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Formalization of Orthogonal Complements of Normed Spaces

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Summary. In this study we are formalizing the optimization theory in Mizar. It is well known that geometric principles of linear vector space theory play fundamental roles in optimization. This article focuses on formalization of definitions and some theorems about dual spaces: we formalize orthogonal complements of real normed spaces, then we deal with minimum norm problems.

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Introduction

This article is the next one in the series developing the theory of normed spaces in Mizar [3], [11] (for similar developments in another theorem provers, see [4] in Isabelle/HOL, [1], [2] in Coq or [5] in Lean 4).

We introduce the fundamentals of the optimization theory, as it is well known that geometric principles of linear vector space theory play fundamental roles in optimization [8]. Furthermore, any optimization problem can be viewed from either of two perspectives: the primal problem or the dual one. Our work focuses on the formalization of definitions and some theorems about dual spaces [12]. In the first section, we formalize orthogonal complements of real normed spaces, which may be seen as a continuation of [9] and [10]. Section 2 is a collection of more or less standard properties of complements while in the last section the encoding of minimum norm problems is contained, following the lines of [6] and [8].

1. Orthogonal Complements of Normed Spaces

Let V be a real normed space and W be a subspace of V. The functor RSubNormSpace(W) yielding a strict real normed space is defined by

(Def. 1) the RLS structure of *it* = the RLS structure of W and the norm of *it* = (the norm of V)$\restriction$(the carrier of *it*).

Now we state the proposition:

(1) Let us consider a real normed space V, and a subspace W of V. Then RSubNormSpace(W) is a subreal normal space of V.

Let V be a real normed space, x be a point of V, and y be a point of DualSp V. The functor $(x|y)$ yielding a real number is defined by the term

(Def. 2) $y(x)$.

Now we state the proposition:

(2) Let us consider a real normed space V, a point x of V, and a point y of DualSp V. Then $|(x|y)| \leqslant \|y\| \cdot \|x\|$.

Let V be a real normed space, x be a point of V, and y be a point of DualSp V. We say that x, y are orthogonal if and only if

(Def. 3) $(x|y) = 0$.

Now we state the propositions:

(3) Let us consider a real normed space V, a point x of V, and points y, z of DualSp V. Then $(x|(y + z)) = (x|y) + (x|z)$.

(4) Let us consider a real normed space V, a point x of V, a point y of DualSp V, and a real number a. Then $(x|a \cdot y) = a \cdot (x|y)$.

(5) Let us consider a real normed space V, points x, y of V, and a point z of DualSp V. Then $((x + y)|z) = (x|z) + (y|z)$.

(6) Let us consider a real normed space V, a point x of V, a point y of DualSp V, and a real number a. Then $(a \cdot x|y) = a \cdot (x|y)$.

(7) Let us consider a real normed space V, a point x of V, points y, z of DualSp V, and real numbers a, b. Then $(x|(a \cdot y + b \cdot z)) = a \cdot (x|y) + b \cdot (x|z)$.

(8) Let us consider a real normed space V, points y, z of V, a point x of DualSp V, and real numbers a, b. Then $((a \cdot y + b \cdot z)|x) = a \cdot (y|x) + b \cdot (z|x)$.
The theorem is a consequence of (5) and (6).

2. Selected Properties of Orthogonality

Let us consider a real normed space V, a point x of V, and a point y of DualSp V. Now we state the propositions:

(9) $(x|(-y)) = -(x|y)$.

(10) $((-x)|y) = -(x|y)$. The theorem is a consequence of (6).

(11) $((-x)|(-y)) = (x|y)$. The theorem is a consequence of (10) and (9).

(12) Let us consider a real normed space V, a point x of V, and points y, z of DualSp V. Then $(x|(y-z)) = (x|y) - (x|z)$. The theorem is a consequence of (9).

(13) Let us consider a real normed space V, points y, z of V, and a point x of DualSp V. Then $((y-z)|x) = (y|x) - (z|x)$. The theorem is a consequence of (5) and (10).

