sciendo
Control and Cybernetics
vol. **51** (2022) No. 1
pages: 71-88
DOI: 10.2478/candc-2022-0005

Sufficient optimality condition and duality of nondifferentiable minimax ratio constraint problems under (p, r)-ρ-(η, θ)-invexity*

by

Navdeep Kailey, Sonali Sethi and Shivani Saini

Thapar Institute of Engineering and Technology,
Patiala, Punjab 147004, India
nkailey@thapar.edu
sethisonali22@gmail.com
ssaini_phd18@thapar.edu

Abstract: There are several classes of decision-making problems that explicitly or implicitly prompt fractional programming problems. Portfolio selection problems, agricultural planning, information transfer, numerical analysis of stochastic processes, and resource allocation problems are just a few examples. The huge number of applications of minimax fractional programming problems inspired us to work on this topic. This paper is concerned with a nondifferentiable minimax fractional programming problem. We study a parametric dual model, corresponding to the primal problem, and derive the sufficient optimality condition for an optimal solution to the considered problem. Further, we obtain the various duality results under (p, r)-ρ-(η, θ)-invexity assumptions. Also, we identify a function lying exclusively in the class of $(-1, 1)$-ρ-(η, θ)-invex functions but not in the class of $(1, -1)$-invex functions and convex function already existing in the literature. We have given a non-trivial model of nondifferentiable minimax problem and obtained its optimal solution using optimality results derived in this paper.

Keywords: minimax fractional programming, optimality conditions, duality, generalized invexity

1. Intoduction

In the field of mathematical programming, minimax problems are optimization problems that involve both minimization and maximization processes. Minimax

*Submitted: July 2021; Accepted: January 2022.

is a decision rule used to minimise the possible loss in the worst-case (maximum loss) scenario. Engineering design, circuit design, and optimal control provide the examples of minimax problems. Fractional programming is an intriguing topic, in these types of problems the objective function is generally presented as a ratio function. A variety of decision-making problems lead directly or indirectly to fractional programming problems. There has been extensive research into the properties of fractional programming problems, see Ahmad and Husain (2006), Du and Pardalos (1995), Husain et al. (2009), Lai and Lee (2002). Some examples of fractional programming are related to system efficiency measures, portfolio selection problems, agricultural planning, information transfer, numerical analysis of stochastic processes, resource allocation problems, cargo loading problems, etc. In physics, maximization of signal-to-noise ratio of a spectral filter gives rise to concave quadratic fractional program, which was studied by Falk (1969). This large number of applications motivated us to work on the optimality and duality conditions of minimax fractional programming problems.

In the line of the above, during the last period, much attention has been paid to minimax fractional programming problems. Earlier, Schmitendorf (1977) obtained necessary and sufficient conditions for static minimax problems. See also Ahmad (2003), Jayswal (2002), Long and Quan (2011) for further information on the respective optimality and duality theorems for minimax fractional programming problems. Tanimoto (1981) applied these conditions to a dual problem and proved various duality results. Liu and Wu (1998) established the sufficient conditions, parametric dual and parameter free dual for generalized fractional programming under (F, ρ)-convex functions. Zheng and Cheng (2007) studied the KKT type sufficient optimality conditions and established duality results for parametric dual and parameter free duals, corresponding to a non differentiable minimax fractional problem, with inequality constraints under $(\mathfrak{F}, \rho, \theta)$-$d$-univex function. A parametric dual model for minimax fractional programming problem was studied by Ahmad et al. (2011). Lai and Liu (2011) employed the elementary method and technique to prove the optimality conditions for nondifferentiable minimax fractional programming problem, involving convexity, and further formulated a parametric dual. Khan and Al-Solamy (2015) obtained sufficient optimality conditions and duality relations for nondifferentiable minimax fractional programming problem under (H_p, r)-invexity. More recently, Antczak, Mishra and Upadhyay (2018) established optimality conditions and duality results for generalized fractional minimax programming problems. Dubey and Mishra (2020) considered a nondifferentiable multiobjective fractional programming problem over cone constraints and further formulated a higher-order symmetric dual, establishing various duality results. Boufi and Roubi (2019) studied the duality results and the dual bundle methods for minimax fractional programs. Finally, Son and Kim (2021) proposed a dual scheme for solving linear countable semi-infinite fractional programming problems.

In this paper, we consider the following nondifferentiable minimax fractional programming problem:

$$(NFP) \qquad \text{Minimize } \psi(s) = \sup_{y\in Y} \frac{d(s,y) + (s^T Ls)^{1/2}}{e(s,y) - (s^T Ns)^{1/2}},$$
$$\text{subject to } h(s) \leq 0,$$

where Y is a compact subset of $R^{l'}$, $d(\cdot,\cdot) : R^n \times R^{l'} \to R$, $e(\cdot,\cdot) : R^n \times R^{l'} \to R$ are C^1 functions on $R^n \times R^{l'}$ and $h(\cdot) : R^n \to R^m$ is C^1 function on R^n. L and N are $n \times n$ positive semidefinite matrices. We assume that $e(s,y) - (s^T Ns)^{1/2} > 0$ and $d(s,y) + (s^T Ls)^{1/2} \geq 0$ for each $(s,y) \in S \times Y$, where $S = \{s \in R^n : h(s) \leq 0\}$ denotes the set of all feasible solutions of (NFP).

Antczak (2001) introduced p-invex sets and (p,r)-invex functions and established sufficient conditions for a nonlinear programming problem under (p,r)-invexity assumptions. Further generalization of (p,r)-invex functions was studied by Mandal and Nahak (2011), leading to (p,r)-ρ-(η,θ)-invex functions. Recently, higher-order duality results for nondifferentiable minimax fractional programming problem were derived by Sonali et al. (2020) using generalized B-(p,r)-invex functions. In this article, we derive duality theorems in order to relate nondifferentiable minimax fractional programming problem and its parametric dual model under (p,r)-ρ-(η,θ)-invexity assumptions.

This paper is structured as follows. Section 2 reviews (p,r)-ρ-(η,θ)-invex functions and sufficient optimality conditions for optimal solution to the nondifferentiable minimax fractional programming problem. Along with this, we have formulated an example, which is $(-1,1)$-ρ-(η,θ)-invex but not $(1,-1)$-invex and not convex. Further, a parametric dual model and duality results are discussed under aforesaid assumptions in Section 3. Also in Section 3, we have given the non-trivial example of the given model, validating our results.

