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Bertrand's Ballot Theorem[1]

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Summary. In this article we formalize the Bertrand's Ballot Theorem based on [17]. Suppose that in an election we have two candidates: A that receives n votes and B that receives k votes, and additionally $n \geqslant k$. Then this theorem states that the probability of the situation where A maintains more votes than B throughout the counting of the ballots is equal to $(n-k)/(n+k)$.

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The notation and terminology used in this paper have been introduced in the following articles: [24], [1], [14], [15], [18], [4], [5], [10], [21], [6], [12], [3], [11], [25], [26], [16], [8], [13], [23], and [9].

1. Preliminaries

From now on D, D_1, D_2 denote non empty sets, d, d_1, d_2 denote finite 0-sequences of D, and n, k, i, j denote natural numbers.

Now we state the propositions:

(1) $\mathrm{XFS2FS}(d{\restriction}n) = \mathrm{XFS2FS}(d){\restriction}n$.

(2) $\mathrm{rng}\, d = \mathrm{rng}\, \mathrm{XFS2FS}(d)$.

(3) Let us consider a finite 0-sequence d_1 of D_1 and a finite 0-sequence d_2 of D_2. If $d_1 = d_2$, then $\mathrm{XFS2FS}(d_1) = \mathrm{XFS2FS}(d_2)$.

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(4) If $\mathrm{XFS2FS}(d_1) = \mathrm{XFS2FS}(d_2)$, then $d_1 = d_2$. Proof: For every i such that $i < \operatorname{len} d_1$ holds $d_1(i) = d_2(i)$ by [2, (13), (11)]. □

(5) Let us consider a finite sequence d of elements of D. Then $\mathrm{XFS2FS}(\mathrm{FS2XFS}(d)) = d$.

(6) Let us consider a finite sequence f and objects x, y. Suppose

(i) $\operatorname{rng} f \subseteq \{x, y\}$, and

(ii) $x \neq y$.

Then $\overline{\overline{f^{-1}(\{x\})}} + \overline{\overline{f^{-1}(\{y\})}} = \operatorname{len} f$.

(7) Let us consider functions f, g. Suppose f is one-to-one. Let us consider an object x. If $x \in \operatorname{dom} f$, then $\mathrm{Coim}(f \cdot g, f(x)) = \mathrm{Coim}(g, x)$. Proof: Set $f_3 = f \cdot g$. $\mathrm{Coim}(f_3, f(x)) \subseteq \mathrm{Coim}(g, x)$ by [6, (11), (12)]. □

(8) Let us consider a real number r and a real-valued finite sequence f. Suppose $\operatorname{rng} f \subseteq \{0, r\}$. Then $\sum f = r \cdot \overline{\overline{f^{-1}(\{r\})}}$. Proof: Define $\mathcal{P}$[natural number] $\equiv$ for every real-valued finite sequence f such that $\operatorname{len} f = \$_1$ and $\operatorname{rng} f \subseteq \{0, r\}$ holds $\sum f = r \cdot \overline{\overline{f^{-1}(\{r\})}}$. $\mathcal{P}[0]$ by [8, (72)]. For every n such that $\mathcal{P}[n]$ holds $\mathcal{P}[n+1]$ by [22, (55)], [8, (74)], [25, (70)], [2, (11)]. For every n, $\mathcal{P}[n]$ from [2, Sch. 2]. □

2. Properties of Elections

In the sequel A, B denote objects, v denotes an element of $\{A, B\}^{n+k}$, and f, g denote finite sequences.

Let us consider A, n, B, and k. The functor $\mathrm{Election}(A, n, B, k)$ yielding a subset of $\{A, B\}^{n+k}$ is defined by

(Def. 1) $v \in it$ if and only if $\overline{\overline{v^{-1}(\{A\})}} = n$.

