a fascinating fusion between two classes of special functions: hypergeometric Bernoulli polynomials and Gegenbauer polynomials.

A significant amount of research has been conducted on various generalizations and analogs of the Bernoulli polynomials and the Bernoulli numbers. For a comprehensive treatment of the diverse aspects, including summation formulas and applications, interested readers can refer to recent works [1–5]. Inspired by recent articles [6–10] where authors explore analytic and numerical aspects of hypergeometric Bernoulli polynomials, hypergeometric Euler polynomials, generalized mixed-type Bernoulli-Gegenbauer polynomials, and Lagrange-based hypergeometric Bernoulli polynomials, in [11] the authors focus their attention on algebraic and differential properties of the polynomials $\left\{\mathscr{V}_n^{[m-1,\alpha]}(x)\right\}_{n\geq 0}$. These properties include their explicit expressions, derivative formulas, matrix representations, matrix-inversion formulas, and other relationships connecting them with hypergeometric Bernoulli polynomials. In this paper we add to these properties the differential equation satisfied by this family of polynomials.

This paper is a written version of the talk given by one of the authors on the occasion of the second Meeting Gruppo di Attività ANA&A-SIMAI, Rome, April 2024, and it is organized as follows. Section 2 provides some notations and preliminary results about hypergeometric Bernoulli polynomials and Gegenbauer polynomials. Section 3 is dedicated to collect some results of [11] (see Theorems 3.1-3.4, and Proposition 3.2), as well as, a new result Theorem 3.5.

2. Background and Previous Results

Throughout this paper, let $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{C}$ denote, respectively, the sets of natural numbers, non-negative integers, integers, real numbers, and complex numbers. As usual, we always use the principal branch for complex powers, in particular, $1^\alpha = 1$ for $\alpha \in \mathbb{C}$. Furthermore, the convention $0^0 = 1$ is adopted.

For $\lambda \in \mathbb{C}$ and $k \in \mathbb{Z}$, we use the notations $\lambda^{(k)}$ and $(\lambda)_k$ for the rising and falling factorials, respectively, i.e.,

$$\lambda^{(k)} = \begin{cases} 1, & \text{if } k = 0, \\ \prod_{i=1}^{k}(\lambda + i - 1), & \text{if } k \geq 1, \\ 0, & \text{if } k < 0, \end{cases}$$

and

$$(\lambda)_k = \begin{cases} 1, & \text{if } k = 0, \\ \prod_{i=1}^{k}(\lambda - i + 1), & \text{if } k \geq 1, \\ 0, & \text{if } k < 0. \end{cases}$$

From now on, we denote by $\mathbb{P}_n$ the linear space of polynomials with real coefficients and a degree less than or equal to n. Moreover, to present some of our results, we require the use of the generalized multinomial theorem (cf. [12,13] and the references therein).

2.1. *Hypergeometric Bernoulli Polynomials and Gegenbauer Polynomials*

For a fixed $m \in \mathbb{N}$, the hypergeometric Bernoulli polynomials are defined by means of the following generating function [8,14–21]:

$$\frac{z^m e^{xz}}{e^z - \sum_{l=0}^{m-1} \frac{z^l}{l!}} = \sum_{n=0}^{\infty} B_n^{[m-1]}(x) \frac{z^n}{n!}, \quad |z| < 2\pi, \tag{3}$$

and the hypergeometric Bernoulli numbers are defined by $B_n^{[m-1]} := B_n^{[m-1]}(0)$ for all $n \geq 0$. The hypergeometric Bernoulli polynomials also are called generalized Bernoulli polynomials of level m [8,9]. It is clear that if $m = 1$ in (3), then we obtain the definition of the classical Bernoulli polynomials $B_n(x)$ and classical Bernoulli numbers, respectively, i.e., $B_n(x) = B_n^{[0]}(x)$ and $B_n = B_n^{[0]}$, respectively, for all $n \geq 0$.

The first four hypergeometric Bernoulli polynomials are as follows:

$$\begin{aligned}
&B_0^{[m-1]}(x) = m!, \\
&B_1^{[m-1]}(x) = m!\left(x - \tfrac{1}{m+1}\right), \\
&B_2^{[m-1]}(x) = m!\left(x^2 - \tfrac{2}{m+1}x + \tfrac{2}{(m+1)^2(m+2)}\right), \\
&B_3^{[m-1]}(x) = m!\left(x^3 - \tfrac{3}{m+1}x^2 + \tfrac{6}{(m+1)^2(m+2)}x + \tfrac{6(m-1)}{(m+1)^3(m+2)(m+3)}\right).
\end{aligned}$$

The following results summarize some properties of the hypergeometric Bernoulli polynomials (cf. [8, 9,14,19,22]).

Proposition 2.1. *[8, Proposition 1], For a fixed $m \in \mathbb{N}$, let $\left\{B_n^{[m-1]}(x)\right\}_{n\geq 0}$ be the sequence of hypergeometric Bernoulli polynomials. Then the following statements hold:*

(a) Summation formula. For every $n \geq 0$,

$$B_n^{[m-1]}(x) = \sum_{k=0}^{n} \binom{n}{k} B_k^{[m-1]} \, x^{n-k}. \tag{4}$$

(b) Differential relations (Appell polynomial sequences). For $n, j \geq 0$ with $0 \leq j \leq n$, we have

$$[B_n^{[m-1]}(x)]^{(j)} = \frac{n!}{(n-j)!} \, B_{n-j}^{[m-1]}(x). \tag{5}$$

(c) Inversion formula. ([19], Equation (2.6)) For every $n \geq 0$,

$$x^n = \sum_{k=0}^{n} \binom{n}{k} \frac{k!}{(m+k)!} B_{n-k}^{[m-1]}(x). \tag{6}$$

(d) Recurrence relation. ([19], Lemma 3.2) For every $n \geq 1$,

$$B_n^{[m-1]}(x) = \left(x - \frac{1}{m+1}\right) B_{n-1}^{[m-1]}(x) - \frac{1}{n(m-1)!} \sum_{k=0}^{n-2} \binom{n}{k} B_{n-k}^{[m-1]} B_k^{[m-1]}(x).$$

(e) Integral formulas.

$$\begin{aligned}\int_{x_0}^{x_1} B_n^{[m-1]}(x)dx &= \frac{1}{n+1}\left[B_{n+1}^{[m-1]}(x_1) - B_{n+1}^{[m-1]}(x_0)\right] \\ &= \sum_{k=0}^{n} \frac{1}{n-k+1}\binom{n}{k} B_k^{[m-1]}((x_1)^{n-k+1} - (x_0)^{n-k+1}).\end{aligned}$$

$$B_n^{[m-1]}(x) = n\int_0^x B_{n-1}^{[m-1]}(t)dt + B_n^{[m-1]}.$$

(f) ([19], Theorem 3.1) Differential equation. For every $n \geq 1$, the polynomial $B_n^{[m-1]}(x)$ satisfies the following differential equation

$$\frac{B_n^{[m-1]}}{n!} y^{(n)} + \frac{B_{n-1}^{[m-1]}}{(n-1)!} y^{(n-1)} + \cdots + \frac{B_2^{[m-1]}}{2!} y'' + (m-1)!\left(\frac{1}{m+1} - x\right) y' + n(m-1)! y = 0. \tag{7}$$

As a straightforward consequence of the inversion Formula (6), the following expected algebraic property is obtained.