2. Notations and preliminaries

For each $(s,y) \in S \times Y$ and $M = \{1,2,\ldots,m\}$, we define

$$J(s) = \{j \in M : h_j(s) = 0\},$$

$$Y(s) = \left\{ y \in Y : \frac{d(s,y) + (s^T Ls)^{1/2}}{e(s,y) - (s^T Ns)^{1/2}} = \sup_{b\in Y} \frac{d(s,b) + (s^T Ls)^{1/2}}{e(s,b) - (s^T Ns)^{1/2}} \right\},$$

$$K(s) = \Bigg\{ (q,\xi,\bar{y}) \in N \times R^q_+ \times R^{l'} : 1 \leq q \leq n+1,\ \xi = (\xi_1,\xi_2,\ldots,\xi_q) \in R^q_+,$$
$$\sum_{i=1}^{q} \xi_i = 1,\ \bar{y} = (\bar{y}_1,\bar{y}_2,\ldots,\bar{y}_q),\ \bar{y}_i \in Y(s), i = 1,2,\ldots,q \Bigg\}.$$

Since d and e are continuously differentiable and Y is compact in $R^{l'}$, it follows that for each $s^* \in S$, $Y(s^*) \neq \phi$, and for any $\bar{y}_i \in Y(s^*)$, we have a positive constant

$$k_0 = \psi(s^*) = \frac{d(s^*,\bar{y}_i) + (s^{*T} Ls^*)^{1/2}}{e(s^*,\bar{y}_i) - (s^{*T} Ns^*)^{1/2}}.$$

Definition 1 (Mandal and Nahak, 2011) *Let $\varphi : X \to R$ (where $X \subset R^n$) be a differentiable function and let p, r be arbitrary real numbers. Then, φ is said to be (p,r)-ρ-(η,θ)-invex (strictly (p,r)-ρ-(η,θ)-invex) with respect to η and θ at $u \in X$ on X if there exist $\eta, \theta : X \times X \to R^n$ and $\rho \in R$ such that for all $s \in X$, the following inequalities hold*

$$\Big[\frac{1}{r}(e^{r(\varphi(s)-\varphi(u))}-1)\Big] \geq \frac{1}{p}\nabla\varphi(u)(e^{p\eta(s,u)}-\mathbf{1}) + \rho||\theta(s,u)||^2$$
$$(> \ \text{if } s \neq u) \text{ for } p \neq 0, r \neq 0,$$
$$\Big[\frac{1}{r}(e^{r(\varphi(s)-\varphi(u))}-1)\Big] \geq \big[\nabla\varphi(u)\eta(s,u)\big] + \rho||\theta(s,u)||^2$$
$$(> \ \text{if } s \neq u) \text{ for } p = 0, r \neq 0,$$
$$(\varphi(s)-\varphi(u)) \geq \frac{1}{p}[\nabla\varphi(u)(e^{p\eta(s,u)}-\mathbf{1})] + \rho||\theta(s,u)||^2$$
$$(> \ \text{if } s \neq u) \text{ for } p \neq 0, r = 0,$$
$$(\varphi(s)-\varphi(u)) \geq \nabla\varphi(u)\eta(s,u) + \rho||\theta(s,u)||^2 \quad (> \ \text{if } s \neq u) \text{ for } p = 0, r = 0.$$

Let us consider an example to understand the importance of the above given functions.

Example 1 Let $X = [3.5,\ 5.5] \subset R$. Consider the function $f : X \to R$, defined by $f(x) = \log(\log(x))$. Let $\eta(x,u) = 10u$ and $\theta(x,u) = x + u$ for $r = 1, p = -1$ and $\rho = -\frac{1}{2}$.

We are going to show that

$$\frac{1}{r}[e^{r(f(x)-f(u))}-1] - \frac{1}{p}\nabla f(u)(e^{p\eta(x,u)}-1) - \rho\,\|\theta(x,\ u)\|^2 \geq 0$$

at $r = 1, p = -1$ and $\rho = -\frac{1}{2}$. Indeed, we have

$$[e^{(\log(\log(x))-\log(\log(u)))}-1] + \frac{1}{u\log(u)}(e^{-10u}-1) + \frac{1}{2}(x+u)$$
$$= \Big[\frac{\log(x)}{\log(u)}-1\Big] + \frac{1}{u\log(u)}(e^{-10u}-1) + \frac{1}{2}(x+u) \geq 0 \quad \forall x, u \in X.$$

Hence, f is $(-1,1)$-ρ-(η,θ)-invex.

Further, for $x = 5.5$ and $u = 3.5$, we have

$$f(x) - f(u) - (x-u)^T\nabla f(u) = \log(\log(x)) - \log(\log(u)) - (x-u)\frac{1}{u\log(u)}$$
$$= -0.14806876 < 0.$$

Thus, f is not a convex function on x.

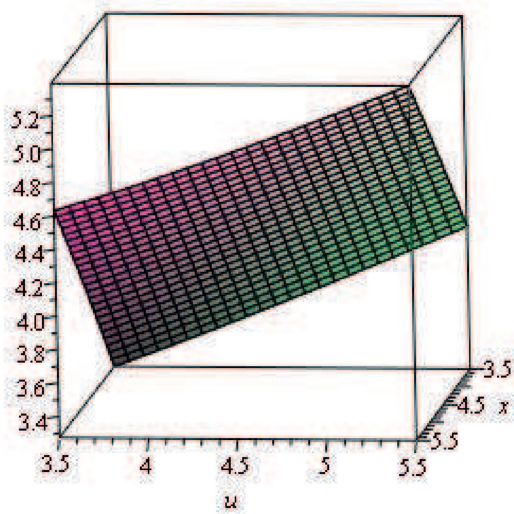

Figure 1. Example 1

Moreover, in order to show that the considered function is not (p, r)-invex for $r = 1$ and $p = -1$, that means, to demonstrate that

$$\frac{1}{r}e^{rf(x)} - \frac{1}{r}e^{rf(u)}\Big[1 + \frac{r}{p}\nabla f(u)(e^{p\eta(x,\ u)} - 1)\Big] < 0$$

for $r = 1$ and $p = -1$, we calculate

$$= e^{\log(\log(x))} - e^{\log(\log(u))}\Big[1 - \frac{1}{u\log(u)}(e^{-10u} - 1)\Big]$$
$$= \log(x) - \log(u)\Big[1 - \frac{1}{u\log(u)}(e^{-10u} - 1)\Big]$$

at $x = 3.5$ and $u = 5.5$ this yields

$$\frac{1}{r}e^{rf(x)} - \frac{1}{r}e^{rf(u)}\Big[1 + \frac{r}{p}\nabla f(u)(e^{p\eta(x,u)} - 1)\Big] = -0.6338033 < 0.$$

Thus, f is not $(1, -1)$-invex.

Thus, the function f is $(-1,1)$-ρ-(η,θ)-invex but not $(1,-1)$-invex or convex.

Lemma 1 (Generalized Schwartz inequality) *Let G be a semidefinite matrix of order n. Then, for all $s, w \in R^n$,*

$$s^T Gw \leq (s^T Gs)^{1/2}(w^T Gw)^{1/2}. \tag{1}$$

The equality holds if $Gs = \lambda Gw$ for some $\lambda \geq 0$.

