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Formal Proof of Transcendence of the Number e. Part I

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Summary. In this article, we prove the transcendence of the number e using the Mizar formalism, following Hurwitz's proof. This article prepares the necessary definitions and lemmas. The main body of the proof will be presented separately.

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Introduction

In this article, we prove that the number e is transcendental [11] using the Mizar formalism [2]. We encode Hurwitz's proof [7], which in simplification of Hermite's idea [6]. This subject (considered challenging as 67th item in Freek Wiedijk's "Top 100 Mathematical Theorems" [19]) has been implemented over the past decade within many theorem provers, such as HOL Light [4], Coq [3], and also Isabelle [5] (Metamath [12] and Lean [13] versions are also available). For formalized fundamentals of transcendental number theory [1] in Mizar (in connection to Liouville numbers), see [10]. Here we formulate and prove some auxiliary definitions and facts needed to go smoothly through the Hurwitz's proof.

As the proof goes by contradiction, at the beginning we should formulate the assumption that e is algebraic, so a polynomial over $\mathbb{Z}$ admits e as a root. It corresponds to the equation (3) of [7] and obviously, this is also rephrased in

Mizar language, but just in the next article in the series (see E_TRANS2:41), as otherwise we could have here a lemma which is redundant in the repository as a whole.

So, we define a polynomial transformation $\mathcal{F}$. For a polynomial f over $\mathbb{Q}$ with degree r, we introduce a functor (see E_TRANS1:def 11):

$$\mathcal{F} : f(x) \mapsto f(x) + f'(x) + f''(x) + \cdots + f^{(r)}(x)$$

In Hurwitz's proof he defines $\mathcal{F}(x) = f(x) + f'(x) + f''(x) + \cdots + f^{(r)}(x)$ as equation (1) in [7]. In the actual formalization for constructing $\mathcal{F}$ we generate a finite sequence of polynomials defined by $\mathcal{G} = \{f^{(i)}(x)\}$, then $\mathcal{F}$ is formalized as the summation of it, namely $\mathcal{F} = \mathrm{Sum}\,\mathcal{G}$. Since higher order derivations for a ring have been implemented in [18], we are able to formalize i^{th} component of $\mathcal{G}$. Then we apply the mean value theorem to $-e^x F(x)$ on an interval (as F can be considered the acting of transformation $\mathcal{F}$ on a polynomial) and formalize the following formula quoted as equation (2) in [7] (see E_TRANS1:34):

$$F(x) - e^x F(0) = -xe^{(1-\vartheta)x} f(\vartheta x).$$

The rest of the section is devoted to preparing lemmas to define the particular polynomial $f(x) = \frac{1}{(p-1)!} x^{p-1}(1-x)^p(2-x)^p \cdots (n-x)^p$ which plays an important role of the main proof.

1. Preliminaries

From now on n, k denote natural numbers, L denotes a commutative ring, R denotes an integral domain, and x_0 denotes a positive real number.

The functor $\frac{1}{\exp_R}$ yielding a function from $\mathbb{R}$ into $\mathbb{R}$ is defined by the term

(Def. 1) $\frac{1}{\text{the function exp}}$.

One can verify that $\frac{1}{\exp_R}$ is differentiable as a function from $\mathbb{R}$ into $\mathbb{R}$ and the function exp is differentiable as a function from $\mathbb{R}$ into $\mathbb{R}$. Now we state the propositions:

(1) Let us consider natural numbers n, m, and an element b of R. Then $(n \cdot m) \cdot b = n \cdot (m \cdot b)$.

Proof: Define $\mathcal{P}[\text{natural number}] \equiv (\$_1 \cdot m) \cdot b = \$_1 \cdot (m \cdot b)$. For every natural number n such that $\mathcal{P}[n]$ holds $\mathcal{P}[n+1]$. For every natural number n, $\mathcal{P}[n]$. □

(2) Let us consider finite sequences F, G of elements of $\mathbb{R}_{\mathrm{F}}$. Suppose $\mathrm{len}\,F = \mathrm{len}\,G$ and for every natural number i such that $i \in \mathrm{dom}\,F$ holds $F(i) \leqslant G(i)$. Then $\sum F \leqslant \sum G$.

(3) Let us consider an ideal I of L, and a finite sequence F of elements of L. Suppose for every natural number i such that $i \in \operatorname{dom} F$ holds $F(i) \in I$. Then $\sum F \in I$.

