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Ascoli-Arzelà Theorem (Metric Space Version)

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Summary. We formulate and prove in Mizar the Ascoli-Arzelà's theorem, which gives necessary and sufficient conditions for a collection of continuous functions to be compact. We use the metric space setting, and the notions of equicontinuousness and equiboundedness of a set of continuous functions are utilized.

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Introduction

The Ascoli-Arzelà theorem is an important result in real analysis [3], which states that a family of continuous functions on a compact space has a uniformly convergent subsequence if and only if the family is pointwise bounded and equicontinuous [9], [13], [14]. The theorem is widely used in the study of functional analysis and has important applications in various fields such as differential equations [12], topology (where is also an important result on its own [4]), and approximation theory. It was proven by Ascoli in 1883 (in its weaker form) [2], and then by Arzelà in 1895 [1], and it is also known in the literature as Arzelà-Ascoli theorem (under such name it is also present, e.g. in Lean's mathematical library `mathlib` [6]).

In Section 1 we formulate the notions of equicontinuousness and equiboundedness of a set of continuous functions [16]. Second section deals with totally bounded metric spaces, and in Section 3 we formally prove some properties of the metric space of continuous functions. The final section contains the formulation and the proof of the main theorem, where we reuse the formal apparatus of metric spaces [5], [7], [10]. Previously proven version (in its topological setting) of the theorem [15] was reused recently to enrich the Mizar Mathematical Library [11] by properties of Feed-forward Neural Network in [8].

1. Equicontinuousness and Equiboundedness of Continuous Functions

Now we state the propositions:

(1) Let us consider a non empty metric space T, and a subset A of T. Then $A \subseteq \overline{A}$.

(2) Let us consider a non empty topological space S, a non empty metric space T, a function f from S into $T_{\rm top}$, and a point x of S. Then f is continuous at x if and only if for every real number e such that $0 < e$ there exists a subset H of S such that H is open and $x \in H$ and for every point y of S such that $y \in H$ holds $\rho(f(x)(\in T), f(y)(\in T)) < e$.
Proof: For every subset G of $T_{\rm top}$ such that G is open and $f(x) \in G$ there exists a subset H of S such that H is open and $x \in H$ and $f^{\circ}H \subseteq G$. □

Let S, T be non empty metric spaces and F be a subset of (the carrier of T)$^{(\text{the carrier of } S)}$. We say that F is equibounded if and only if

(Def. 1) there exists a subset K of T such that K is bounded and for every function f from the carrier of S into the carrier of T such that $f \in F$ for every element x of S, $f(x) \in K$.

Let x_0 be a point of S. We say that F is equicontinuous at x_0 if and only if

(Def. 2) for every real number e such that $0 < e$ there exists a real number d such that $0 < d$ and for every function f from the carrier of S into the carrier of T such that $f \in F$ for every point x of S such that $\rho(x, x_0) < d$ holds $\rho(f(x), f(x_0)) < e$.

We say that F is equicontinuous if and only if

(Def. 3) for every real number e such that $0 < e$ there exists a real number d such that $0 < d$ and for every function f from the carrier of S into the carrier of T such that $f \in F$ for every points x_1, x_2 of S such that $\rho(x_1, x_2) < d$ holds $\rho(f(x_1), f(x_2)) < e$.

2. On Totally Bounded Spaces

Now we state the proposition:

(3) Let us consider a non empty metric space Z, and a non empty subset F of Z. If Z is complete, then $Z\restriction\overline{F}$ is complete.
PROOF: Set $N = Z\restriction\overline{F}$. Reconsider $S_1 = S2$ as a sequence of Z. For every real number r such that $r > 0$ there exists a natural number k such that for every natural numbers n, m such that $n \geqslant k$ and $m \geqslant k$ holds $\rho(S_1(n), S_1(m)) < r$. Consider H being a subset of $Z_{\rm top}$ such that $H = F$ and $\overline{F} = \overline{H}$. For every natural number n, $S_1(n) \in \overline{H}$. Reconsider $L = \lim S_1$ as a point of N. For every real number r such that $0 < r$ there exists a natural number m such that for every natural number n such that $m \leqslant n$ holds $\rho(S2(n), L) < r$. □