Theorem 1 (Lai et al., 1999) *[Necessary Condition] If s^* is an optimal solution of the problem (NFP) satisfying $s^{*T}Ls^* > 0$, $s^{*T}Ns^* > 0$, and $\nabla h_j(s^*)$, $j \in J(s^*)$, are linearly independent, then there exist $(q, \xi^*, \bar{y}) \in K(s^*)$, $k_0 \in R_+, w, v \in R^n$ and $\zeta^* \in R_+^m$ such that*

$$\sum_{i=1}^{q} \xi_i^* \{\nabla d(s^*, \bar{y}_i) + Lw - k_0(\nabla e(s^*, \bar{y}_i) - Nv)\} + \nabla \sum_{j=1}^{m} \zeta_j^* h_j(s^*) = 0, \tag{2}$$

$$d(s^*, \bar{y}_i) + (s^{*T}Ls^*)^{1/2} - k_0(e(s^*, \bar{y}_i) - (s^{*T}Ns^*)^{1/2}) = 0, i = 1, 2, \ldots, q, \tag{3}$$

$$\sum_{j=1}^{m} \zeta_j^* h_j(s^*) = 0, \tag{4}$$

$$\xi_i^* \geq 0 \ (i = 1, 2, \ldots, q), \quad \sum_{i=1}^{q} \xi_i^* = 1, \tag{5}$$

$$w^T Lw \leq 1, \quad v^T Nv \leq 1, \quad (s^{*T}Ls^*)^{1/2} = s^{*T}Lw, \quad (s^{*T}Ns^*)^{1/2} = s^{*T}Nv. \tag{6}$$

In the above theorem, both matrices L and N are positive definite. If one of $s^{*T}Ls^*$ and $s^{*T}Ns^*$ is zero or both of them are zero, then the functions involved in the objective of the problem (NFP) are not differentiable. To derive the necessary conditions under this situation, for $(q, \xi^*, \bar{y}) \in K(s^*)$, we define

$$Z_{\bar{y}}(s^*) = \big\{z \in R^n : z^T \nabla h_j(s^*) \leq 0, j \in J(s^*), \text{ with anyone of the following conditions, (i)-(iii), holding}\big\}:$$

(i) $s^{*T}Ls^* > 0, \quad s^{*T}Ns^* = 0.$

$$\Rightarrow z^T \left(\sum_{i=1}^{q} \xi_i^* \left\{ \nabla d(s^*, \bar{y}_i) + \frac{Ls^*}{(s^{*T}Ls^*)^{1/2}} - k_0 \nabla e(s^*, \bar{y}_i) \right\} \right) + (z^T(k_0^2 N)z)^{1/2} < 0,$$

(ii) $s^{*T}Ls^* = 0, \quad s^{*T}Ns^* > 0$

$$\Rightarrow z^T \left(\sum_{i=1}^{q} \xi_i^* \left\{ \nabla d(s^*, \bar{y}_i) - k_0 \left(\nabla e(s^*, \bar{y}_i) - \frac{Ns^*}{(s^{*T}Ns^*)^{1/2}} \right) \right\} \right) + (z^T Lz)^{1/2} < 0,$$

(iii) $s^{*T}Ls^* = 0, \quad s^{*T}Ns^* = 0$

$$\Rightarrow z^T \left(\sum_{i=1}^{q} \xi_i^* \left\{ \nabla d(s^*, \bar{y}_i) - k_0 \nabla e(s^*, \bar{y}_i) \right\} \right) + (z^T(k_0^2 N)z)^{1/2} + (z^T Lz)^{1/2} < 0,$$

If, in addition, we insert the condition $Z_{\bar{y}}(s^*) = \phi$, then the result of Theorem 1 still holds.

Remark 1 *All the results in this paper will be given only for the case when* $p \neq 0,\ r \neq 0$. *The proofs for the other cases are easier than for this one. This follows from the form of inequalities, which are given in Definition 1. Moreover, without limiting the generality considerations, we shall assume that* $r > 0$.

Theorem 2 *(Sufficient Condition) Let* s^* *be a feasible solution of (NFP) and there exist: a positive integer* $q,\ 1 \leq q \leq n+1,\ \xi^* \in R_+^q,\ \bar{y}_i \in Y(s^*)(i = 1, 2, \ldots, q),\ k_0 \in R_+,\ w,\ v \in R^n$ *and* $\zeta^* \in R_+^m$ *satisfying the relations (2)-(6). Assume that*

(i) $\sum_{i=1}^{q} \xi_i^*(d(\cdot, \bar{y}_i) + (\cdot)^T Lw - k_0(e(\cdot, \bar{y}_i) - (\cdot)^T Nv))$ *is* (p, r)-ρ^1-(η, θ)*-invex function at* s^* *with respect to* η *and* θ *for all* $s \in S$,

(ii) $\sum_{j=1}^{m} \zeta_j^* h_j(\cdot)$ *is* (p, r)-ρ^2-(η, θ)*-invex function at* s^* *with respect to the same function* η *and* θ,

(iii) $\rho^1 + \rho^2 \geq 0$.

Then, s^* *is an optimal solution of problem (NFP).*

Proof Suppose, to the contrary, that s^* is not an optimal solution of (NFP). Then there exists an $\bar{s} \in S$ such that

$$\sup_{\bar{y} \in Y} \frac{d(\bar{s}, \bar{y}) + (\bar{s}^T L \bar{s})^{1/2}}{e(\bar{s}, \bar{y}) - (\bar{s}^T N \bar{s})^{1/2}} < \sup_{\bar{y} \in Y} \frac{d(s^*, \bar{y}) + (s^{*T} L s^*)^{1/2}}{e(s^*, \bar{y}) - (s^{*T} N s^*)^{1/2}}.$$

We note that

$$\sup_{\bar{y} \in Y} \frac{d(s^*, \bar{y}) + (s^{*T} L s^*)^{1/2}}{e(s^*, \bar{y}) - (s^{*T} N s^*)^{1/2}} = \frac{d(s^*, \bar{y}_i) + (s^{*T} L s^*)^{1/2}}{e(s^*, \bar{y}_i) - (s^{*T} N s^*)^{1/2}} = k_0,$$

for any $\bar{y}_i \in Y(s^*), i = 1, 2, \ldots, q$ and

$$\frac{d(\bar{s}, \bar{y}_i) + (\bar{s}^T L \bar{s})^{1/2}}{e(\bar{s}, \bar{y}_i) - (\bar{s}^T N \bar{s})^{1/2}} \leq \sup_{\bar{y} \in Y} \frac{d(\bar{s}, \bar{y}) + (\bar{s}^T L \bar{s})^{1/2}}{e(\bar{s}, \bar{y}) - (\bar{s}^T N \bar{s})^{1/2}}.$$

Therefore, we have

$$\frac{d(\bar{s}, \bar{y}_i) + (\bar{s}^T L \bar{s})^{1/2}}{e(\bar{s}, \bar{y}_i) - (\bar{s}^T N \bar{s})^{1/2}} < k_0.$$