Proof: Define $\mathcal{P}$[natural number] $\equiv$ for every finite sequence F of elements of L such that $\operatorname{len} F = \$_1$ and for every natural number i such that $i \in \operatorname{dom} F$ holds $F(i) \in I$ holds $\sum F \in I$. $\mathcal{P}[0]$. For every natural number n such that $\mathcal{P}[n]$ holds $\mathcal{P}[n+1]$. For every natural number n, $\mathcal{P}[n]$. □

(4) Let us consider an element a of L, and a non empty finite sequence p of elements of the carrier of L. Suppose for every natural number j such that $j \in \operatorname{dom} p$ holds $a \mid p_{/j}$. Then $a \mid \sum p$.

Proof: For every natural number i such that $i \in \operatorname{dom} p$ holds $p(i) \in \{a\}$–ideal by [9, (18)]. □

Let k, j be natural numbers. The functor $\eta_{k,j}$ yielding an element of $\mathbb{N}$ is defined by the term

(Def. 2) $\frac{k!}{(k -' j)!}$.

Now we state the proposition:

(5) Let us consider natural numbers k, j. If $j \leqslant k$, then $j! \cdot \binom{k}{j} = \eta_{k,j}$.

Let R be a $(\mathbb{Z}^{\mathrm{R}})$-extending commutative ring and i be an integer. One can check that $i(\in R)$ reduces to i. Now we state the propositions:

(6) Let us consider a natural number n, and an element f of the carrier of Polynom-Ring $\mathbb{F}_{\mathbb{Q}}$. Then $n \cdot f = n(\in \mathbb{F}_{\mathbb{Q}}) \cdot f$.

Proof: Define $\mathcal{P}$[natural number] $\equiv \$_1 \cdot f = \$_1(\in \mathbb{F}_{\mathbb{Q}}) \cdot f$. $\mathcal{P}[0]$. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$. For every natural number k, $\mathcal{P}[k]$. □

(7) Let us consider a natural number n, and elements f, g of L. If $f \mid g$, then $f \mid n \cdot g$.

2. Casting Functions between Polynomials and Elements of Rings

Let R be an add-associative, right zeroed, right complementable, distributive, non empty double loop structure and f be an element of the carrier of Polynom-Ring R. The functor R2P(f) yielding a polynomial over R is defined by the term

(Def. 3) f.

Let p be a polynomial over R. The functor P2R(p) yielding an element of the carrier of Polynom-Ring R is defined by the term

(Def. 4) p.

Observe that there exists a finite sequence of elements of $\mathbb{F}_{\mathbb{Q}}$ which is $\mathbb{Z}$-valued and $\mathbf{0}.\mathbb{F}_{\mathbb{Q}}$ is $\mathbb{Z}$-valued and $\mathbf{1}.\mathbb{F}_{\mathbb{Q}}$ is $\mathbb{Z}$-valued and there exists a polynomial over $\mathbb{F}_{\mathbb{Q}}$ which is monic and $\mathbb{Z}$-valued. Now we state the proposition:

(8) Let us consider an element f of the carrier of Polynom-Ring R. Then $\operatorname{rng} f = f^{\circ}(\operatorname{Support} f) \cup \{0_R\}$.
PROOF: For every object y such that $y \in f^{\circ}(\mathbb{N} \setminus (\operatorname{Support} f))$ holds $y \in \{0_R\}$. For every object y such that $y \in \{0_R\}$ holds $y \in f^{\circ}(\mathbb{N} \setminus (\operatorname{Support} f))$. □

Let f be an element of the carrier of Polynom-Ring $\mathbb{F}_{\mathbb{Q}}$. The functor $\operatorname{denomi}_{\text{set}}(f)$ yielding a non empty, finite subset of $\mathbb{N}$ is defined by the term

(Def. 5) $(\text{TRANQN})^{\circ}(\operatorname{rng} f)$.

The functor $\operatorname{denomi}_{\text{seq}}(f)$ yielding a non empty finite sequence of elements of $\mathbb{N}$ is defined by the term

(Def. 6) $\text{CFS}(\operatorname{denomi}_{\text{set}}(f))$.

Now we state the propositions:

(9) Let us consider an element f of the carrier of Polynom-Ring $\mathbb{F}_{\mathbb{Q}}$. Then $\prod \operatorname{denomi}_{\text{seq}}(f)$ is not zero.

(10) Let us consider an element f of the carrier of Polynom-Ring $\mathbb{F}_{\mathbb{Q}}$, and a natural number i. Then

(i) $\operatorname{den} f(i) \in \operatorname{denomi}_{\text{set}}(f)$, and

(ii) there exists an integer z such that $z \cdot (\operatorname{den} f(i)) = \prod \operatorname{denomi}_{\text{seq}}(f)$.