(14) Let us consider a real normed space V, and a point x of V. Then $(x|0_{\mathrm{DualSp}\,V}) = 0$.

(15) Let us consider a real normed space V, and a point x of DualSp V. Then $(0_V|x) = 0$. The theorem is a consequence of (6).

Let V be a real normed space, x be a point of V, and y be a point of DualSp V. We say that x, y are parallel if and only if

(Def. 4) $(x|y) = \|x\| \cdot \|y\|$.

Let W be a subspace of V. The functor OrtComp(W) yielding a strict subspace of DualSp V is defined by

(Def. 5) the carrier of it = $\{v$, where v is a vector of DualSp V : for every vector w of V such that $w \in W$ holds w, v are orthogonal$\}$.

Let W be a subspace of DualSp V. The functor OrtComp(W) yielding a strict subspace of V is defined by

(Def. 6) the carrier of $it = \{v$, where v is a vector of V : for every vector w of DualSp V such that $w \in W$ holds v, w are orthogonal$\}$.

Now we state the propositions:

(16) Let us consider a real normed space V, a subspace M of V, a vector v of DualSp V, and a vector m of V. If $v \in$ OrtComp(M) and $m \in M$, then $(m|v) = 0$.

(17) Let us consider a real normed space V, a subspace M of DualSp V, a vector v of V, and a vector m of DualSp V. If $v \in$ OrtComp(M) and $m \in M$, then $(v|m) = 0$.

3. Minimum Norm Problems

Let us consider a real normed space X, a point x of X, and a non empty subspace M of X. Now we state the propositions:

(18) $\{\|x-m\|$, where m is a point of $X : m \in M\}$ is a non empty, lower bounded, real-membered set.

(19) $\{(x|y)$, where y is a point of $\mathrm{DualSp}\,X : y \in \mathrm{OrtComp}(M)$ and $\|y\| \leqslant 1\}$ is a non empty, upper bounded, real-membered set.
PROOF: Set $B = \{(x|y)$, where y is a point of $\mathrm{DualSp}\,X : y \in \mathrm{OrtComp}(M)$ and $\|y\| \leqslant 1\}$. $B \subseteq \mathbb{R}$. B is upper bounded by [7, (26)]. $\square$

(20) Let us consider a real normed space X, a point x of X, a non empty subspace M of X, finite sequences F, K of elements of the carrier of X, and a finite sequence G of elements of $\mathbb{R}$. Suppose $\mathrm{len}\,G = \mathrm{len}\,F$ and $\mathrm{len}\,K = \mathrm{len}\,F$ and for every natural number i such that $i \in \mathrm{dom}\,F$ holds $F(i) \in x + M$ and for every natural number i such that $i \in \mathrm{dom}\,K$ holds $K(i) = G_{/i} \cdot (F_{/i})$. Then $\sum K \in \{a \cdot x + m$, where a is a real number, m is a point of $X : m \in M\}$.
PROOF: Define $\mathcal{P}$[natural number] $\equiv$ for every finite sequences F, K of elements of the carrier of X for every finite sequence G of elements of $\mathbb{R}$ such that $\mathrm{len}\,F = \$_1$ and $\mathrm{len}\,G = \mathrm{len}\,F$ and $\mathrm{len}\,K = \mathrm{len}\,F$ and for every natural number i such that $i \in \mathrm{dom}\,F$ holds $F(i) \in x + M$ and for every natural number i such that $i \in \mathrm{dom}\,K$ holds $K(i) = G_{/i} \cdot (F_{/i})$ holds $\sum K \in \{a \cdot x + m$, where a is a real number, m is a point of $X : m \in M\}$. $\mathcal{P}[0]$. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$. For every natural number k, $\mathcal{P}[k]$. $\square$