Let us consider a non empty metric space Z and a non empty subset H of Z. Now we state the propositions:

(4) $Z\restriction H$ is totally bounded if and only if $Z\restriction\overline{H}$ is totally bounded.
PROOF: Consider D being a subset of $Z_{\rm top}$ such that $D = H$ and $\overline{H} = \overline{D}$. $Z\restriction H$ is totally bounded. □

(5) If Z is complete and $Z\restriction H$ is totally bounded, then $\overline{H}$ is sequentially compact and $Z\restriction\overline{H}$ is compact. The theorem is a consequence of (3) and (4).

(6) Suppose Z is complete. Then

(i) $Z\restriction H$ is totally bounded iff $\overline{H}$ is sequentially compact, and

(ii) $Z\restriction H$ is totally bounded iff $Z\restriction\overline{H}$ is compact.

The theorem is a consequence of (3) and (4).

3. Continuous Functions Revisited

Let S be a non empty topological space and T be a non empty metric space. The continuous functions of S and T yielding a non empty set is defined by the term

(Def. 4) $\{f$, where f is a function from S into $T_{\rm top}$: f is continuous$\}$.

Now we state the propositions:

(7) Let us consider a metric space X, and elements x, y, v, w of X. Then $|\rho(x, y) - \rho(v, w)| \leqslant \rho(x, v) + \rho(y, w)$.

(8) Let us consider a non empty topological space S, a non empty metric space T, and functions f, g from S into $T_{\rm top}$. Suppose f is continuous and

g is continuous. Let us consider a real map D_1 of S. Suppose for every point x of S, $D_1(x) = \rho(f(x)(\in T), g(x)(\in T))$. Then D_1 is continuous. The theorem is a consequence of (2) and (7).

(9) Let us consider a non empty, compact topological space S, a non empty metric space T, and functions f, g from S into $T_{\rm top}$. Suppose f is continuous and g is continuous. Let us consider a real map D_1 of S. Suppose for every point x of S, $D_1(x) = \rho(f(x)(\in T), g(x)(\in T))$. Then

(i) $\operatorname{rng} D_1 \neq \emptyset$, and

(ii) $\operatorname{rng} D_1$ is upper bounded and lower bounded.

The theorem is a consequence of (8).

(10) Let us consider a non empty topological space S, and a non empty metric space T. Then there exists a function F from (the continuous functions of S and T) $\times$ (the continuous functions of S and T) into $\mathbb{R}$ such that for every functions f, g from S into $T_{\rm top}$ such that f, $g \in$ the continuous functions of S and T there exists a real map D_1 of S such that for every point x of S, $D_1(x) = \rho(f(x)(\in T), g(x)(\in T))$ and $F(f, g) = \sup \operatorname{rng} D_1$.
Proof: Set $F_1 =$ the continuous functions of S and T. Define $\mathcal{P}$[object, object, object] $\equiv$ there exist functions f, g from S into $T_{\rm top}$ and there exists a real map D_1 of S such that $\$_1 = f$ and $\$_2 = g$ and for every point t of S, $D_1(t) = \rho(f(t)(\in T), g(t)(\in T))$ and $\$_3 = \sup \operatorname{rng} D_1$. For every objects x, y such that x, $y \in F_1$ there exists an object z such that $z \in \mathbb{R}$ and $\mathcal{P}[x, y, z]$. Consider F being a function from $F_1 \times F_1$ into $\mathbb{R}$ such that for every objects x, y such that x, $y \in F_1$ holds $\mathcal{P}[x, y, F(x, y)]$. □

Let S be a non empty topological space and T be a non empty metric space. The functor $\operatorname{distFunc}(S, T)$ yielding a function from (the continuous functions of S and T) $\times$ (the continuous functions of S and T) into $\mathbb{R}$ is defined by

(Def. 5) for every functions f, g from S into $T_{\rm top}$ such that f, $g \in$ the continuous functions of S and T there exists a real map D_1 of S such that for every point x of S, $D_1(x) = \rho(f(x)(\in T), g(x)(\in T))$ and $it(f, g) = \sup \operatorname{rng} D_1$.