(11) Let us consider fields K, L, and an element w of L. Suppose K is a subring of L and w is integral over K. Then $\operatorname{AnnPoly}(w, K)$ is maximal.

(12) Let us consider an element f of Polynom-Ring $\mathbb{F}_{\mathbb{Q}}$, and a non zero natural number n. If f is irreducible, then $n \cdot f$ is irreducible. The theorem is a consequence of (7) and (6).

(13) Let us consider an element x of $\mathbb{R}_{\text{F}}$. Suppose x is irrational. Let us consider a non zero polynomial g over $\mathbb{F}_{\mathbb{Q}}$. If $\operatorname{ExtEval}(g, x) = 0$, then $\deg(g) \geqslant 2$.

3. More on Derivation of Polynomials

Now we state the propositions:

(14) Let us consider a polynomial g over $\mathbb{F}_{\mathbb{Q}}$. Suppose $\deg(g) \geqslant 2$ and $\text{P2R}(g)$ is irreducible. Then $g(0) \neq 0_{\mathbb{F}_{\mathbb{Q}}}$.
PROOF: Reconsider $g_1 = \text{NormPoly}\,\text{P2R}(g)$ as a polynomial over $\mathbb{F}_{\mathbb{Q}}$. $g_1(0) \neq 0_{\mathbb{F}_{\mathbb{Q}}}$ by [17, (30), (37)]. □

(15) Let us consider a non degenerated integral domain L, a non zero natural number n, and a non zero element a of L. If $\mathrm{char}(L) = 0$, then $n \cdot a \neq 0_L$.

(16) Let us consider a commutative ring R, an element f of the carrier of Polynom-Ring R, and a natural number i. Suppose $i \geqslant 1$ and the length of f is at most i and $f(i-1) \neq 0_R$. Then $\mathrm{len}\, f = i$.
PROOF: For every natural number i such that $i \geqslant 1$ and the length of f is at most i and $f(i-1) \neq 0_R$ holds $\mathrm{len}\, f = i$. □

(17) Let us consider an integral domain R, and an element f of the carrier of Polynom-Ring R. Suppose $\mathrm{len}\, f > 1$ and $\mathrm{char}(R) = 0$.
Then $\mathrm{len}(\mathrm{Der1}(R))(f) = \mathrm{len}\, f - 1$.
PROOF: Reconsider $l_1 = \mathrm{len}\, f - 1$ as a natural number. For every natural number i such that $i \geqslant l_1$ holds $(\mathrm{Der1}(R))(f)(i) = 0_R$. □

(18) Let us consider an integral domain L, a derivation D of L, an element f of the carrier of L, and natural numbers j, n. Then $D^n(j \cdot f) = j \cdot D^n(f)$.
PROOF: For every element f of the carrier of L and for every natural numbers j, n, $D^n(j \cdot f) = j \cdot D^n(f)$ by [14, (18)], [18, (9), (6)]. □

(19) Let us consider a natural number k, and an element f of the carrier of Polynom-Ring $\mathbb{Z}^{\mathrm{R}}$. Suppose $(\mathrm{Der1}(\mathbb{Z}^{\mathrm{R}}))^1(f^1) = 1_{\text{Polynom-Ring}\, \mathbb{Z}^{\mathrm{R}}}$. Let us consider a natural number j. Suppose $1 \leqslant j \leqslant k$. Then $(\mathrm{Der1}(\mathbb{Z}^{\mathrm{R}}))^j(f^k) = \eta_{k,j} \cdot f^{k -' j}$.
PROOF: Set $D = \mathrm{Der1}(\mathbb{Z}^{\mathrm{R}})$. Define $\mathcal{P}[\text{natural number}] \equiv$ for every natural number j such that $1 \leqslant j \leqslant \$_1$ holds $D^j(f^{\$_1}) = \eta_{\$_1,j} \cdot f^{\$_1 -' j}$. For every natural number k such that for every natural number n such that $n < k$ holds $\mathcal{P}[n]$ holds $\mathcal{P}[k]$. For every natural number k, $\mathcal{P}[k]$. □

(20) Let us consider a natural number k, and an element f of the carrier of Polynom-Ring $\mathbb{Z}^{\mathrm{R}}$. Suppose $(\mathrm{Der1}(\mathbb{Z}^{\mathrm{R}}))^1(f^1) = 1_{\text{Polynom-Ring}\, \mathbb{Z}^{\mathrm{R}}}$. Then $(\mathrm{Der1}(\mathbb{Z}^{\mathrm{R}}))^k(f^k) = k! \cdot (1_{\text{Polynom-Ring}\, \mathbb{Z}^{\mathrm{R}}})$. The theorem is a consequence of (19).