(21) Let us consider a real normed space V, a point x of V, and a non empty subspace M of V. Suppose $x \notin M$. Then there exists a non empty, lower bounded, real-membered set L and there exists a non empty, upper bounded, real-membered set U such that $L = \{\|x-m\|$, where m is a point of $V : m \in M\}$ and $U = \{(x|y)$, where y is a point of $\mathrm{DualSp}\,V : y \in \mathrm{OrtComp}(M)$ and $\|y\| \leqslant 1\}$ and $\inf L = \sup U$ and $\sup U \in U$ and if $0 < \inf L$, then there exists a point v of $\mathrm{DualSp}\,V$ such that $\|v\| = 1$ and $v \in \mathrm{OrtComp}(M)$ and $(x|v) = \inf L$ and for every point m_0 of V such that $m_0 \in M$ and $\|x - m_0\| = \inf L$ for every point v of $\mathrm{DualSp}\,V$ such that $\|v\| = 1$ and $v \in \mathrm{OrtComp}(M)$ and $(x|v) = \inf L$ holds $x - m_0$, v are parallel.
PROOF: Reconsider $L = \{\|x-m\|$, where m is a point of $V : m \in M\}$ as a non empty, lower bounded, real-membered set. Reconsider $U = \{(x|y)$, where y is a point of $\mathrm{DualSp}\,V : y \in \mathrm{OrtComp}(M)$ and $\|y\| \leqslant 1\}$ as a non empty, upper bounded, real-membered set. Set $d = \inf L$. For

every real number r such that $r \in U$ holds $r \leqslant d$. Reconsider $x_1 = x + M$ as a subset of (V **qua** real linear space). Reconsider $L_2 = \mathrm{Lin}(x_1)$ as a subspace of V. Set $S = \{a \cdot x + m$, where a is a real number, m is a point of $V : m \in M\}$. For every object z, $z \in S$ iff $z \in$ the carrier of L_2. Reconsider $L_1 = \mathrm{RSubNormSpace}(L_2)$ as a subreal normal space of V. For every real numbers a_1, a_2 and for every points m_1, m_2 of V such that $m_1, m_2 \in M$ and $a_1 \cdot x + m_1 = a_2 \cdot x + m_2$ holds $a_1 = a_2$ and $m_1 = m_2$. Define $\mathcal{Q}[\text{object}, \text{object}] \equiv$ there exists a point m of V and there exists a real number a such that $m \in M$ and $\$_1 = a \cdot x + m$ and $\$_2 = a \cdot d$. For every element s of the carrier of L_1, there exists an element y of $\mathbb{R}$ such that $\mathcal{Q}[s, y]$.

Consider f being a function from the carrier of L_1 into $\mathbb{R}$ such that for every element x of the carrier of L_1, $\mathcal{Q}[x, f(x)]$. For every points s, t of L_1, $f(s + t) = f(s) + f(t)$. For every point s of L_1 and for every real number r, $f(r \cdot s) = r \cdot f(s)$. $0 \leqslant d$. For every vector s of L_1, $|f(s)| \leqslant 1 \cdot \|s\|$. Reconsider $p_1 = f$ as a point of $\mathrm{DualSp}\, L_1$. Consider g being a Lipschitzian linear functional in V, p_2 being a point of $\mathrm{DualSp}\, V$ such that $g = p_2$ and $g{\restriction}(\text{the carrier of } L_1) = f$ and $\|p_2\| = \|p_1\|$. Consider m being a point of V, a being a real number such that $m \in M$ and $x = a \cdot x + m$ and $f(x) = a \cdot d$. For every vector m of V such that $m \in M$ holds m, p_2 are orthogonal. For every real number s such that $0 < s$ there exists a real number r such that $r \in U$ and $d - s < r$. If $0 < \inf L$, then there exists a point p_2 of $\mathrm{DualSp}\, V$ such that $\|p_2\| = 1$ and $p_2 \in \mathrm{OrtComp}(M)$ and $(x|p_2) = \inf L$. □

(22) Let us consider a real normed space V, points x, m_0 of V, and a non empty subspace M of V. Suppose $x \notin M$ and $m_0 \in M$. Then for every point m of V such that $m \in M$ holds $\|x - m_0\| \leqslant \|x - m\|$ if and only if there exists a point p of $\mathrm{DualSp}\, V$ such that $p \in \mathrm{OrtComp}(M)$ and $p \neq 0_{\mathrm{DualSp}\, V}$ and $x - m_0$, p are parallel. The theorem is a consequence of (21), (13), (16), and (2).