Proposition 2.2. *[8, Proposition 2]. For a fixed $m \in \mathbb{N}$ and each $n \geq 0$, the set $\left\{B_0^{[m-1]}(x), B_1^{[m-1]}(x), \ldots, B_n^{[m-1]}(x)\right\}$ is a basis for $\mathbb{P}_n$, i.e.,*

$$\mathbb{P}_n = \operatorname{span}\left\{B_0^{[m-1]}(x), B_1^{[m-1]}(x), \ldots, B_n^{[m-1]}(x)\right\}.$$

Let $\zeta(s)$ be the Riemann zeta function defined by

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^{s}},\quad \Re(s)>1.$$

The following result provides a formula for evaluating $\zeta(2r)$ in terms of the hypergeometric Bernoulli numbers.

Proposition 2.3. *[9, Theorem 3.3]. For a fixed $m\in\mathbb{N}$ and any $r\in\mathbb{N}$, the following identity holds.*

$$\zeta(2r)=\frac{(-1)^{r-1}2^{2r-1}\pi^{2r}B_{2r}^{[m-1]}}{m!(2r)!}+\Delta_r^{[m-1]},$$

where

$$\Delta_r^{[m-1]}=\frac{(-1)^{r-1}2^{2r-1}\pi^{2r}}{m!}\left[\frac{B_{2r}^{[m-1]}(1)-B_{2r}^{[m-1]}}{2(2r)!}-\frac{B_{2r+1}^{[m-1]}(1)-B_{2r+1}^{[m-1]}}{(2r+1)!}-\sum_{j=1}^{r-1}\frac{\left(B_{2r-2j+1}^{[m-1]}(1)-B_{2r-2j+1}^{[m-1]}\right)}{(2r-2j+1)!}\frac{B_{2j}}{(2j)!}\right].$$

With respect to Gegenbauer polynomials, we recall that for $\alpha>-\frac{1}{2}$, we denote by $\{\hat{C}_n^{(\alpha)}(x)\}_{n\geq 0}$ the sequence of Gegenbauer polynomials, orthogonal on $[-1,1]$ with respect to the measure $d\mu(x)=(1-x^2)^{\alpha-\frac{1}{2}}\,dx$ (cf. [23], Chapter IV), normalized by

$$\hat{C}_n^{(\alpha)}(1)=\frac{\Gamma(n+2\alpha)}{n!\Gamma(2\alpha)}.$$

More precisely,

$$\int_{-1}^{1}\hat{C}_n^{(\alpha)}(x)\hat{C}_m^{(\alpha)}(x)\,d\mu(x)=\int_{-1}^{1}\hat{C}_n^{(\alpha)}(x)\hat{C}_m^{(\alpha)}(x)(1-x^2)^{\alpha-\frac{1}{2}}\,dx=M_n^{\alpha}\,\delta_{n,m},\quad n,m\geq 0,$$

where the constant M_n^{α} is positive. It is clear that the normalization above does not allow α to be zero or a negative integer. Nevertheless, the following limits exist for every $x\in[-1,1]$ (see [23], (4.7.8))

$$\lim_{\alpha\to 0}\hat{C}_0^{(\alpha)}(x)=T_0(x),\quad \lim_{\alpha\to 0}\frac{\hat{C}_n^{(\alpha)}(x)}{\alpha}=\frac{2}{n}T_n(x),$$

where $T_n(x)$ is the nth Chebyshev polynomial of the first kind. In order to avoid confusing notation, we define the sequence $\{\hat{C}_n^{(0)}(x)\}_{n\geq 0}$ as follows:

$$\hat{C}_0^{(0)}(1)=1,\quad \hat{C}_n^{(0)}(1)=\frac{2}{n},\quad \hat{C}_n^{(0)}(x)=\frac{2}{n}T_n(x),\quad n\geq 1.$$

We denote the nth monic Gegenbauer orthogonal polynomial by

$$C_n^{(\alpha)}(x)=(k_n^{\alpha})^{-1}\hat{C}_n^{(\alpha)}(x),$$

where the constant k_n^{α} (cf. [23], Formula (4.7.31)) is given by

$$k_n^{\alpha}=\frac{2^n\Gamma(n+\alpha)}{n!\Gamma(\alpha)},\quad \alpha\neq 0,$$
$$k_n^{0}=\lim_{\alpha\to 0}\frac{k_n^{\alpha}}{\alpha}=\frac{2^n}{n},\quad n\geq 1.$$

Then for $n\geq 1$, we have

$$C_n^{(0)}(x)=\lim_{\alpha\to 0}(k_n^{\alpha})^{-1}\hat{C}_n^{(\alpha)}(x)=\frac{1}{2^{n-1}}T_n(x). \tag{8}$$

As is well known, for $\alpha > -\frac{1}{2}$ the monic Gegenbauer orthogonal polynomials $C_n^{(\alpha)}(x)$ admit other different definitions [23–26]. In order to deal with the definitions (1) and (2) of the HBG polynomials, we are interested in the definition of these polynomials by means of the following generating functions:

$$\left(1-\frac{xz}{\pi}+\frac{z^2}{4\pi^2}\right)^{-\alpha}=\sum_{n=0}^{\infty}\frac{\Gamma(n+\alpha)}{\pi^n\Gamma(\alpha)}C_n^{(\alpha)}(x)\frac{z^n}{n!},\quad |z|<2\pi,\ |x|\leq 1,\ \alpha\in(-1/2,\infty)\setminus\{0\}, \tag{9}$$

and

$$\frac{2\pi-xz}{1-\frac{xz}{\pi}+\frac{z^2}{4\pi^2}}=\sum_{n=0}^{\infty}\frac{1}{\pi^{n-1}}C_n^{(0)}(x)z^n=\sum_{n=0}^{\infty}\frac{\Gamma(n+1)}{\pi^{n-1}}C_n^{(0)}(x)\frac{z^n}{n!},\quad |z|<2\pi,\ |x|\leq 1. \tag{10}$$