The functor $\operatorname{ContFuncs}_{\rm metr}(S, T)$ yielding a metric structure is defined by the term

(Def. 6) $\langle$the continuous functions of S and $T, \operatorname{distFunc}(S, T)\rangle$.

Let S be a non empty, compact topological space. One can check that $\operatorname{ContFuncs}_{\rm metr}(S, T)$ is reflexive, discernible, symmetric, and triangle. Let S be a non empty topological space. Let us observe that $\operatorname{ContFuncs}_{\rm metr}(S, T)$ is non empty and strict and the continuous functions of S and T is non empty and functional. Let S be a non empty, compact topological space. One can check that $\operatorname{ContFuncs}_{\rm metr}(S, T)$ is constituted functions. Let f be an element of

$\mathrm{ContFuncs}_{\mathrm{metr}}(S,T)$ and v be a point of S. One can verify that the functor $f(v)$ yields a point of T_{top}. Now we state the propositions:

(11) Let us consider a non empty, compact topological space S, a non empty metric space T, points f, g of $\mathrm{ContFuncs}_{\mathrm{metr}}(S,T)$, and a point t of S. Then $\rho(f(t)(\in T), g(t)(\in T)) \leqslant \rho(f,g)$. The theorem is a consequence of (9).

(12) Let us consider a non empty, compact topological space S, a non empty metric space T, points f, g of $\mathrm{ContFuncs}_{\mathrm{metr}}(S,T)$, functions f_1, g_1 from S into T, and a real number e. Suppose $f = f_1$ and $g = g_1$ and for every point t of S, $\rho(f_1(t), g_1(t)) \leqslant e$. Then $\rho(f,g) \leqslant e$. The theorem is a consequence of (9).

(13) Let us consider a non empty, compact topological space S, and a non empty metric space T. If T is complete, then $\mathrm{ContFuncs}_{\mathrm{metr}}(S,T)$ is complete. The theorem is a consequence of (11), (2), and (12).

(14) Let us consider a non empty, compact topological space S, and a non empty metric space T. Suppose T is complete. Let us consider a non empty subset H of $\mathrm{ContFuncs}_{\mathrm{metr}}(S,T)$. Then $\overline{H}$ is sequentially compact if and only if $\mathrm{ContFuncs}_{\mathrm{metr}}(S,T)\restriction H$ is totally bounded. The theorem is a consequence of (13), (3), and (4).

Let us consider a non empty metric space M, a non empty, compact topological space S, a non empty metric space T, a subset G of (the carrier of $T)^{(\text{the carrier of } M)}$, and a non empty subset H of $\mathrm{ContFuncs}_{\mathrm{metr}}(S,T)$. Now we state the propositions:

(15) If $S = M_{\mathrm{top}}$, then if $G = H$ and $\mathrm{ContFuncs}_{\mathrm{metr}}(S,T)\restriction H$ is totally bounded, then G is equicontinuous.

Proof: Set $Z = \mathrm{ContFuncs}_{\mathrm{metr}}(S,T)$. Set $M_2 = Z\restriction H$. Define $\mathcal{Q}$[object, object] $\equiv$ there exists a point w of M_2 such that $\$_2 = w$ and $\$_1 = \mathrm{Ball}(w,1)$. For every real number e such that $0 < e$ there exists a real number d such that $0 < d$ and for every function f from the carrier of M into the carrier of T such that $f \in G$ for every points x_1, x_2 of M such that $\rho(x_1,x_2) < d$ holds $\rho(f(x_1), f(x_2)) < e$. □

(16) Suppose $S = M_{\mathrm{top}}$. Then suppose $G = H$ and $\mathrm{ContFuncs}_{\mathrm{metr}}(S,T)\restriction H$ is totally bounded. Then

  (i) for every point x of S and for every non empty subset H_1 of T such that $H_1 = \{f(x)$, where f is a function from S into $T : f \in H\}$ holds $T\restriction H_1$ is totally bounded, and

  (ii) G is equicontinuous.