Let us consider a real normed space X, a point x of $\mathrm{DualSp}\, X$, and a non empty subspace M of X. Now we state the propositions:

(23) $\{\|x - m\|$, where m is a point of $\mathrm{DualSp}\, X : m \in \mathrm{OrtComp}(M)\}$ is a non empty, lower bounded, real-membered set.

(24) $\{(y|x)$, where y is a point of $X : y \in M$ and $\|y\| \leqslant 1\}$ is a non empty, upper bounded, real-membered set.

Proof: Set $B = \{(y|x)$, where y is a point of $X : y \in M$ and $\|y\| \leqslant 1\}$. $B \subseteq \mathbb{R}$. B is upper bounded. □

(25) Let us consider a real normed space V, a point x of $\mathrm{DualSp}\, V$, a non empty subspace M of V, and a subreal normal space S_2 of V. Suppose

$S_2 = \mathrm{RSubNormSpace}(M)$. Then there exists a non empty, lower bounded, real-membered set L and there exists a non empty, upper bounded, real-membered set U such that $L = \{\|x - m\|$, where m is a point of $\mathrm{DualSp}\,V : m \in \mathrm{OrtComp}(M)\}$ and $U = \{(y|x)$, where y is a point of $V : y \in M$ and $\|y\| \leqslant 1\}$ and there exists a point m_0 of $\mathrm{DualSp}\,V$ and there exists a Lipschitzian linear functional f_2 in V and there exists a point x_1 of $\mathrm{DualSp}\,S_2$ such that $x = f_2$ and $x_1 = f_2\restriction$(the carrier of S_2) and $m_0 \in \mathrm{OrtComp}(M)$ and $\|x - m_0\| = \inf L$ and $\inf L \in L$ and $\|x - m_0\| = \|x_1\|$ and for every point y of V such that $\|y\| = 1$ and $y \in M$ and $f_2(y) = \|x - m_0\|$ holds y, $x - m_0$ are parallel.

Proof: Reconsider $L = \{\|x - m\|$, where m is a point of $\mathrm{DualSp}\,V$: $m \in \mathrm{OrtComp}(M)\}$ as a non empty, lower bounded, real-membered set. Reconsider $f_2 = x$ as a Lipschitzian linear functional in V. Set $f_3 = f_2\restriction$(the carrier of S_2). For every points s, t of S_2, $f_3(s+t) = f_3(s) + f_3(t)$. For every point s of S_2 and for every real number r, $f_3(r \cdot s) = r \cdot f_3(s)$. For every vector s of S_2, $|f_3(s)| \leqslant \|x\| \cdot \|s\|$. Reconsider $x_1 = f_3$ as a point of $\mathrm{DualSp}\,S_2$.

Consider f_4 being a Lipschitzian linear functional in V, y being a point of $\mathrm{DualSp}\,V$ such that $f_4 = y$ and $f_4\restriction$(the carrier of S_2) $= f_3$ and $\|y\| = \|x_1\|$. Set $m_0 = x - y$. For every point t of V such that $t \in M$ holds t, m_0 are orthogonal. For every real number r such that $r \in L$ holds $\|x_1\| \leqslant r$. For every real number s such that $0 < s$ there exists a real number r such that $r \in L$ and $r < \|x_1\| + s$. $(y|(x - m_0)) = (y|x) - (y|m_0)$. $\square$

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