Remark 2.1. Note that (9) and (10) are suitable modifications of the generating functions for the Gegenbauer polynomials $\hat{C}_n^{(\alpha)}(x)$:

$$\left(1-2xz+z^2\right)^{-\alpha}=\sum_{n=0}^{\infty}\hat{C}_n^{(\alpha)}(x)z^n,\quad |z|<1,\ |x|\leq 1,\ \alpha\in(-1/2,\infty)\setminus\{0\},$$
$$\frac{1-xz}{1-xz+z^2}=1+\sum_{n=1}^{\infty}\frac{n}{2}\hat{C}_n^{(0)}(x)z^n,\quad |z|<1,\ |x|\leq 1.$$

Proposition 2.4. *[27, cf. Proposition 2.1]. Let $\{C_n^{(\alpha)}\}_{n\geq 0}$ be the sequence of monic Gegenbauer orthogonal polynomials. Then the following statements hold.*

(a) Three-term recurrence relation.

$$xC_n^{(\alpha)}(x)=C_{n+1}^{(\alpha)}(x)+\gamma_n^{(\alpha)}C_{n-1}^{(\alpha)}(x),\quad \alpha>-\frac{1}{2},\ \alpha\neq 0, \tag{11}$$

with initial conditions $C_{-1}^{(\alpha)}(x)=0$, $C_0^{(\alpha)}(x)=1$ and recurrence coefficients $\gamma_0^{(\alpha)}\in\mathbb{R}$, $\gamma_n^{(\alpha)}=\frac{n(n+2\alpha-1)}{4(n+\alpha)(n+\alpha-1)}$, $n\in\mathbb{N}$.

(b) For every $n\in\mathbb{N}$ (see [23], (4.7.15))

$$h_n^{\alpha}:=\|C_n^{(\alpha)}\|_{\mu}^2=\int_{-1}^{1}[C_n^{(\alpha)}(x)]^2d\mu(x)=\pi 2^{1-2\alpha-2n}\frac{n!\Gamma(n+2\alpha)}{\Gamma(n+\alpha+1)\Gamma(n+\alpha)}. \tag{12}$$

(c) Rodrigues formula.

$$(1-x^2)^{\alpha-\frac{1}{2}}C_n^{(\alpha)}(x)=\frac{(-1)^n\Gamma(n+2\alpha)}{\Gamma(2n+2\alpha)}\frac{d^n}{dx^n}\left[(1-x^2)^{n+\alpha-\frac{1}{2}}\right],\quad x\in(-1,1).$$

(d) Structure relation (see [23], (4.7.29)). For every $n\geq 2$

$$C_n^{(\alpha-1)}(x)=C_n^{(\alpha)}(x)+\xi_{n-2}^{(\alpha)}C_{n-2}^{(\alpha)}(x),$$

where

$$\xi_n^{(\alpha)}=\frac{(n+2)(n+1)}{4(n+\alpha+1)(n+\alpha)},\quad n\geq 0.$$

(e) For every $n\in\mathbb{N}$ (see [23], Formula (4.7.14))

$$\frac{d}{dx}C_n^{(\alpha)}(x)=nC_{n-1}^{(\alpha+1)}(x).$$

(f) For every $n\in\mathbb{N}$ (see [28], Proposition 2.1)

$$\frac{d}{dx}C_n^{(0)}(x)=\frac{n}{2}C_{n-1}^{(1)}(x).$$

The interested readers are referred to [29–32] for detailed explanations and examples of integral representations of the Gegenbauer polynomials in terms of the Gould-Hopper polynomials, and the study of other classical special functions within the context of exponential operators.

3. The HBG polynomials and their properties

The following properties of the HBG polynomials have been recently showed. We omit their proofs, and refer interested readers to [11] for the details of them.

Proposition 3.1. *For $\alpha\in(-1/2,\infty)$, let $\left\{\mathscr{V}_n^{[m-1,\alpha]}(x)\right\}_{n\geq 0}$ be the sequence of HBG polynomials of order α. Then the following explicit formulas hold.*

$$\mathscr{V}_n^{[m-1,\alpha]}(x)=\sum_{k=0}^{n}\binom{n}{k}\frac{\Gamma(k+\alpha)}{\pi^k\Gamma(\alpha)}C_k^{(\alpha)}(x)B_{n-k}^{[m-1]}(x),\quad n\geq 0,\ \alpha\neq 0, \tag{13}$$

$$\mathscr{V}_n^{[m-1,0]}(x)=\sum_{k=0}^{n}\binom{n}{k}\frac{k!}{\pi^{k-1}}C_k^{(0)}(x)B_{n-k}^{[m-1]}(x),\quad n\geq 0. \tag{14}$$

Thus, the suitable use of (11) and (13) allow us to check that for $\alpha\in(-1/2,\infty)\setminus\{0\}$, the first five HBG polynomials are:

$$\mathscr{V}_0^{[m-1,\alpha]}(x)=m!\,v_0(\alpha),$$

$$\mathscr{V}_1^{[m-1,\alpha]}(x)=m!\left[v_1(\alpha)x-\frac{1}{m+1}\right],$$

$$\mathscr{V}_2^{[m-1,\alpha]}(x)=m!\left[v_2(\alpha)x^2-\frac{2(\pi+\alpha)}{\pi(m+1)}x+\frac{4\pi^2(\alpha+1)+\alpha(m+1)^2(m+2)}{2\pi^2(m+1)^2(m+2)(1+\alpha)}\right],$$

$$\begin{aligned}\mathscr{V}_3^{[m-1,\alpha]}(x)=m!\Bigg[&v_3(\alpha)x^3-\frac{3}{m+1}v_2(\alpha)x^2+3\left(\frac{2}{(m+1)^2(m+2)}\left(1+\frac{\alpha}{\pi}\right)-\frac{\alpha}{2\pi^2}\left(1+\frac{(1+\alpha)}{\pi}\right)\right)x\\&+3\left(\frac{2(m-1)}{(m+1)^3(m+2)(m+3)}-\frac{\alpha}{2\pi^2(m+1)}\right)\Bigg],\end{aligned}$$