Also from $\xi_i^* \geq 0, i = 1, 2, \ldots, q, \quad \xi^* \neq 0$ and $\bar{y}_i \in Y(s^*)$, we get

$$\sum_{i=1}^{q} \xi_i^* \big[d(\bar{s}, \bar{y}_i) + (\bar{s}^T L \bar{s})^{1/2} - k_0\big(e(\bar{s}, \bar{y}_i) - (\bar{s}^T N \bar{s})^{1/2}\big)\big] < 0. \tag{7}$$

Using (1), (3), (6) and (7), we obtain

$$\begin{aligned}
&\sum_{i=1}^{q}\xi_i^*\big[d(\bar{s},\bar{y}_i)+\bar{s}^TLw-k_0\big(e(\bar{s},\bar{y}_i)-\bar{s}^TNv\big)\big]\\
&\le\sum_{i=1}^{q}\xi_i^*\big[d(\bar{s},\bar{y}_i)+(\bar{s}^TL\bar{s})^{1/2}-k_0\big(e(\bar{s},\bar{y}_i)-(\bar{s}^TN\bar{s})^{1/2}\big)\big]\\
&<0=\sum_{i=1}^{q}\xi_i^*\big[d(s^*,\bar{y}_i)+(s^{*T}Ls^*)^{1/2}-k_0\big(e(s^*,\bar{y}_i)-(s^{*T}Ns^*)^{1/2}\big)\big]\\
&=\sum_{i=1}^{q}\xi_i^*\big[d(s^*,\bar{y}_i)+s^{*T}Lw-k_0\big(e(s^*,\bar{y}_i)-s^{*T}Nv\big)\big].
\end{aligned}$$

It follows that

$$\begin{aligned}
&\sum_{i=1}^{q}\xi_i^*\big[d(\bar{s},\bar{y}_i)+\bar{s}^TLw-k_0\big(e(\bar{s},\bar{y}_i)-\bar{s}^TNv\big)\big]\\
&<\sum_{i=1}^{q}\xi_i^*\big[d(s^*,\bar{y}_i)+s^{*T}Lw-k_0\big(e(s^*,\bar{y}_i)-s^{*T}Nv\big)\big].
\end{aligned}\tag{8}$$

As

$$\sum_{i=1}^{q}\xi_i^*\big[d(\cdot,\bar{y}_i)+(\cdot)^TLw-k_0\big(e(\cdot,\bar{y}_i)-(\cdot)^TNv\big)\big]$$

is (p,r)-ρ^1-(η,θ)-invex at s^* on S with respect to η and θ, we have that

$$\begin{aligned}
&\frac{1}{r}(e^{r(A-B)}-1)\ge\\
&\ge\frac{1}{p}\Big(\sum_{i=1}^{q}\xi_i^*\big(\nabla d(s^*,\bar{y}_i)+Lw-k_0\big(\nabla e(s^*,\bar{y}_i)-Nv\big)\big)\Big)\big(e^{p\eta(s,s^*)}-\mathbf{1}\big)+\rho^1||\theta(s,s^*)||^2
\end{aligned}$$

where

$$A=\sum_{i=1}^{q}\xi_i^*(d(s,\bar{y}_i)+s^TLw-k_0\big(e(s,\bar{y}_i)-s^TNv\big))$$

$$B=\sum_{i=1}^{q}\xi_i^*\big(d(s^*,\bar{y}_i)+s^{*T}Lw-k_0\big(e(s^*,\bar{y}_i)-s^{*T}Nv\big)\big)$$

holds for all $s\in S$, and so for $s=\bar{s}$. Using (8), together with the inequality above, we get

$$\begin{aligned}
&\frac{1}{p}\Big[\Big(\sum_{i=1}^{q}\xi_i^*\big[\nabla d(s^*,\bar{y}_i)+Lw-k_0\big(\nabla e(s^*,\bar{y}_i)-Nv\big)\big]\Big)\big(e^{p\eta(\bar{s},s^*)}-\mathbf{1}\big)\Big]\\
&\qquad+\rho^1||\theta(\bar{s},s^*)||^2<0.
\end{aligned}\tag{9}$$

From the feasibility of $\bar{s}$, together with $\zeta_j^* \geq 0,\ j \in J$, we have

$$\sum_{j=1}^{m} \zeta_j^* h_j(\bar{s}) \leq 0. \tag{10}$$

By (p,r)-ρ^2-(η,θ)-invexity of $\sum\limits_{j=1}^{m} \zeta_j^* h_j(\cdot)$ at s^* on S with respect to the same function η and θ, we get

$$\begin{aligned}&\frac{1}{r}\left(e^{r\left(\sum\limits_{j=1}^{m}\zeta_j^* h_j(\bar{s})-\sum\limits_{j=1}^{m}\zeta_j^* h_j(s^*)\right)}-1\right)\\ &\geq \frac{1}{p}\left[\sum_{j=1}^{m}\nabla\zeta_j^* h_j(s^*)\left(e^{p\eta(\bar{s},s^*)}-\mathbf{1}\right)\right]+\rho^2||\theta(\bar{s},s^*)||^2.\end{aligned}$$

Using (4), (10) and the above inequality, we get

$$\frac{1}{p}\left[\sum_{j=1}^{m}\nabla\zeta_j^* h_j(s^*)\left(e^{p\eta(\bar{s},s^*)}-\mathbf{1}\right)\right]+\rho^2||\theta(\bar{s},s^*)||^2 \leq 0. \tag{11}$$

By summing up (9) and (11), we obtain

$$\begin{aligned}&\frac{1}{p}\Bigg[\Big(\sum_{i=1}^{q}\left[\nabla d(s^*,\bar{y}_i)+Lw-k_0\left(\nabla e(s^*,\bar{y}_i)-Nv\right)\right]\\ &+\sum_{j=1}^{m}\nabla\zeta_j^* h_j(s^*)\Big)\left(e^{p\eta(\bar{s},s^*)}-\mathbf{1}\right)\Bigg]+(\rho^1+\rho^2)||\theta(\bar{s},s^*)||^2<0.\end{aligned}$$

Using (2), we get

$$(\rho^1+\rho^2)||\theta(\bar{s},s^*)||^2<0,$$

which contradicts hypothesis (iii). Hence the result. □

3. Duality results

In this section, we consider the following dual to (NFP):

(NFD) $\max\limits_{(q,\xi,\bar{y})\in K(c)} \ \sup\limits_{(c,\zeta,k,w,v)\in H_1(q,\xi,\bar{y})} k,$

where $H_1(q,\xi,\bar{y})$ denotes the set of all $(c,\zeta,k,w,v)\in R^n\times R_+^m\times R_+\times R^n\times R^n$

satisfying

$$\sum_{i=1}^{q} \xi_i \{\nabla d(c, \bar{y}_i) + Lw - k(\nabla e(c, \bar{y}_i) - Nv)\} + \nabla \sum_{j=1}^{m} \zeta_j h_j(c) = 0, \tag{12}$$

$$\sum_{i=1}^{q} \xi_i \{d(c, \bar{y}_i) + c^T Lw - k(e(c, \bar{y}_i) - c^T Nv)\} \geq 0, \tag{13}$$

$$\sum_{j=1}^{m} \zeta_j h_j(c) \geq 0, \tag{14}$$

$$(q, \xi, \bar{y}) \in K(c), \tag{15}$$

$$w^T Lw \leq 1, \quad v^T Nv \leq 1. \tag{16}$$

If for a triplet $(q, \xi, \bar{y}) \in K(c)$, the set $H_1(q, \xi, \bar{y}) = \phi$, then we define the supremum over it to be $-\infty$.