Let us note that $\mathrm{Election}(A, n, B, k)$ is finite. Now we state the propositions:

(9) $\mathrm{Election}(A, n, A, 0) = \{n \mapsto A\}$. Proof: $\mathrm{Election}(A, n, A, 0) \subseteq \{n \mapsto A\}$ by [19, (29)], [9, (33)], [21, (9)]. □

(10) If $k > 0$, then $\mathrm{Election}(A, n, A, k)$ is empty.

Let us consider A and n. Let k be a non empty natural number. Let us observe that $\mathrm{Election}(A, n, A, k)$ is empty. Now we state the proposition:

(11) $\mathrm{Election}(A, n, B, k) = \mathrm{Choose}(\mathrm{Seg}(n+k), n, A, B)$. Proof: $\mathrm{Election}(A, n, B, k) \subseteq \mathrm{Choose}(\mathrm{Seg}(n+k), n, A, B)$ by [7, (2)]. □

Let us assume that $A \neq B$. Now we state the propositions:

(12) $v \in \mathrm{Election}(A, n, B, k)$ if and only if $\overline{\overline{v^{-1}(\{B\})}} = k$. The theorem is a consequence of (6).

(13) $\overline{\overline{\mathrm{Election}(A, n, B, k)}} = \binom{n+k}{n}$. The theorem is a consequence of (11).

3. Properties of Dominated Elections

Let us consider A, n, B, and k. Let v be a finite sequence. We say that v is an (A, n, B, k)-dominated-election if and only if

(Def. 2) (i) $v \in \text{Election}(A, n, B, k)$, and

(ii) for every i such that $i > 0$ holds $\overline{\overline{(v{\restriction}i)^{-1}(\{A\})}} > \overline{\overline{(v{\restriction}i)^{-1}(\{B\})}}$.

Let us assume that f is an (A, n, B, k)-dominated-election. Now we state the propositions:

(14) $A \neq B$.

(15) $n > k$. The theorem is a consequence of (14) and (12).

Now we state the propositions:

(16) If $A \neq B$ and $n > 0$, then $n \mapsto A$ is an (A, n, B, 0)-dominated-election.

(17) If f is an (A, n, B, k)-dominated-election and $i < n-k$, then $f \frown (i \mapsto B)$ is an (A, n, B, $(k+i)$)-dominated-election. The theorem is a consequence of (14) and (12).

(18) Suppose f is an (A, n, B, k)-dominated-election and g is an (A, i, B, j)-dominated-election. Then $f \frown g$ is an (A, $(n+i)$, B, $(k+j)$)-dominated-election. The theorem is a consequence of (14), (12), and (15).

Let us consider A, n, B, and k. The functor DominatedElection(A, n, B, k) yielding a subset of Election(A, n, B, k) is defined by

(Def. 3) $f \in it$ if and only if f is an (A, n, B, k)-dominated-election.

(19) If $A = B$ or $n \leqslant k$, then DominatedElection(A, n, B, k) is empty. The theorem is a consequence of (14) and (15).

(20) If $n > k$ and $A \neq B$, then $n \mapsto A \frown (k \mapsto B) \in$ DominatedElection(A, n, B, k). The theorem is a consequence of (17) and (16).

(21) If $A \neq B$, then $\overline{\overline{\text{DominatedElection}(A, n, B, k)}} = \overline{\overline{\text{DominatedElection}(0, n, 1, k)}}$. Proof: Set $T = [A \longmapsto 0, B \longmapsto 1]$. Define $\mathcal{P}[\text{object}, \text{object}] \equiv$ for every f such that $f = \$_1$ holds $T \cdot f = \$_2$. For every object x such that $x \in$ DominatedElection(A, n, B, k) there exists an object y such that $y \in$ DominatedElection$(0, n, 1, k)$ and $\mathcal{P}[x, y]$ by [25, (27), (26)], [5, (92)], (7). Consider C being a function from DominatedElection(A, n, B, k) into DominatedElection$(0, n, 1, k)$ such that for every object x such that $x \in$ DominatedElection(A, n, B, k) holds $\mathcal{P}[x, C(x)]$ from [7, Sch. 1]. DominatedElection$(0, n, 1, k) \subseteq \text{rng}\, C$ by [25, (27), (26)], [5, (92)], (7). □