$$\begin{aligned}\mathscr{V}_4^{[m-1,\alpha]}(x)=m!\Bigg[&v_4(\alpha)x^4-\frac{4}{m+1}v_3(\alpha)x^3+3\left(\frac{m-2}{(m+1)(m+2)}+\frac{8\alpha}{\pi(m+1)^2(m+2)}-\frac{\alpha}{\pi^2}-\frac{2(1+\alpha)\alpha}{\pi^3}\right.\\&\left.-\frac{(2+\alpha)(1+\alpha)\alpha}{\pi^4}\right)x^2+6\left(\frac{5-m}{(m+1)^2(m+2)(m+3)}+\frac{4(m-1)\alpha}{\pi(m+1)^3(m+2)(m+3)}+\frac{\alpha}{\pi^2(m+1)}\right.\\&\left.+\frac{(1+\alpha)\alpha}{\pi^3(m+1)}\right)x^2-\frac{6(m^3-3m^2-6m+36)}{(m+1)^2(m+2)^2(m+3)(m+4)}+\frac{6(1+2\alpha)\alpha}{\pi^2(m+1)^2(m+2)}+\frac{3(1+\alpha)\alpha}{4\pi^4}\Bigg],\end{aligned}$$

where $v_n(\alpha)=\sum_{k=0}^{n}\binom{n}{k}\frac{\alpha^{(k)}}{\pi^k}$, $0\leq n\leq 4$.

In contrast to the hypergeometric Bernoulli polynomials and Gegenbauer polynomials, the HBG polynomials neither satisfy a Hanh condition nor an Appell condition. More precisely, we have the following result.

Theorem 3.1. *For $\alpha \in (-1/2, \infty)$, let $\left\{\mathscr{V}_n^{[m-1,\alpha]}(x)\right\}_{n\geq 0}$ be the sequence of HBG polynomials of order α. Then we have*

$$\frac{d}{dx}\mathscr{V}_{n+1}^{[m-1,\alpha]}(x) = (n+1)\left[\frac{\alpha}{\pi}\mathscr{V}_n^{[m-1,\alpha+1]}(x) + \mathscr{V}_n^{[m-1,\alpha]}(x)\right], \quad \alpha \neq 0, \tag{15}$$

$$\frac{d}{dx}\mathscr{V}_{n+1}^{[m-1,0]}(x) = (n+1)\left[\mathscr{V}_n^{[m-1,0]}(x) + \frac{1}{2}\sum_{k=0}^{n}\binom{n}{k}\frac{(k+1)!}{\pi^k}C_k^{(1)}(x)B_{n-k}^{[m-1]}(x)\right], \quad \alpha = 0. \tag{16}$$

From Theorem 3.1 we can conclude that a general methodology involving operational methods could fail for this family of polynomials (see for instance, [8]).

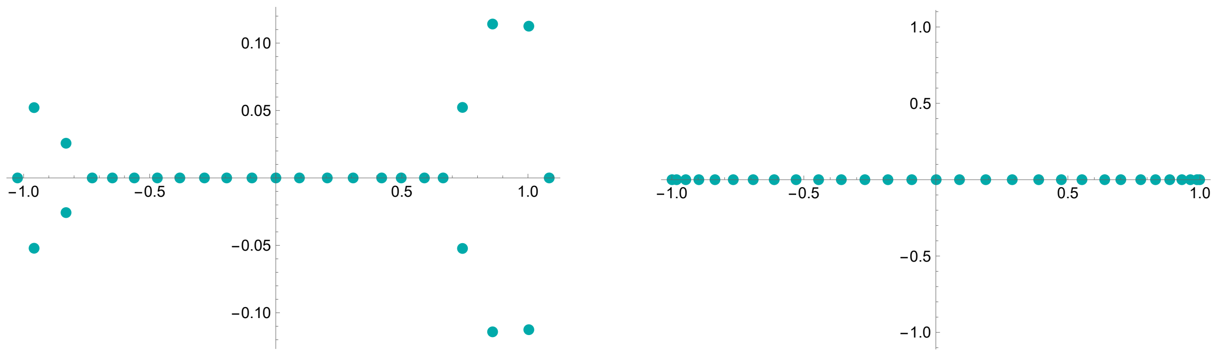
On the other hand, it is possible to establish an integral formula connecting the HBG polynomials with the monic Gegenbauer polynomials. This integral formula allows us to deduce a concise expression for the Fourier coefficients of the HBG polynomials.

Lemma 3.1. *For $\alpha \in (-1/2, \infty)$, let $\left\{\mathscr{V}_n^{[m-1,\alpha]}(x)\right\}_{n\geq 0}$ be the sequence of HBG polynomials of order α. Then, the following formula holds.*

$$\int_{-1}^{1}\mathscr{V}_n^{[m-1,\alpha]}(x)C_n^{(\alpha)}(x)d\mu(x) = \begin{cases} \frac{m!n!\Gamma(n+2\alpha)}{\pi^{2\alpha+2n}\Gamma(n+\alpha+1)\Gamma(n+\alpha)}\sum_{k=0}^{n}\binom{n}{k}\frac{\Gamma(k+\alpha)}{\pi^{k-1}\Gamma(\alpha)}, & \alpha \neq 0, \\ \frac{m!\pi}{2^n}\sum_{k=0}^{n}\binom{n}{k}\frac{k!}{\pi^{k-1}}, & \alpha = 0, \end{cases} \tag{17}$$

whenever $n \geq 0$.

Regarding the zero distribution of these polynomials, the numerical evidence indicates that this distribution does not align with the behavior of either Bernoulli hypergeometric polynomials or Gegenbauer polynomials. For instance, in Figure 1, the plots for the zeros of $\mathscr{V}_{28}^{[m-1,\alpha]}(x)$ and $\mathscr{V}_{30}^{[m-1,\alpha]}(x)$ are shown for $m=2$ and $\alpha = -\frac{1}{4}$.



(a) Zeros of $\mathscr{V}_{28}^{[1,-\frac{1}{4}]}(x)$. (b) Zeros of $\mathscr{V}_{30}^{[1,-\frac{1}{4}]}(x)$.

Figure 1: Zeros of $\mathscr{V}_{28}^{[1,-\frac{1}{4}]}(x)$ and $\mathscr{V}_{30}^{[1,-\frac{1}{4}]}(x)$.

As expected, the symmetry property of Gegenbauer polynomials is not inherited by the HBG polynomials. For instance, Figure 2 displays the induced mesh of $\mathscr{V}_j^{[m-1,\alpha]}(x)$ for $m=2$, $\alpha=1$, and $j=1,\ldots,21$. Each point on this mesh takes the form $(x_j^{[m-1,\alpha]}, j)$, $j=1,\ldots,21$. In contrast, Figure 3 displays the induced mesh of $C_j^{(\alpha)}(x)$ for $\alpha=1$, and $j=1,\ldots,19$.