Now we derive the following weak, strong and strict converse duality theorems.

Theorem 3 (Weak duality) *Let s be a feasible solution of (NFP) and $(c, \zeta, k, w, v, q, \xi, \bar{y})$ be a feasible solution of (NFD). Assume that*

(i) $\sum_{i=1}^{q} \xi_i(d(\cdot, \bar{y}_i) + (\cdot)^T Lw - k(e(\cdot, \bar{y}_i) - (\cdot)^T Nv))$ *is (p, r)-ρ^1-(η, θ)-invex at c with respect to η and θ,*

(ii) $\sum_{j=1}^{m} \zeta_j h_j(\cdot)$ *is (p, r)-ρ^2-(η, θ)-invex at c with respect to the same η and θ,*

(iii) $\rho^1 + \rho^2 \geq 0$.

Then,

$$\sup_{y \in Y} \frac{d(s, y) + (s^T Ls)^{1/2}}{e(s, y) - (s^T Ns)^{1/2}} \geq k. \tag{17}$$

Proof Suppose, contrary to the above hypothesis, that

$$\sup_{y \in Y} \frac{d(s, y) + (s^T Ls)^{1/2}}{e(s, y) - (s^T Ns)^{1/2}} < k.$$

From the above, we have

$$d(s, \bar{y}_i) + (s^T Ls)^{1/2} - k(e(s, \bar{y}_i) - (s^T Ns)^{1/2}) < 0, \text{ for all } \bar{y}_i \in Y.$$

Using (5), we get

$$\xi_i(d(s, \bar{y}_i) + (s^T Ls)^{1/2} - k(e(s, \bar{y}_i) - (s^T Ns)^{1/2})) < 0, \tag{18}$$

with at least one strict inequality, since $\xi = (\xi_1, \xi_2, \ldots, \xi_q) \neq 0$.

From (1), (13), (16), and (18), we have

$$\begin{aligned}
&\sum_{i=1}^{q}\xi_i\big[d(s,\bar{y}_i)+s^TLw-k\big(e(s,\bar{y}_i)-s^TNv\big)\big]\\
&\le \sum_{i=1}^{q}\xi_i\big[d(s,\bar{y}_i)+(s^TLs)^{1/2}-k\big(e(s,\bar{y}_i)-(s^TNs)^{1/2}\big)\big]\\
&< 0 \le \sum_{i=1}^{q}\xi_i\big[d(c,\bar{y}_i)+c^TLw-k\big(e(c,\bar{y}_i)-c^TNv\big)\big].
\end{aligned}$$

Hence

$$\begin{aligned}
&\sum_{i=1}^{q}\xi_i\big[d(s,\bar{y}_i)+s^TLw-k\big(e(s,\bar{y}_i)-s^TNv\big)\big]\\
&< \sum_{i=1}^{q}\xi_i\big[d(c,\bar{y}_i)+c^TLw-k\big(e(c,\bar{y}_i)-c^TNv\big)\big].
\end{aligned} \tag{19}$$

Since

$$\sum_{i=1}^{q}\xi_i\big(d(\cdot,\bar{y}_i)+(\cdot)^TLw-k(e(\cdot,\bar{y}_i)-(\cdot)^TNv)\big)$$

is (p,r)-ρ^1-(η,θ)-invex at c with respect to η and θ, we have

$$\begin{aligned}
&\frac{1}{r}(e^{r(C-D)}-1)\ge\\
&\ge \frac{1}{p}\Bigg[\Big(\sum_{i=1}^{q}\xi_i\big[\nabla d(c,\bar{y}_i)+Lw-k\big(\nabla e(c,\bar{y}_i)-Nv\big)\big]\Big)\big(e^{p\eta(s,c)}-\mathbf{1}\big)\Bigg]+\rho^1||\theta(s,c)||^2.
\end{aligned}$$

where

$$C=\sum_{i=1}^{q}\xi_i\big(d(s,\bar{y}_i)+s^TLw-k\big(e(s,\bar{y}_i)-s^TNv\big)\big)$$

$$D=\sum_{i=1}^{q}\xi_i\big(d(c,\bar{y}_i)+c^TLw-k\big(e(c,\bar{y}_i)-c^TNv\big)\big).$$

Using (19), together with the inequality above, we get

$$\begin{aligned}
&\frac{1}{p}\Bigg[\Big(\sum_{i=1}^{q}\xi_i\big[\nabla d(c,\bar{y}_i)+Lw-k\big(\nabla e(c,\bar{y}_i)-Nv\big)\big]\Big)\big(e^{p\eta(s,c)}-\mathbf{1}\big)\Bigg]+\rho^1||\theta(s,c)||^2\\
&< 0.
\end{aligned} \tag{20}$$

From the feasibility of s, together with $\zeta_j \geq 0,\ j \in J$, we obtain

$$\sum_{j=1}^{m} \zeta_j h_j(s) \leq 0. \tag{21}$$

By (p,r)-ρ^2-(η,θ)-invexity of $\sum_{j=1}^{m} \zeta_j h_j(\cdot)$ at c with respect to the same function η and θ, we have

$$\frac{1}{r}\big(e^{r\big(\sum_{j=1}^{m}\zeta_j h_j(s)-\sum_{j=1}^{m}\zeta_j h_j(c)\big)}-1\big) \geq \frac{1}{p}\sum_{j=1}^{m}\nabla\zeta_j h_j(c)\big(e^{p\eta(s,c)}-\mathbf{1}\big)+\rho^2||\theta(s,c)||^2.$$

Now, from (14) and (21), we get

$$\frac{1}{p}\sum_{j=1}^{m}\nabla\zeta_j h_j(c)\big(e^{p\eta(s,c)}-\mathbf{1}\big)+\rho^2||\theta(s,c)||^2 \leq 0. \tag{22}$$

By adding (3) and (22), we obtain

$$\frac{1}{p}\Bigg[\Bigg(\sum_{i=1}^{q}\xi_i\big[\nabla d(c,\bar{y}_i)+Lw-k\big(\nabla e(c,\bar{y}_i)-Nv\big)\big]+\sum_{j=1}^{m}\nabla\zeta_j h_j(c)\Bigg)\big(e^{p\eta(s,c)}-\mathbf{1}\big)\Bigg]$$
$$+(\rho^1+\rho^2)||\theta(s,c)||^2 < 0.$$