(22) Let us consider a finite sequence p of elements of $\mathbb{N}$. Then p is a (0, n, 1, k)-dominated-election if and only if p is an $(n+k)$-tuple of $\{0, 1\}$ and $n > 0$ and $\sum p = k$ and for every i such that $i > 0$ holds $2 \cdot \sum (p{\restriction}i) < i$. Proof: If p is a (0, n, 1, k)-dominated-election, then p is an $(n+k)$-tuple of $\{0, 1\}$

and $n > 0$ and $\sum p = k$ and for every i such that $i > 0$ holds $2 \cdot \sum(p \restriction i) < i$ by (8), (12), (15), [25, (70)]. $1 \cdot \overline{\overline{p^{-1}(\{1\})}} = k$. $\overline{\overline{p^{-1}(\{1\})}} + \overline{\overline{p^{-1}(\{0\})}} = \operatorname{len} p$. $1 \cdot \overline{\overline{(p \restriction i)^{-1}(\{1\})}} = \sum(p \restriction i)$. $\overline{\overline{(p \restriction i)^{-1}(\{1\})}} + \overline{\overline{(p \restriction i)^{-1}(\{0\})}} = \operatorname{len}(p \restriction i)$. □

(23) If f is an (A, n, B, k)-dominated-election, then $f(1) = A$. The theorem is a consequence of (15).

(24) Let us consider a finite 0-sequence d of $\mathbb{N}$. Then $d \in \mathrm{Domin}_0(n + k, k)$ if and only if $\langle 0 \rangle \frown \mathrm{XFS2FS}(d) \in \mathrm{DominatedElection}(0, n + 1, 1, k)$. Proof: Set $X_1 = \mathrm{XFS2FS}(d)$. Set $Z = \langle 0 \rangle$. Set $Z_1 = Z \frown X_1$. Reconsider $D = d$ as a finite 0-sequence of $\mathbb{R}$. $\mathrm{XFS2FS}(d) = \mathrm{XFS2FS}(D)$. If $d \in \mathrm{Domin}_0(n+k, k)$, then $Z_1 \in \mathrm{DominatedElection}(0, n + 1, 1, k)$ by [15, (20)], (2), [4, (31), (22)]. Z_1 is an $(n + 1 + k)$-tuple of $\{0, 1\}$. For every k such that $k \leqslant \operatorname{dom} d$ holds $2 \cdot \sum(d \restriction k) \leqslant k$ by [20, (14)], [8, (76)], (1), (3). d is dominated by 0. $\sum d = k$. □

(25) $\overline{\overline{\mathrm{Domin}_0(n + k, k)}} = \overline{\overline{\mathrm{DominatedElection}(0, n + 1, 1, k)}}$. Proof: Set $D = \mathrm{Domin}_0(n + k, k)$. Set $B = \mathrm{DominatedElection}(0, n + 1, 1, k)$. Set $Z = \langle 0 \rangle$. Define $\mathcal{F}[\text{object}, \text{object}] \equiv$ for every finite 0-sequence d of $\mathbb{N}$ such that $d = \$_1$ holds $\$_2 = Z \frown \mathrm{XFS2FS}(d)$. For every object x such that $x \in D$ there exists an object y such that $y \in B$ and $\mathcal{F}[x, y]$. Consider f being a function from D into B such that for every object x such that $x \in D$ holds $\mathcal{F}[x, f(x)]$ from [7, Sch. 1]. □