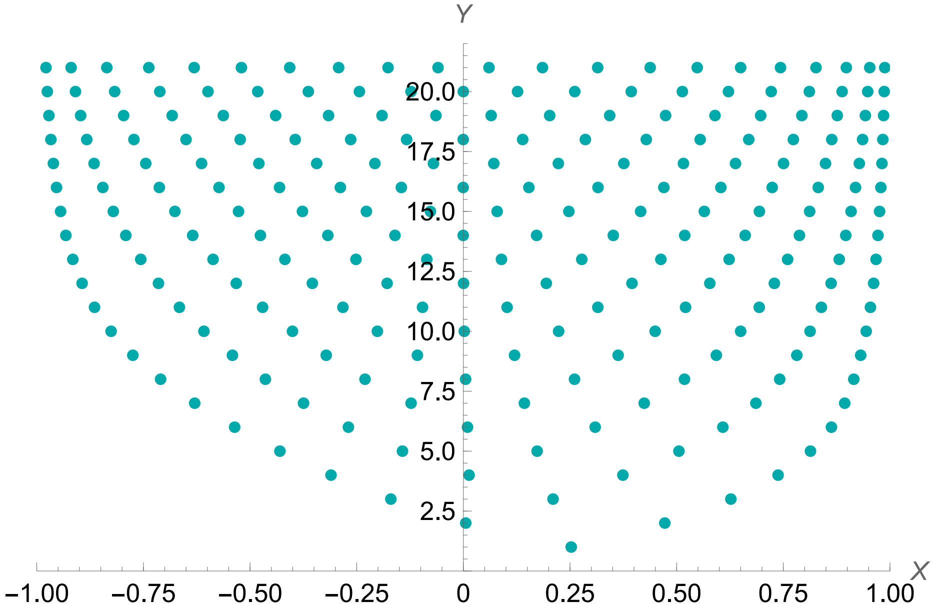


Figure 2: Induced mesh of $\mathscr{V}_j^{[m-1,\alpha]}(x)$ for $m=2$, $\alpha=1$, and $j=1,\dots,21$.

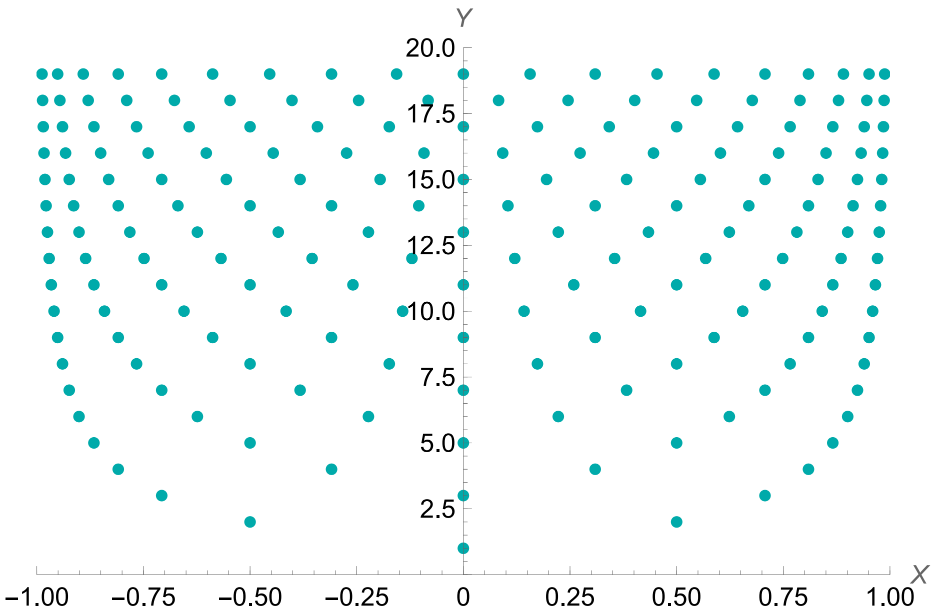


Figure 3: Induced mesh of $C_j^{(\alpha)}(x)$ for $\alpha=1$, and $j=1,\dots,19$.

For any $\alpha\in(-1/2,\infty)$, it is possible to deduce interesting relations connecting the HBG polynomials $\mathscr{V}_n^{[m-1,\alpha]}(x)$ and the hypergeometric Bernoulli polynomials $B_n^{[m-1]}(x)$. The following two results concern these relations.

Proposition 3.2. *For a fixed $m\in\mathbb{N}$, let $\mathscr{V}_n^{[m-1,\alpha]}(x)$ be the nth HBG polynomial of order $\alpha\in(-1/2,\infty)\setminus\{0\}$. Then, the following relation is satisfied:*

$$\sum_{n=0}^{\infty} B_n^{[m-1]}(x)\frac{z^n}{n!}=\sum_{n=0}^{\infty}\sum_{0\le j,k\le|\alpha|}\frac{(-1)^j}{2^{2k}\pi^{2k+j}}\binom{\alpha}{j,k}x^j\mathscr{V}_n^{[m-1,\alpha]}(x)\frac{z^{n+2k+j}}{n!}. \tag{18}$$

Theorem 3.2. *For a fixed $m\in\mathbb{N}$, the HBG polynomials $\mathscr{V}_n^{[m-1,0]}(x)$ are related with the hypergeometric Bernoulli polynomials $B_n^{[m-1]}(x)$ by means of the following identities.*

$$
(19)\qquad
\begin{aligned}
&2\pi B_0^{[m-1]}(x) = \mathscr{V}_0^{[m-1,0]}(x),\\
&2\pi B_1^{[m-1]}(x) - xB_0^{[m-1]}(x) = \mathscr{V}_1^{[m-1,0]}(x) - \tfrac{x}{\pi}\mathscr{V}_0^{[m-1,0]}(x),\\
&2\pi B_n^{[m-1]}(x) - nxB_{n-1}^{[m-1]}(x) = \mathscr{V}_n^{[m-1,0]}(x) - \tfrac{nx}{\pi}\mathscr{V}_{n-1}^{[m-1,0]}(x) + \tfrac{n(n-1)}{4\pi^2}\mathscr{V}_{n-2}^{[m-1,0]}(x), \quad n \geq 2.
\end{aligned}
$$

Remark 3.1. When $\alpha = r \in \mathbb{N}$, by multinomial Theorem we have

$$
\left(1 - \frac{xz}{\pi} + \frac{z^2}{4\pi^2}\right)^r = \sum_{j+k=r} \frac{(-1)^j}{2^{2k}\pi^{2k+j}} \binom{r}{j,k} x^j z^{2k+j}.
$$