Consequently, (12) and the above inequality yield

$$(\rho^1+\rho^2)||\theta(s,c)||^2 < 0,$$

which contradicts the fact that $\rho^1+\rho^2 \geq 0$. Hence the result. □

Theorem 4 (Strong duality) *Let s^* be an optimal solution for (NFP) and let $\nabla h_j(s^*), j \in J(s^*)$ be linearly independent.*
Then, there exist $(q^*,\xi^*,\bar{y}^*) \in K(s^*)$ and $(s^*,\zeta^*,k^*,w^*,v^*) \in H_1(q^*,\xi^*,\bar{y}^*)$, such that $(s^*,\zeta^*,k^*,w^*,v^*,q^*,\xi^*,\bar{y}^*)$ is a feasible solution of (NFD).
If, in addition, the assumptions of Theorem 3 hold for all feasible solutions $(c,\zeta,k,w,v,q,\xi,\bar{y})$ of (NFD), then $(s^*,\zeta^*,k^*,w^*,v^*,q^*,\xi^*,\bar{y}^*)$ is an optimal solution of (NFD) and the two objectives have the same optimal values.

Proof Since s^* is an optimal solution of (NFP) and $\nabla h_j(s^*), j \in J(s^*)$ are linearly independent, then, by Theorem 1, there exist $(q^*,\xi^*,\bar{y}^*) \in K(s^*)$ and $(s^*,\zeta^*,k^*,w^*,v^*) \in H_1(q^*,\xi^*,\bar{y}^*)$ such that $(s^*,\zeta^*,k^*,w^*,v^*,q^*,\xi^*,\bar{y}^*)$ is a feasible solution of (NFD) and the two objectives have the same values as

$$k^* = \frac{d(s^*,\bar{y}_i^*)+(s^*Ls^*)^{1/2}}{e(s^*,\bar{y}_i^*)-(s^*Ns^*)^{1/2}}.$$

Optimality of $(s^*,\zeta^*,k^*,w^*,v^*,q^*,\xi^*,\bar{y}^*)$ for (NFD) thus follows from Theorem 3. □

Theorem 5 (Strict converse duality) *Let s^* and $(\bar{c}, \zeta^*, k^*, w^*, v^*, q^*, \xi^*, \bar{y}^*)$ be the optimal solutions of (NFP) and (NFD), respectively, and let $\nabla h_j(s^*), j \in J(s^*)$ be linearly independent. Suppose that*

(i) $\sum_{i=1}^{q^*} \xi_i^*(d(\cdot, \bar{y_i}^*) + (\cdot)^T Lw^* - k^*(e(\cdot, \bar{y_i}^*) - (\cdot)^T Nv^*))$ *is strictly (p, r)-ρ^1-(η, θ)-invex at $\bar{c}$ with respect to η and θ,*

(ii) $\sum_{j=1}^{m} \zeta_j^* h_j(\cdot)$ *is (p, r)-ρ^2-(η, θ)-invex at $\bar{c}$ with respect to same η and θ,*

(iii) $\rho^1 + \rho^2 \geq 0$.

Then, $s^* = \bar{c}$, that is, $\bar{c}$ is an optimal point in (NFP) and

$$\sup_{\bar{y}^* \in Y} \frac{d(\bar{c}, \bar{y}^*) + (\bar{c}^T L \bar{c})^{1/2}}{e(\bar{c}, \bar{y}^*) - (\bar{c}^T N \bar{c})^{1/2}} = k^*.$$

Proof We shall assume that $s^* \neq \bar{c}$ and reach a contradiction. From the strong duality theorem (Theorem 4), it follows that

$$\sup_{\bar{y}^* \in Y} \frac{d(s^*, \bar{y}^*) + (s^{*T} L s^*)^{1/2}}{e(s^*, \bar{y}^*) - (s^{*T} N s^*)^{1/2}} = k^*. \tag{23}$$

By the feasibility of s^*, together with $\zeta_j^* \geq 0, \ j \in J$, we get

$$\sum_{j=1}^{m} \zeta_j^* h_j(s^*) \leq 0. \tag{24}$$

Now, from (14) and (24), we have

$$\frac{1}{r}\big(e^{r\left(\sum_{j=1}^{m} \zeta_j^* h_j(s^*) - \sum_{j=1}^{m} \zeta_j^* h_j(\bar{c})\right)} - 1\big) \leq 0.$$

From hypothesis (ii) and the above, we have

$$\frac{1}{p}\Big[\big(\sum_{j=1}^{m} \nabla \zeta_j^* h_j(\bar{c})\big)\big(e^{p\eta(s^*, \bar{c})} - \mathbf{1}\big)\Big] + \rho^2 ||\theta(s^*, \bar{c})||^2 \leq 0,$$

that is

$$\frac{1}{p}\Big[\big(\sum_{j=1}^{m} \nabla \zeta_j^* h_j(\bar{c})\big)\big(e^{p\eta(s^*, \bar{c})} - \mathbf{1}\big)\Big] \leq -\rho^2 ||\theta(s^*, \bar{c})||^2. \tag{25}$$

Now, using (12), (25) and the assumption $\rho^1 + \rho^2 \geq 0$, we get

$$\frac{1}{p}\Big[\big(\sum_{i=1}^{q^*} \xi_i^* \left[\nabla d(\bar{c}, \bar{y}_i^*) + Lw^* - k^*\left(\nabla e(\bar{c}, \bar{y}_i^*) - Nv^*\right)\right]\big)\big(e^{p\eta(s^*, \bar{c})} - \mathbf{1}\big)\Big] \geq$$
$$\geq -\rho^1 ||\theta(s^*, \bar{c})||^2. \tag{26}$$

Since

$$\sum_{i=1}^{q^*} \xi_i^* (d(\cdot, \bar{y}_i{}^*) + (\cdot)^T Lw^* - k^*(e(\cdot, \bar{y}_i^*) - (\cdot)^T Nv^*))$$

is strictly (p, r)-ρ^1-(η, θ)-invex at $\bar{c}$ with respect to η and θ, therefore, using (3), we have

$$\frac{1}{r}(e^{r(E-F)} - 1) > 0,$$

where

$$E = \sum_{i=1}^{q^*} \xi_i^* \big(d(s^*, \bar{y}_i^*) + s^{*T} Lw^* - k^*\big(e(s^*, \bar{y}_i^*) - s^{*T} Nv^*\big)\big)$$

$$F = \sum_{i=1}^{q^*} \xi_i^* \big(d(\bar{c}, \bar{y}_i^*) + \bar{c}^T Lw^* - k^*\big(e(\bar{c}, \bar{y}_i^*) - \bar{c}^T Nv^*\big)\big).$$