Proof: For every point x of S and for every non empty subset H_1 of T

such that $H_1 = \{f(x)$, where f is a function from S into $T : f \in H\}$ holds $T \restriction H_1$ is totally bounded. □

(17) Suppose $S = M_{\rm top}$ and T is complete and $G = H$. Then $\mathrm{ContFuncs}_{\rm metr}(S, T) \restriction H$ is totally bounded if and only if G is equicontinuous and for every point x of S and for every non empty subset H_1 of T such that $H_1 = \{f(x)$, where f is a function from S into $T : f \in H\}$ holds $T \restriction \overline{H_1}$ is compact.

PROOF: Set $Z = \mathrm{ContFuncs}_{\rm metr}(S, T)$. Set $M_2 = Z \restriction H$. For every real number e such that $e > 0$ there exists a family L of subsets of M_2 such that L is finite and the carrier of $M_2 = \bigcup L$ and for every subset C of M_2 such that $C \in L$ there exists an element w of M_2 such that $C = \mathrm{Ball}(w, e)$. □

4. Ascoli-Arzelà Theorem

Now we state the proposition:

(18) Let us consider a non empty metric space M, a non empty, compact topological space S, a non empty metric space T, a subset G of (the carrier of $T)^{\alpha}$, and a non empty subset H of $\mathrm{ContFuncs}_{\rm metr}(S, T)$. Suppose $S = M_{\rm top}$ and T is complete and $G = H$. Then $\overline{H}$ is sequentially compact if and only if G is equicontinuous and for every point x of S and for every non empty subset H_1 of T such that $H_1 = \{f(x)$, where f is a function from S into $T : f \in H\}$ holds $T \restriction \overline{H_1}$ is compact, where α is the carrier of M. The theorem is a consequence of (14) and (17).

Let us consider a non empty metric space M, a non empty, compact topological space S, a non empty metric space T, a non empty subset F of $\mathrm{ContFuncs}_{\rm metr}(S, T)$, and a subset G of (the carrier of $T)^{(\text{the carrier of } M)}$. Now we state the propositions:

(19) Suppose $S = M_{\rm top}$ and T is complete and $G = F$. Then $\mathrm{ContFuncs}_{\rm metr}(S, T) \restriction \overline{F}$ is compact if and only if G is equicontinuous and for every point x of S and for every non empty subset F_2 of T such that $F_2 = \{f(x)$, where f is a function from S into $T : f \in F\}$ holds $T \restriction \overline{F_2}$ is compact. The theorem is a consequence of (14) and (17).

(20) Suppose $S = M_{\rm top}$ and T is complete and $G = F$. Then $\mathrm{ContFuncs}_{\rm metr}(S, T) \restriction \overline{F}$ is compact if and only if for every point x of M, G is equicontinuous at x and for every point x of S and for every non empty subset F_2 of T such that $F_2 = \{f(x)$, where f is a function from S into $T : f \in F\}$ holds $T \restriction \overline{F_2}$ is compact. The theorem is a consequence of (19).

(21) Let us consider a non empty metric space M, a non empty, compact topological space S, a non empty metric space T, a compact subset U of $T_{\rm top}$, a non empty subset F of $\mathrm{ContFuncs}_{\rm metr}(S,T)$, and a subset G of (the carrier of $T)^{\alpha}$. Suppose $S = M_{\rm top}$ and T is complete and $G = F$ and for every function f such that $f \in F$ holds $\mathrm{rng}\, f \subseteq U$. Then $\mathrm{ContFuncs}_{\rm metr}(S,T){\restriction}\overline{F}$ is compact if and only if G is equicontinuous, where α is the carrier of M.

Proof: Set $Z = \mathrm{ContFuncs}_{\rm metr}(S,T)$. $\overline{F}$ is sequentially compact iff $Z{\restriction}F$ is totally bounded. For every point x of S and for every non empty subset F_2 of T such that $F_2 = \{f(x)$, where f is a function from S into $T : f \in F\}$ holds $T{\restriction}\overline{F_2}$ is compact. $\square$

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