(26) $\overline{\overline{\mathrm{Domin}_0(n + k, k)}} = \overline{\overline{\mathrm{DominatedElection}(0, n + 1, 1, k)}}$. Proof: Set $D = \mathrm{Domin}_0(n + k, k)$. Set $B = \mathrm{DominatedElection}(0, n + 1, 1, k)$. Set $Z = \langle 0 \rangle$. Define $\mathcal{F}[\text{object}, \text{object}] \equiv$ for every finite 0-sequence d of $\mathbb{N}$ such that $d = \$_1$ holds $\$_2 = Z \frown \mathrm{XFS2FS}(d)$. For every object x such that $x \in D$ there exists an object y such that $y \in B$ and $\mathcal{F}[x, y]$. Consider f being a function from D into B such that for every object x such that $x \in D$ holds $\mathcal{F}[x, f(x)]$ from [7, Sch. 1]. □

(27) If $A \neq B$ and $n > k$, then $\overline{\overline{\mathrm{DominatedElection}(A, n, B, k)}} = \frac{n-k}{n+k} \cdot \binom{n+k}{k}$. The theorem is a consequence of (21) and (26).

4. Main Theorem

(28) Bertrand's Ballot Theorem:
If $A \neq B$ and $n \geqslant k$, then $\mathrm{P}(\mathrm{DominatedElection}(A, n, B, k)) = \frac{n-k}{n+k}$. The theorem is a consequence of (13), (19), and (27).

References

[1] Grzegorz Bancerek. Cardinal numbers. *Formalized Mathematics*, 1(**2**):377–382, 1990.

[2] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(**1**):41–46, 1990.

[3] Grzegorz Bancerek. The ordinal numbers. *Formalized Mathematics*, 1(**1**):91–96, 1990.
[4] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.
[5] Czesław Byliński. Finite sequences and tuples of elements of a non-empty sets. *Formalized Mathematics*, 1(**3**):529–536, 1990.
[6] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**): 55–65, 1990.
[7] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.
[8] Czesław Byliński. The sum and product of finite sequences of real numbers. *Formalized Mathematics*, 1(**4**):661–668, 1990.
[9] Czesław Byliński. Some basic properties of sets. *Formalized Mathematics*, 1(**1**):47–53, 1990.
[10] Agata Darmochwał. Finite sets. *Formalized Mathematics*, 1(**1**):165–167, 1990.
[11] Artur Korniłowicz. On the real valued functions. *Formalized Mathematics*, 13(**1**):181–187, 2005.
[12] Rafał Kwiatek. Factorial and Newton coefficients. *Formalized Mathematics*, 1(**5**):887–890, 1990.
[13] Beata Padlewska. Families of sets. *Formalized Mathematics*, 1(**1**):147–152, 1990.
[14] Karol Pąk. Cardinal numbers and finite sets. *Formalized Mathematics*, 13(**3**):399–406, 2005.
[15] Karol Pąk. The Catalan numbers. Part II. *Formalized Mathematics*, 14(**4**):153–159, 2006. doi:10.2478/v10037-006-0019-7.
[16] Jan Popiołek. Introduction to probability. *Formalized Mathematics*, 1(**4**):755–760, 1990.
[17] M. Renault. Four proofs of the ballot theorem. *Mathematics Magazine*, 80(5):345–352, December 2007.
[18] Andrzej Trybulec. Domains and their Cartesian products. *Formalized Mathematics*, 1(**1**): 115–122, 1990.
[19] Andrzej Trybulec. Enumerated sets. *Formalized Mathematics*, 1(**1**):25–34, 1990.
[20] Andrzej Trybulec. On the decomposition of finite sequences. *Formalized Mathematics*, 5 (**3**):317–322, 1996.
[21] Andrzej Trybulec. Binary operations applied to functions. *Formalized Mathematics*, 1 (**2**):329–334, 1990.
[22] Wojciech A. Trybulec. Non-contiguous substrings and one-to-one finite sequences. *Formalized Mathematics*, 1(**3**):569–573, 1990.
[23] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.
[24] Tetsuya Tsunetou, Grzegorz Bancerek, and Yatsuka Nakamura. Zero-based finite sequences. *Formalized Mathematics*, 9(**4**):825–829, 2001.
[25] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1 (**1**):73–83, 1990.
[26] Edmund Woronowicz. Relations defined on sets. *Formalized Mathematics*, 1(**1**):181–186, 1990.

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