This further gives

$$\sum_{i=1}^{q^*} \xi_i^* \big[d(s^*, \bar{y}_i^*) + s^{*T} Lw^* - k^*\big(e(s^*, \bar{y}_i^*) - s^{*T} Nv^*\big)\big]$$
$$- \sum_{i=1}^{q^*} \xi_i^* \big[d(\bar{c}, \bar{y}_i^*) + \bar{c}^T Lw^* - k^*\big(e(\bar{c}, \bar{y}_i^*) - \bar{c}^T Nv^*\big)\big] > 0.$$

Therefore, from (13),

$$\sum_{i=1}^{q^*} \xi_i^* \big[d(s^*, \bar{y}_i^*) + s^{*T} Lw^* - k^*\big(e(s^*, \bar{y}_i^*) - s^{*T} Nv^*\big)\big] > 0.$$

Since $\xi_i^* \geq 0$ and $\xi^* \neq 0$, therefore there exists i such that

$$d(s^*, \bar{y}_i^*) + s^{*T} Lw^* - k^*\big(e(s^*, \bar{y}_i^*) - s^{*T} Nv^*\big) > 0.$$

Hence, we get

$$\frac{d(s^*, \bar{y}_i^*) + s^{*T} Lw^*}{e(s^*, \bar{y}_i^*) - s^{*T} Nv^*} > k^*,$$

which contradicts (23). Hence, the proof is completed. □

Now, we illustrate an example of a nondifferentiable minimax model and the way to obtain its optimal solution using optimality results.

EXAMPLE 2 Let $n = l' = 1,\ m = 2$ and $Y = [0,\ 1]$.
Let $d(s,\ y) = s^2 + y + 9,\ e(s,\ y) = s^2 + 12,\ L = N = 1,\ h_1(s) = s - 4$ and $h_2(s) = -s + 2$. Also $h(s) \leq 0 \Rightarrow s - 4 \leq 0$ and $-s + 2 \leq 0, \Rightarrow 2 \leq s \leq 4$.

Therefore

$$S = \{s \in R | 2 \leq s \leq 4\}.$$

Now

$$d(s,\ y) + (s^T L s)^{\frac{1}{2}} = s^2 + y + 9 + |s| > 0$$

and

$$e(s,\ y) - (s^T N s)^{\frac{1}{2}} = s^2 + 12 - |s| > 0 \quad \forall (s,\ y) \in S \times Y,$$

where

$$S \times Y = \{(s,\ y) | 2 \leq s \leq 4,\ 0 \leq y \leq 1\}.$$

Now, the problem NFP becomes

$$\begin{aligned} \min \varphi(s) = \quad & \sup_{y \in Y} \tfrac{s^2+y+9+|s|}{s^2+12-|s|} \\ \text{s.t.} \quad & s - 4 \leq 0, \\ & -s + 2 \leq 0, \\ \text{and} \quad & y \in [0,\ 1], \end{aligned}$$

with $Y(s) = \{1\}$ and $K(s) = \{(1,\ 1,\ 1)\}$.

In order to find a minimax solution of NFP for $s^* \in [2,\ 4]$, we have considered the following cases.

Case 1. Take $s^* = 2$ from (3), (5) and (6), so that we get $k_0 = 8/7,\ w = v = 1$ and $\xi_1^* = 1$.

Since $h_1(s^*) \neq 0 \Rightarrow \zeta_1^* = 0$ for $s^* = 2$, and also since $h_2(s^*) = 0$, therefore from (2) $\zeta_2^* = 11/7 > 0$. Now, using $w = v = 1$ along with $\xi_1^* = 1,\ k_0 = 8/7,\ \zeta_1^* = 0$ and $\zeta_2^* = 11/7$, we can check that all the necessary conditions for a minimax solution are satisfied.

Case 2. Take $2 < s^* < 4$. Using (4), we obtain

$$\zeta_1^*(s^* - 4) + \zeta_2^*(-s^* + 2) = 0$$

$$\Rightarrow \zeta_2^* = -\frac{(s^* - 4)\zeta_1^*}{(-s^* + 2)}.$$

From (5), $\xi_1^* = 1$, and from (6), we get $w^2 \leq 1,\ v^2 \leq 1,\ |s^*| = s^*w$ and $|s^*| = s^*v$, which implies $w = v = 1$. By using values of $\zeta_2^*,\ \xi_1^*,\ w$ and v in (2) and (3), we get

$$(2s^*+1) - \frac{(s^{*2}+1+9+|s|)}{(s^{*2}-|s^*|+12)}(2s^*-1) + \frac{2\zeta_1^*}{(s^*-2)}$$

$$(2s^*+1) - \frac{(s^{*2}+10+|s^*|)}{(s^{*2}-|s^*|+12)}(2s^*-1) = -\frac{2\zeta_1^*}{(s^*-2)}.$$

Since $\frac{-2\zeta_1}{(s^*-2)} \leq 0$ for $s^* \in (2,4)$, therefore

$$(2s^*+1) - \frac{(s^{*2}+10+|s^*|)}{(s^{*2}-|s^*|+12)}(2s^*-1) \leq 0,$$

that is,

$$(2s^*+1)(s^{*2}-|s^*|+12) - (s^{*2}+10+|s^*|)(2s^*-1) \leq 0,$$

that is, $-2s^{*2}+4s^*+22 \leq 0$, which is not possible for any $s^* \in (2,4)$. Therefore $s^* \in (2,4)$ cannot be a minimax solution.

Case 3. Take $s^* = 4$. Using (3), (5) and (6), we get $k_0 = 5/4$, $w = v = 1$ and $\zeta_1^* = -1/4$.

Since $\zeta_1^* = -0.25 < 0$ does not satisfy the condition that $\zeta_1^* \in R_+$, therefore $s^* = 4$ cannot be a minimax solution.

Now we will justify that $s^* = 2$ is an optimal solution of NFP. For this, we will show that $\psi(s)$ and $\phi(s)$ is a (p,r)-ρ-(η,θ)-invex function at s^*. Here

$$\begin{aligned}
\psi(s) &= \sum_{i=1}^{q} \xi_i^* \Big(d(s,\bar{y}_i) + (s)^T Lw - k_0(e(s,\bar{y}_i) - (s)^T Nv) \Big) \\
&= [(s)^2 + 1 + 9 + (s)w - k_0((s)^2 + 12 - (s)v)] \\
\phi(s) &= \sum_{j=1}^{m} \zeta_j^* h_j(s).
\end{aligned}$$

Using $w = v = 1,\ k_0 = 1.1428,\ \zeta_1^* = 0$ and $\zeta_2^* = 1.5714$, we get

$$\begin{aligned}
\psi(s) &= [(s)^2 + 1 + 9 + (s) - k_0((s)^2 + 12 - (s))] \\
&= -\frac{1}{7}(s)^2 + \frac{15}{7}(s) - \frac{26}{7} \\
\phi(s) &= \zeta_2^* h_2(s) \\
&= \frac{11}{7}(-(s) + 2).
\end{aligned}$$