Thus, for $r = 1$ we can combine the above identity with (18), and obtain the following connecting relations:

$$
(20)\qquad
\begin{aligned}
&B_0^{[m-1]}(x) = \mathscr{V}_0^{[m-1,1]}(x)\\
&B_1^{[m-1]}(x) = \mathscr{V}_1^{[m-1,1]}(x) - \frac{x}{\pi}\mathscr{V}_0^{[m-1,1]}(x)\\
&B_n^{[m-1]}(x) = \mathscr{V}_n^{[m-1,1]}(x) - \frac{nx}{\pi}\mathscr{V}_{n-1}^{[m-1,1]}(x) + \frac{n(n-1)}{4\pi^2}\mathscr{V}_{n-2}^{[m-1,1]}(x), \quad n \geq 2,
\end{aligned}
$$

Hence, as a straightforward consequence of (18) and (19), the HBG polynomials $\mathscr{V}_n^{[m-1,1]}(x)$ and $\mathscr{V}_n^{[m-1,0]}(x)$ are related by means of the following identities:

$$
(21)\qquad
\begin{aligned}
&2\pi\mathscr{V}_0^{[m-1,1]}(x) = \mathscr{V}_0^{[m-1,0]}(x)\\
&2\pi\mathscr{V}_1^{[m-1,1]}(x) - 3x\mathscr{V}_0^{[m-1,1]}(x) = \mathscr{V}_1^{[m-1,0]}(x) - \frac{x}{\pi}\mathscr{V}_0^{[m-1,0]}(x)\\
&2\pi\mathscr{V}_n^{[m-1,1]}(x) - 3nx\mathscr{V}_{n-1}^{[m-1,1]}(x) + \left(\frac{n(n-1)}{2\pi} + \frac{n(n-1)x^2}{\pi}\right)\mathscr{V}_{n-2}^{[m-1,1]}(x) - \frac{n(n-1)(n-2)x}{4\pi^2}\mathscr{V}_{n-3}^{[m-1,1]}(x)\\
&\quad = \mathscr{V}_n^{[m-1,0]}(x) - \frac{nx}{\pi}\mathscr{V}_{n-1}^{[m-1,0]}(x) + \frac{n(n-1)}{4\pi^2}\mathscr{V}_{n-2}^{[m-1,0]}(x), \quad n \geq 2.
\end{aligned}
$$

Using (13), (14), and employing a matrix approach, we can obtain a matrix representation for $\mathscr{V}_n^{[m-1,\alpha]}(x)$, $n \geq 0$. In order to implement that, we follow some ideas from [7,8].

First of all, we must point out that for $r = 0, 1, \dots, n$, Equations (13) and (14) allow us to deduce the following matrix form of $\mathscr{V}_r^{[m-1,\alpha]}(x)$:

$$
(22)\qquad \mathscr{V}_r^{[m-1,\alpha]}(x) = \mathbf{C}_r^{(\alpha)}(x)\mathbf{B}^{[m-1]}(x), \quad r = 0, 1, \dots, n,
$$

where

$$
\mathbf{C}_r^{(\alpha)}(x) = \begin{cases}
\left[\binom{r}{r}\frac{\Gamma(r+\alpha)}{\pi^r\Gamma(\alpha)}C_r^{(\alpha)}(x)\ \binom{r}{r-1}\frac{\Gamma(r-1+\alpha)}{\pi^{r-1}\Gamma(\alpha)}C_{r-1}^{(\alpha)}(x) \cdots C_0^{(\alpha)}(x)\ 0 \cdots 0\right], & \text{if } \alpha \neq 0,\\[2ex]
\left[\binom{r}{r}\frac{r!}{\pi^{r-1}}C_r^{(0)}(x)\ \binom{r}{r-q}\frac{(r-1)!}{\pi^{r-2}}C_{r-1}^{(0)}(x) \cdots C_0^{(0)}(x)\ 0 \cdots 0\right], & \text{if } \alpha = 0,
\end{cases}
$$

the null entries of the matrix $\mathbf{C}_r^{(\alpha)}(x)$ appear $(n-r)$-times, and the matrix $\mathbf{B}^{[m-1]}(x)$ is given by $\mathbf{B}^{[m-1]}(x) = \left[B_0^{[m-1]}(x)\ B_1^{[m-1]}(x) \cdots B_n^{[m-1]}(x)\right]^T$.

Now, for $\alpha \in (-1/2, \infty)$, let $\mathbf{C}^{(\alpha)}(x)$ be the $(n+1)\times(n+1)$ whose rows are precisely the matrices $\mathbf{C}_r^{(\alpha)}(x)$ for $r = 0, 1, \dots, n$. That is,

$$
\mathbf{C}^{(\alpha)}(x) = \begin{bmatrix}
C_0^{(\alpha)}(x) & 0 & \cdots & 0\\
\binom{1}{1}\frac{\Gamma(1+\alpha)}{\pi\Gamma(\alpha)}C_1^{(\alpha)}(x) & C_0^{(\alpha)}(x) & \cdots & 0\\
\binom{2}{2}\frac{\Gamma(2+\alpha)}{\pi^2\Gamma(\alpha)}C_2^{(\alpha)}(x) & \binom{2}{1}\frac{\Gamma(1+\alpha)}{\pi\Gamma(\alpha)}C_1^{(\alpha)}(x) & \cdots & 0\\
\vdots & \vdots & \ddots & \vdots\\
\binom{n}{n}\frac{\Gamma(n+\alpha)}{\pi^n\Gamma(\alpha)}C_n^{(\alpha)}(x) & \binom{n}{n-1}\frac{\Gamma(n-1+\alpha)}{\pi^{n-1}\Gamma(\alpha)}C_{n-1}^{(\alpha)}(x) & \cdots & C_0^{(\alpha)}(x)
\end{bmatrix}, \quad \alpha > -\frac{1}{2},\ \alpha \neq 0,
$$

and from (8):

$$\mathbf{C}^{(0)}(x)=\begin{bmatrix} 1 & 0 & \cdots 0\\ \binom{1}{1}\pi T_1(x) & 1 & \cdots 0\\ \binom{2}{2}\frac{1}{\pi}T_2(x) & \binom{2}{1}T_1(x) & \cdots 0\\ \vdots & \vdots & \ddots \vdots\\ \binom{n}{n}\frac{n!}{(2\pi)^{n-1}}T_n(x) & \binom{n}{n-1}\frac{(n-1)!}{(2\pi)^{n-2}}T_{n-1}(x) & \cdots 1 \end{bmatrix}.$$

It is clear that the matrix $\mathbf{C}^{(\alpha)}(x)$ is a lower triangular matrix for each $x\in\mathbb{R}$, so that $\det\left(\mathbf{C}^{(\alpha)}(x)\right)=1$. Therefore, $\mathbf{C}^{(\alpha)}(x)$ is a nonsingular matrix for each $x\in\mathbb{R}$ and $\alpha\in(-1/2,\infty)$.