(21) Let us consider a natural number j. Suppose $j > k$. Let us consider an element f of the carrier of Polynom-Ring $\mathbb{Z}^{\mathrm{R}}$. Suppose $(\mathrm{Der1}(\mathbb{Z}^{\mathrm{R}}))^1(f^1) = 1_{\text{Polynom-Ring}\, \mathbb{Z}^{\mathrm{R}}}$. Then $(\mathrm{Der1}(\mathbb{Z}^{\mathrm{R}}))^j(f^k) = 0_{\text{Polynom-Ring}\, \mathbb{Z}^{\mathrm{R}}}$.
PROOF: Set $L = \text{Polynom-Ring}\, \mathbb{Z}^{\mathrm{R}}$. Set $D = \mathrm{Der1}(\mathbb{Z}^{\mathrm{R}})$. For every element f of the carrier of L such that $D^1(f^1) = 1_{\text{Polynom-Ring}\, \mathbb{Z}^{\mathrm{R}}}$ holds $D^j(f^k) = 0_L$. □

(22) Let us consider an integral domain R, an element f of the carrier of Polynom-Ring R, a natural number k, and a natural number i. Then $(\mathrm{Der1}(R))^k(f)(i) = \eta_{i+k,k} \cdot f(i+k)$.
PROOF: Set $D = \mathrm{Der1}(R)$. Define $\mathcal{P}[\text{natural number}] \equiv$ for every natural

number i for every natural number i, $D^{\$_1}(f)(i) = \eta_{i+\$_1,\$_1} \cdot f(i + \$_1)$. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k + 1]$. For every natural number i, $D^0(f)(i) = \eta_{i+0,0} \cdot f(i + 0)$. For every natural number k, $\mathcal{P}[k]$. $\square$

(23) Let us consider a function h from R into R, and a finite sequence s of elements of the carrier of R. If h is additive, then $h(\sum s) = \sum h \cdot s$.

PROOF: Define $\mathcal{P}$[natural number] $\equiv$ for every function h from R into R for every finite sequence s of elements of R such that $\operatorname{len} s = \$_1$ and h is additive holds $h(\sum s) = \sum h \cdot s$. $\mathcal{P}[0]$. For every natural number n such that $\mathcal{P}[n]$ holds $\mathcal{P}[n + 1]$. For every natural number n, $\mathcal{P}[n]$. $\square$

(24) Let us consider an integral domain R, an element f of the carrier of Polynom-Ring R, and a natural number j. Suppose $\operatorname{len} f > j$ and $\operatorname{char}(R) = 0$. Then $\operatorname{len}(\operatorname{Der1}(R))^j(f) = \operatorname{len} f - j$.

PROOF: Reconsider $l_1 = \operatorname{len} f - 1$ as a natural number. Reconsider $l_3 = \operatorname{len} f - j$ as a natural number. Reconsider $l_4 = l_3 - 1$ as a natural number. Reconsider $l_5 = \binom{l_4+j}{l_4} \cdot (j!)$ as a natural number. $\eta_{l_4+j,j} = \binom{l_4+j}{j} \cdot (j!)$. $(\operatorname{Der1}(R))^j(f)(l_4) = l_5 \cdot f(l_1)$. For every natural number i such that $i \geqslant l_3$ holds $(\operatorname{Der1}(R))^j(f)(i) = 0_R$. $\square$

4. Constructing Polynomial Transformation $\mathcal{F}$

Let p be an element of the carrier of Polynom-Ring $\mathbb{Z}^{\mathrm{R}}$. The functor ${}^{@}p$ yielding an element of the carrier of Polynom-Ring $\mathbb{R}_{\mathrm{F}}$ is defined by the term

(Def. 7) p.

Let F be a finite sequence of elements of the carrier of Polynom-Ring $\mathbb{Z}^{\mathrm{R}}$. The functor ${}^{@}F$ yielding a finite sequence of elements of the carrier of Polynom-Ring $\mathbb{R}_{\mathrm{F}}$ is defined by

(Def. 8) $\operatorname{dom} it = \operatorname{dom} F$ and for every natural number i such that $i \in \operatorname{dom} F$ holds $it(i) = {}^{@}F_{/i}$.