For $\theta(s,u) = 1$ and $\eta(s,u) = \sin(s) + 10$ at $r = 1,\ p = -1,\ \rho^1 = -1.5713999$ and $s^* = 2$ we have

$$\begin{aligned} &\tfrac{1}{r}[e^{r(\psi(s)-\psi(2))} - 1] - \tfrac{1}{p}\nabla\psi(2)(e^{p\eta(s,\ u)} - 1) - \rho\,\|\theta(s,\ u)\|^2 \\ &= [e^{\left(\frac{1}{7}(4-(s)^2)+\frac{15}{7}((s)-2)\right)} - 1] + \tfrac{11}{7}(e^{\left(-\sin(s)-10\right)} - 1) + 1.5713999 \geq 0, \quad \forall\ s \in S, \end{aligned}$$

and at $\rho^2 = 1.5713999$

$$\begin{aligned} &\tfrac{1}{r}[e^{r(\phi(s)-\phi(2))} - 1] - \tfrac{1}{p}\nabla\phi(2)(e^{p\eta(s,\ u)} - 1) - \rho\,\|\theta(s,\ u)\|^2 \\ &= [e^{\left(-\frac{11}{7}(s)+\frac{22}{7}\right)} - 1] - \tfrac{11}{7}(e^{\left(-\sin(s)-10\right)} - 1) - 1.5713999 \geq 0, \quad \forall\ s \in S. \end{aligned}$$

Thus, the sufficient conditions of the theorem are easily verified and $s^* = 2$ is a minimax solution.

4. Concluding remarks

In the present work, we have considered a parametric dual model for the nondifferentiable minimax fractional programming problem. We have proven weak, strong and strict converse duality results involving (p,r)-ρ-(η,θ)-invex functions. We have given an example of a non-trivial function to show the existence of the functions, which satisfy the definition of the (p,r)-ρ-(η,θ)-invexity. Also, we have formulated an example of a non-trivial model, validating the necessary and sufficient conditions. Now, the question arises whether or not duality in nondifferentiable minimax fractional programming with (p,r)-ρ-(η,θ)-invexity can be further extended to the second order case.

References

Ahmad, I. (2003) Optimality conditions and duality in fractional minimax programming involving generalized ρ-invexity. *International Journal of Statistics and Systems*, **19**, 165–180.

Ahmad, I., Gupta, S. K., Kailey, N., and Agarwal, R. P. (2011) Duality in nondifferentiable minimax fractional programming with B-(p,r)-invexity. *Journal of Inequalities and Applications*, **2011**(1), 1–14.

Ahmad, I. and Husain, Z. (2006) Optimality conditions and duality in nondifferentiable minimax fractional programming with generalized convexity, *Journal of Optimization Theory Applications*, **129**(2), 255–275.

Antczak, T. (2001) (p,r)-invex sets and functions. *Journal of Mathematical Analysis and Applications*, **263**(2), 355–379.

Antczak, T., Mishra, S. K. and Upadhyay B. B. (2018) Optimality conditions and duality for generalized fractional minimax programming involving locally Lipschitz (b,Ψ,Φ,ρ)-univex functions. *Control and Cybernetics*, **47**(1), 5–32.

Boufi, K. and Roubi, A. (2019) Duality results and dual bundle methods based on the dual method of centers for minimax fractional programs. *SIAM Journal on Optimization*, **29**(2), 1578–1602.

Du, D. Z. and Pardalos P. M. (1995) *Minimax and Applications*. Kluwer Academic Publishers.

Dubey, R. and Mishra, V. N. (2020) Higher-order symmetric duality in nondifferentiable multiobjective fractional programming problem over cone contraints. *Statistics, Optimization and Information Computing*, **8**(1), 187–205.

Falk, J. E. (1969) Maximization of signal-to-noise ratio in an optical filter. *SIAM Journal on Applied Mathematics*, **17**(3), 582–592.

Husain, Z., Ahmad, I. and Sharma S. (2009) Second order duality for minimax fractional programming. *Optimization Letter*, **3**(2), 277–286.

Jayswal, A. (2008) Non-differentiable minimax fractional programming with generalized a-univexity. *Journal of Computational and Applied Mathematics* **214**(1), 121–135.

Khan, M. A. and Al-Solamy, F. R. (2015) Sufficiency and duality in nondifferentiable minimax fractional programming with (H_p, r)-invexity. *Journal of the Egyptian Mathematical Society*, **23**(1), 208–213.

Lai, H. C. and Huang, T. Y. (2012) Nondifferentiable minimax fractional programming in complex spaces with parametric duality. *Journal of Global Optimization*, **53**(2), 243–254.

Lai, H. C. and Lee, J. C. (2002) On duality theorems for a nondifferentiable minimax fractional programming. *Journal of Computational and Applied Mathematics*, **146**(1), 115–126.

Lai, H. C. and Liu J. C. (2011) A new characterization on optimality and duality for nondifferentiable minimax fractional programming problems. *Journal of Nonlinear and Convex Analysis* **12**(1), 69–80.

Lai, H. C., Liu, J. C. and Tanaka, K. (1999) Necessary and sufficient conditions for minimax fractional programming. *Journal of Mathematical Analysis and Applications*, **230**(2), 311–328.

Long, J. C. and Quan, J. (2011) Optimality conditions and duality for minimax fractional programming involving nonsmooth generalized univexity. *Numerical Algebra, Control and Optimization* **1**(3), 361–370.

Liu, J. C. and Wu, C. S. (1998) On minimax fractional optimality conditions with (F, ρ)-convexity. *Journal of Mathematical Analysis and Applications* **219**(1), 36–51.

Mandal, P. and Nahak, C. (2011) Symmetric duality with (p, r)-ρ-(η, θ)-invexity. *Applied Mathematics and Computation*, **217**(21), 8141–8148.

Schmitendorf, W. E. (1977) Necessary conditions and sufficient optimality conditions for static minmax problems. *Journal of Mathematical Analysis and Applications*, **57**(3), 683–693.

Son, T. Q. and Kim, D. S. (2021) A dual scheme for solving linear countable semi-infinite fractional programming problems. *Optimization Letters*, 1–14.

Sonali, S., Sharma, V. and Kailey, N. (2020) Higher-order non-symmetric duality for nondifferentiable minimax fractional programs with square root

terms. *Acta Mathematica Scientia*, **40**(1), 127–140.
TANIMOTO, S. (1981) Duality for a class of nondifferentiable mathematical programming problems. *Journal of Mathematical Analysis and Applications*, **79**(2), 283–294.
ZHENG, X. J. AND CHENG, L. (2007) Minimax fractional programming under nonsmooth generalized (F, ρ, θ)-d-univex. *Journal of Mathematical Analysis and Applications*, **328**(1), 676–689.