Theorem 3.3. *For a fixed $m\in\mathbb{N}$ and any $\alpha\in(-1/2,\infty)$, let $\left\{\mathscr{V}_n^{[m-1,\alpha]}(x)\right\}_{n\geq 0}$ be the sequence of HBG polynomials. Then, the following matrix representation holds.*

$$\mathbf{V}^{[m-1,\alpha]}(x)=\mathbf{C}^{(\alpha)}(x)\mathbf{B}^{[m-1]}(x), \tag{23}$$

where $\mathbf{V}^{[m-1,\alpha]}(x)=\left[\mathscr{V}_0^{[m-1,\alpha]}(x)\ \mathscr{V}_1^{[m-1,\alpha]}(x)\ \cdots\ \mathscr{V}_n^{[m-1,\alpha]}(x)\right]^T$.

The following examples show how Theorem 3.3 can be used.

Example 3.1. Let us consider $m=1$, $n=3$, and $\alpha=1$, then,

$$\mathbf{B}(x)=\left(\mathbf{C}^{(1)}(x)\right)^{-1}\mathbf{V}^{[0,1]}(x)=\begin{bmatrix} 1 & 0 & 0 & 0\\ \frac{x}{\pi} & 1 & 0 & 0\\ \frac{4x^2-1}{2\pi^2} & \frac{2x}{\pi} & 1 & 0\\ \frac{6x^3-3x}{\pi^3} & \frac{3\left(4x^2-1\right)}{2\pi^2} & \frac{3x}{\pi} & 1 \end{bmatrix}^{-1}\mathbf{V}^{[0,1]}(x), \tag{24}$$

where

$$\mathbf{V}^{[0,1]}(x)=\begin{bmatrix} 1\\ \left(1+\frac{1}{\pi}\right)x-\frac{1}{2}\\ \left(1+\frac{2}{\pi}+\frac{1}{\pi^2}\right)x^2-\left(1+\frac{1}{\pi}\right)x+\frac{1}{6}-\frac{1}{2\pi^2}\\ \left(1+\frac{3}{\pi}+\frac{6}{\pi^2}+\frac{6}{\pi^3}\right)x^3-\frac{3}{2}\left(1+\frac{2}{\pi}+\frac{2}{\pi^2}\right)x^2+\frac{1}{2}\left(1+\frac{1}{\pi}-\frac{3}{\pi^2}-\frac{6}{\pi^3}\right)x+\frac{3}{4\pi^2} \end{bmatrix}.$$

Since

$$\left(\mathbf{C}^{(1)}(x)\right)^{-1}=\begin{bmatrix} 1 & 0 & 0 & 0\\ \frac{x}{\pi} & 1 & 0 & 0\\ \frac{4x^2-1}{2\pi^2} & \frac{2x}{\pi} & 1 & 0\\ \frac{6x^3-3x}{\pi^3} & \frac{3\left(4x^2-1\right)}{2\pi^2} & \frac{3x}{\pi} & 1 \end{bmatrix}^{-1}=\begin{bmatrix} 1 & 0 & 0 & 0\\ -\frac{x}{\pi} & 1 & 0 & 0\\ \frac{1}{2\pi^2} & -\frac{2x}{\pi} & 1 & 0\\ 0 & \frac{3}{2\pi^2} & -\frac{3x}{\pi} & 1 \end{bmatrix},$$

then (24) becomes

$$\mathbf{B}(x)=\begin{bmatrix} 1\\ x-\frac{1}{2}\\ x^2-x+\frac{1}{6}\\ x^3-\frac{3}{2}x^2+\frac{1}{2}x \end{bmatrix}.$$

That is, the entries of the matrix $\mathbf{B}(x)$ are the first four classical Bernoulli polynomials.

It is worth noting that for $\alpha = m = 1$, the HBG polynomials $\mathscr{V}_n^{[0,1]}(x)$ coincide with the GBG polynomials $\mathscr{V}_n^{(1)}(x)$, for all $n \geq 0$ (cf. [7]).

Example 3.2. Let $m = n = 3$ and $\alpha = -\frac{1}{4}$. From (23), we obtain

$$\mathbf{C}^{\left(-\frac{1}{4}\right)}(x)\mathbf{B}^{[2]}(x) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\frac{x}{4\pi} & 1 & 0 & 0 \\ -\frac{3}{16\pi^2}\left(x^2-\frac{2}{3}\right) & -\frac{x}{2\pi} & 1 & 0 \\ -\frac{21}{64\pi^3}\left(x^3-\frac{6x}{7}\right) & -\frac{9}{16\pi^2}\left(x^2-\frac{2}{3}\right) & -\frac{3x}{4\pi} & 1 \end{bmatrix} \begin{bmatrix} 6 \\ 6x-\frac{3}{2} \\ 6x^2-3x+\frac{3}{20} \\ 6x^3-\frac{9x^2}{2}+\frac{9x}{20}+\frac{3}{80} \end{bmatrix}$$

$$= \begin{bmatrix} 6 \\ -\frac{3x}{2\pi}+6x-\frac{3}{2} \\ \frac{-45x^2+6\pi^2\left(40x^2-20x+1\right)+30\pi(1-4x)x+30}{40\pi^2} \\ \frac{3\left(-6\pi^2x\left(40x^2-20x+1\right)+15x\left(6-7x^2\right)-15\pi\left(12x^3-3x^2-8x+2\right)+\pi^3\left(320x^3-240x^2+24x+2\right)\right)}{160\pi^3} \end{bmatrix}.$$

Straightforward calculations show that this last matrix coincides with

$$\mathbf{V}^{[2,-\frac{1}{4}]}(x) = \begin{bmatrix} 6 \\ 6\left(1-\frac{1}{4\pi}\right)x-\frac{3}{2} \\ 6\left(1-\frac{1}{2\pi}-\frac{3}{16\pi^2}\right)x^2-3\left(1-\frac{1}{4\pi}\right)x+\frac{3}{20}+\frac{3}{4\pi^2} \\ 6\left(1-\frac{3}{4\pi}-\frac{9}{16\pi^2}-\frac{21}{64\pi^2}\right)x^3-\frac{9}{2}\left(1-\frac{1}{2\pi}-\frac{3}{16\pi^2}\right)x^2+\frac{9}{4}\left(\frac{1}{5}-\frac{1}{20\pi}+\frac{1}{\pi^2}+\frac{3}{4\pi^3}\right)x+\frac{3}{80}-\frac{9}{16\pi^2} \end{bmatrix}.$$

Hence, $\mathbf{C}^{\left(-\frac{1}{4}\right)}(x)\mathbf{B}^{[2]}(x) = \mathbf{V}^{[2,-\frac{1}{4}]}(x)$.