Let L be a commutative ring, F be a finite sequence of elements of the carrier of Polynom-Ring L, and x be an element of L. The functor $\operatorname{eval}(F, x)$ yielding a finite sequence of elements of the carrier of L is defined by

(Def. 9) $\operatorname{dom} it = \operatorname{dom} F$ and for every natural number i such that $i \in \operatorname{dom} F$ holds $it(i) = \operatorname{eval}(\operatorname{R2P}(F_{/i}), x)$.

Now we state the propositions:

(25) Let us consider a natural number N_0, a commutative ring L, a finite sequence F of elements of the carrier of Polynom-Ring L, and an ele-

ment x of L. Suppose $\operatorname{len} F = N_0 + 1$. Then $\operatorname{eval}(F, x) = \operatorname{eval}(F \restriction N_0, x) \frown \langle \operatorname{eval}(\mathrm{R2P}(F_{/\operatorname{len} F}), x) \rangle$.

Proof: For every natural number k such that $1 \leqslant k \leqslant \operatorname{len} \operatorname{eval}(F, x)$ holds $(\operatorname{eval}(F, x))(k) = (\operatorname{eval}(F \restriction N_0, x) \frown \langle \operatorname{eval}(\mathrm{R2P}(F_{/\operatorname{len} F}), x) \rangle)(k)$. □

(26) Let us consider a commutative ring L, a finite sequence F of elements of the carrier of Polynom-Ring L, and an element x of L. Then $\operatorname{eval}(\mathrm{R2P}(\sum F), x) = \sum \operatorname{eval}(F, x)$. The theorem is a consequence of (25).

(27) Let us consider elements p, q of the carrier of Polynom-Ring $\mathbb{Z}^{\mathrm{R}}$. Then

(i) ${}^{@}(p + q) = {}^{@}p + {}^{@}q$, and

(ii) ${}^{@}(p \cdot q) = ({}^{@}p) \cdot ({}^{@}q)$.

5. The Formal Counterpart of the Transformation (1) in [7]

Let f be an element of the carrier of Polynom-Ring $\mathbb{Z}^{\mathrm{R}}$. The functor $\mathcal{G}(f)$ yielding a finite sequence of elements of the carrier of Polynom-Ring $\mathbb{Z}^{\mathrm{R}}$ is defined by

(Def. 10) $\operatorname{len} it = \operatorname{len} f$ and for every natural number i such that $i \in \operatorname{dom} it$ holds $it(i) = (\mathrm{Der1}(\mathbb{Z}^{\mathrm{R}}))^{i -' 1}(f)$.

Now we state the propositions:

(28) Let us consider a finite sequence F of elements of the carrier of Polynom-Ring $\mathbb{Z}^{\mathrm{R}}$, an element x of $\mathbb{Z}^{\mathrm{R}}$, and an element x_1 of $\mathbb{R}_{\mathrm{F}}$. If $x = x_1$, then $\operatorname{eval}({}^{@}F, x_1) = \operatorname{eval}(F, x)$.

Proof: For every natural number i such that $i \in \operatorname{dom}(\operatorname{eval}({}^{@}F, x_1))$ holds $(\operatorname{eval}({}^{@}F, x_1))(i) = (\operatorname{eval}(F, x))(i)$ by [15, (27)]. □

(29) Let us consider a finite sequence F of elements of the carrier of Polynom-Ring $\mathbb{Z}^{\mathrm{R}}$. Then $\sum {}^{@}F = {}^{@}\sum F$. The theorem is a consequence of (27).

(30) Let us consider an element x_0 of $\mathbb{Z}^{\mathrm{R}}$, an element x of $\mathbb{R}_{\mathrm{F}}$, and a finite sequence F of elements of the carrier of Polynom-Ring $\mathbb{Z}^{\mathrm{R}}$. Suppose $x = x_0$. Then $(\operatorname{Eval}(\mathrm{R2P}({}^{@}\sum F)))(x) = \sum \operatorname{eval}(F, x_0)$. The theorem is a consequence of (28), (29), and (26).

Let f be an element of the carrier of Polynom-Ring $\mathbb{Z}^{\mathrm{R}}$. The functor $\mathcal{F}(f)$ yielding a function from $\mathbb{R}$ into $\mathbb{R}$ is defined by the term

(Def. 11) $\operatorname{Eval}(\mathrm{R2P}({}^{@}\sum \mathcal{G}(f)))$.

6. Formulating Equation (2) in [7]

Now we state the proposition:

(31) Let us consider an element p of the carrier of Polynom-Ring $\mathbb{R}_{\rm F}$. Then $\mathrm{Eval}(\mathrm{R2P}(p))\,`| = \mathrm{Eval}(\mathrm{R2P}((\mathrm{Der1}(\mathbb{R}_{\rm F}))(p)))$.

Proof: Set $D_1 = \mathrm{Der1}(\mathbb{R}_{\rm F})$. Define $\mathcal{P}$[natural number] $\equiv$ for every element p of the carrier of Polynom-Ring $\mathbb{R}_{\rm F}$ such that $\mathrm{len}\,\mathrm{R2P}(p) \leqslant \$_1$ holds $\mathrm{Eval}(\mathrm{R2P}(p))\,`| = \mathrm{Eval}(\mathrm{R2P}(D_1(p)))$. $\mathcal{P}[0]$ by [16, (58)], [8, (52), (54)]. If $\mathcal{P}[n]$, then $\mathcal{P}[n+1]$ by [8, (36), (37), (55), (14)]. $\mathcal{P}[n]$. □

Let f be an element of the carrier of Polynom-Ring $\mathbb{Z}^{\rm R}$. The functor $\Phi(f)$ yielding a function from $\mathbb{R}$ into $\mathbb{R}$ is defined by the term

(Def. 12) $\frac{1}{\exp_R} \cdot \mathcal{F}(f)$.

Note that $\mathcal{F}(f)$ is differentiable as a function from $\mathbb{R}$ into $\mathbb{R}$.

Let us consider an element f of the carrier of Polynom-Ring $\mathbb{Z}^{\rm R}$. Now we state the propositions:

(32) $(\frac{1}{\exp_R} \cdot \mathcal{F}(f)) \restriction [0, x_0]$ is continuous.

Proof: Set $f_1 = \frac{1}{\text{the function exp}}$. Set $f_2 = \mathcal{F}(f)$. For every real number r such that $r \in \mathrm{dom}((f_1 \cdot f_2) \restriction [0, x_0])$ holds $(f_1 \cdot f_2) \restriction [0, x_0]$ is continuous in r. □

(33) $\frac{1}{\exp_R} \cdot \mathcal{F}(f)$ is differentiable on $]0, x_0[$.

(34) The formal version of the equation (2) in [7]:
Let us consider an element f of the carrier of Polynom-Ring $\mathbb{Z}^{\rm R}$, and a positive real number x_0. Suppose $\mathrm{len}\, f > 0$. Then there exists a real number s such that

(i) $0 < s < 1$, and

(ii) $(\mathcal{F}(f))(x_0) - (\text{the function exp})(x_0) \cdot (\mathcal{F}(f))(0) =$
$-x_0 \cdot (\text{the function exp})(x_0 \cdot (1 - s)) \cdot (\mathrm{Eval}(\mathrm{R2P}(^{@}f)))(s \cdot x_0)$.

7. On Some Ring and Domain Ring Extensions

Now we state the propositions:

(35) Let us consider an integral domain F, a ring extension E of F, a polynomial p over F, a polynomial q over E, an element a of F, and an element b of E. If $p = q$ and $a = b$, then $a \cdot p = b \cdot q$.

(36) Let us consider an integral domain F, a domain ring extension E of F, a polynomial p over F, an element a of F, and elements x, b of E. If $b = a$, then $\mathrm{ExtEval}(a \cdot p, x) = b \cdot (\mathrm{ExtEval}(p, x))$. The theorem is a consequence of (35).

(37) Let us consider a non degenerated commutative ring L, a non empty finite sequence F of elements of the carrier of Polynom-Ring L, and an element x of L. Then $\mathrm{eval}(\mathrm{R2P}(\prod F), x) = \prod \mathrm{eval}(F, x)$.

Proof: For every non zero natural number k such that $\mathrm{len}\, F = k$ holds $\mathrm{eval}(\mathrm{R2P}(\prod F), x) = \prod \mathrm{eval}(F, x)$. $\square$

(38) Let us consider a non empty finite sequence F of elements of the carrier of Polynom-Ring $\mathbb{Z}^{\mathrm{R}}$, and an element x of $\mathbb{R}_{\mathrm{F}}$. Then $\mathrm{eval}(\mathrm{R2P}({}^{@}\prod F), x) = \prod \mathrm{eval}({}^{@}F, x)$. The theorem is a consequence of (37).

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