We can now proceed as outlined in [8]. From the summation Formula (4) it follows

$$B_r^{[m-1]}(x) = \mathbf{M}_r^{[m-1]}\mathbf{T}(x), \quad r = 0, 1, \ldots, n,$$

where

$$\mathbf{M}_r^{[m-1]} = \left[\binom{r}{r}B_r^{[m-1]} \ \binom{r}{r-1}B_{r-1}^{[m-1]} \cdots \binom{r}{0}B_0^{[m-1]} \ 0 \cdots 0\right], \tag{25}$$

the null entries of the matrix $\mathbf{M}_r^{[m-1]}$ appear $(n-r)$-times, and $\mathbf{T}(x) = \left[1 \ x \cdots x^n\right]^T$.

Analogously, by (25) the matrix $\mathbf{B}^{[m-1]}(x)$, can be expressed as follows:

$$\mathbf{B}^{[m-1]}(x) = \mathbf{M}^{[m-1]}\mathbf{T}(x)$$
$$= \begin{bmatrix} B_0^{[m-1]} & 0 & \cdots & 0 \\ \binom{1}{1}B_1^{[m-1]} & \binom{1}{0}B_0^{[m-1]} & \cdots & 0 \\ \binom{2}{2}B_2^{[m-1]} & \binom{2}{1}B_1^{[m-1]} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \binom{n}{n}B_n^{[m-1]} & \binom{n}{n-1}B_{n-1}^{[m-1]} & \cdots & \binom{n}{0}B_0^{[m-1]} \end{bmatrix}\mathbf{T}(x). \tag{26}$$

Notice that according to (25) the rows of the matrix $\mathbf{M}^{[m-1]}$ are precisely the matrices $\mathbf{M}_r^{[m-1]}$ for $r = 0, \dots, n$. Furthermore, the matrix $\mathbf{M}^{[m-1]}$ is a lower triangular matrix, so that $\det\left(\mathbf{M}^{[m-1]}\right) = (m!)^{n+1}$. Therefore, $\mathbf{M}^{[m-1]}$ is a nonsingular matrix.

Another interesting algebraic property of the HBG polynomials is related with the following matrix-inversion formula.

Theorem 3.4. *For a fixed $m \in \mathbb{N}$ and any $\alpha \in (-1/2, \infty)$, let $\left\{\mathscr{V}_n^{[m-1,\alpha]}(x)\right\}_{n\geq 0}$ be the sequence of HBG polynomials. Then, the following formula holds.*

$$\mathbf{T}(x) = \left(\mathbf{Q}^{[m-1,\alpha]}(x)\right)^{-1}\mathbf{V}^{[m-1,\alpha]}(x), \tag{27}$$

where $\mathbf{Q}^{[m-1,\alpha]}(x) = \mathbf{C}^{(\alpha)}(x)\mathbf{M}^{[m-1]}$.

A simple and important consequence of Theorem 3.4 is:

Corollary 3.1. *For a fixed $m \in \mathbb{N}$ and any $\alpha \in (-1/2, \infty)$ the set $\left\{\mathscr{V}_0^{[m-1,\alpha]}(x), \dots, \mathscr{V}_n^{[m-1,\alpha]}(x)\right\}$ is a basis for $\mathbb{P}_n$, $n \geq 0$, i.e.,*

$$\mathbb{P}_n = \operatorname{span}\left\{\mathscr{V}_0^{[m-1,\alpha]}(x), \mathscr{V}_1^{[m-1,\alpha]}(x), \dots, \mathscr{V}_n^{[m-1,\alpha]}(x)\right\}.$$

We finish this section with the following new result:

Theorem 3.5. *For a fixed $m \in \mathbb{N}$ and any $\alpha = r \in \mathbb{N}$, the polynomials $y = A_{j,k}(x)\mathscr{V}_{n-k-r}^{[m-1,r]}(x)$, with $A_{j,k}(x) = \frac{(-1)^j}{2^k\pi^{k+r}}\binom{r}{j,k}x^j$, satisfy the following ordinary differential equation:*

$$\begin{aligned}\sum_{j+k=r}\Bigg(&\frac{B_n^{[m-1]}}{(n-k-r)!}y^{(n)} + \frac{nB_{n-1}^{[m-1]}}{(n-k-r)!}y^{(n-1)} + \cdots + \frac{n(n-1)\cdots 3}{(n-k-r)!}B_2^{[m-1]}y'' \\ &+(m-1)!\left(\frac{1}{m+1} - x\right)\frac{n!}{(n-k-r)!}y' + m(m-1)!\frac{n!}{(n-k-r)!}y\Bigg) = 0.\end{aligned} \tag{28}$$

Proof. Using (18) and taking $\mathscr{V}_{n-k-r}^{[m-1,r]}(x) = 0$ for $n-k-r < 0$ we can deduce that

$$\begin{aligned}B_n^{[m-1]}(x) &= \sum_{j+k=r}\frac{(-1)^j}{2^k\pi^{k+r}}\binom{r}{j,k}x^j\frac{n!}{(n-k-r)!}\mathscr{V}_{n-k-r}^{[m-1,r]}(x) \\ &= \sum_{j+k=r}A_{j,k}(x)\frac{n!}{(n-k-r)!}\mathscr{V}_{n-k-r}^{[m-1,r]}(x).\end{aligned}$$

Hence the substitution of this last identity into (7) yields (28). □

4. Concluding remarks

In the present paper, we collect some recent results concerning mixed-type hypergeometric Bernoulli-Gegenbauer polynomials and use some them to deduce an ordinary differential equation satisfied by these polynomials (Theorem 3.5). Since the HBG polynomials do not fulfill either Hanh or Appell conditions (see Theorem 3.1) we can conclude that a general methodology involving operational methods could fail for this family of polynomials (see for instance, [8]).

Furthermore, we provided some examples to illustrate that the class of HBG polynomials does not generalize to the classical Bernoulli polynomials, although the latter can be recovered using Theorem 3.3. Unfortunately, the numerical evidence suggests that the zero distribution of the HBG polynomials does not align with the behavior of either Bernoulli hypergeometric polynomials or Gegenbauer polynomials.

Finally, the use of Theorem 3.4 and the differential equation (7) allow to prove that the HBG polynomials satisfy a differential equation of order n (see Theorem 3.5).

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