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Universality of Measure Space[1]

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Summary. This paper deals with the interconversion between Cartesian product types and tuple types and their integration for measures in higher dimensional spaces. We prove the universality between both types and construct a measure (and also underlying integral) based on the set of tuple types.

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Introduction

In this paper we continue the formalization of fundamentals of measure theory [12] in Mizar [2], [3] and we prove the interconversion between Cartesian product types and tuple types and their integration for measures in higher dimensional spaces. In Mizar, two types of representations are mainly used for higher-dimensional sets: those using direct products and those using tuples. The direct product type is suitable for recursively extending from lower dimensions to higher dimensions (and the development of the integral and measure using this language is contained in [7]), but is not suitable for representations of general orders such as n-dimensional. The tuple type compensates for this disadvantage and is also used as the domain of multivariable functions [10]. However,

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the direct relationship (universality) between Cartesian product type and tuple type has not yet been demonstrated within the Mizar Mathematical Library (although the problem solved here is strictly connected with the Mizar choice of the formalization technique; see the outline of the encoding of corresponding topics in Isabelle/HOL [9] or Coq [5]).

For lower dimensions, where we could use enumerated types, the difference is not that important (see, e.g. [8] for $n = 2$). We prove the universality between Cartesian product type and tuple type, and construct a measure [4] on the set of tuple types [1]. We then show that the integral over the Cartesian product type coincides with the integral over the set of tuple types.

1. Universality of Cartesian Product Type Sets and Tuple Type Sets

Now we state the propositions:

(1) Let us consider non empty sets X, Y, and a function f from X into Y. Suppose f is bijective. Then

(i) $^{\circ}f$ is bijective, and

(ii) for every subset s of X, $(^{\circ}f)(s) = f^{\circ}s$.

Proof: For every object y such that $y \in 2^Y$ there exists an object x such that $x \in 2^X$ and $y = (^{\circ}f)(x)$. □

(2) Let us consider non empty sets X, Y, a function f from X into Y, and a field S of subsets of X. If f is bijective, then $(^{\circ}f)^{\circ}S$ is a field of subsets of Y.

Proof: $^{\circ}f$ is bijective. Reconsider $S_1 = (^{\circ}f)^{\circ}S$ as a family of subsets of Y. For every sets A, B such that A, $B \in S_1$ holds $A \cap B \in S_1$. For every subset A of Y such that $A \in S_1$ holds $A^{\mathrm{c}} \in S_1$. □

Let X, Y be non empty sets, f be a function from X into Y, and S be a field of subsets of X. Assume f is bijective. The functor $\mathrm{Field}_{\mathrm{Copy}}(f, S)$ yielding a field of subsets of Y is defined by the term

(Def. 1) $(^{\circ}f)^{\circ}S$.

Now we state the proposition:

(3) Let us consider non empty sets X, Y, a function f from X into Y, and a σ-field S of subsets of X. Suppose f is bijective. Then $(^{\circ}f)^{\circ}S$ is a σ-field of subsets of Y.

Proof: Set $S_1 = (^{\circ}f)^{\circ}S$. $^{\circ}f$ is bijective. For every sequence A_1 of subsets of Y such that $\operatorname{rng} A_1 \subseteq S_1$ holds $\operatorname{Intersection} A_1 \in S_1$. □

Let X, Y be non empty sets, f be a function from X into Y, and S be a σ-field of subsets of X. Assume f is bijective. The functor $\mathrm{Field}_{\mathrm{Copy}}(f, S)$ yielding a σ-field of subsets of Y is defined by the term

(Def. 2) $({}^{\circ}f)^{\circ}S$.

Let us consider non empty sets X, Y, a function f from X into Y, a field S of subsets of X, and a measure M on S. Now we state the propositions:

(4) Suppose f is bijective. Then

(i) there exists a function G from S into $\mathrm{Field}_{\mathrm{Copy}}(f, S)$ such that $G = {}^{\circ}f \restriction S$ and $\mathrm{dom}\, G = S$ and $\mathrm{rng}\, G = \mathrm{Field}_{\mathrm{Copy}}(f, S)$ and G is bijective, and

(ii) there exists a function F from $\mathrm{Field}_{\mathrm{Copy}}(f, S)$ into S such that $F = ({}^{\circ}f \restriction S)^{-1}$ and $\mathrm{rng}\, F = S$ and $\mathrm{dom}\, F = \mathrm{Field}_{\mathrm{Copy}}(f, S)$ and F is bijective.

(5) Suppose f is bijective. Then there exists a measure M_1 on $\mathrm{Field}_{\mathrm{Copy}}(f, S)$ such that

(i) $M_1 = M \cdot (({}^{\circ}f \restriction S)^{-1})$, and

(ii) for every element s of $\mathrm{Field}_{\mathrm{Copy}}(f, S)$, there exists an element t of S such that $s = f^{\circ}t$ and $M_1(s) = M(t)$.

PROOF: Consider F being a function from $\mathrm{Field}_{\mathrm{Copy}}(f, S)$ into S such that $F = ({}^{\circ}f \restriction S)^{-1}$ and $\mathrm{rng}\, F = S$ and $\mathrm{dom}\, F = \mathrm{Field}_{\mathrm{Copy}}(f, S)$ and F is bijective. Consider G being a function from S into $\mathrm{Field}_{\mathrm{Copy}}(f, S)$ such that $G = {}^{\circ}f \restriction S$ and $\mathrm{dom}\, G = S$ and $\mathrm{rng}\, G = \mathrm{Field}_{\mathrm{Copy}}(f, S)$ and G is bijective. Reconsider $M_1 = M \cdot F$ as a function from $\mathrm{Field}_{\mathrm{Copy}}(f, S)$ into $\overline{\mathbb{R}}$. $({}^{\circ}f \restriction S)(\emptyset) = f^{\circ}\emptyset$. For every element s of $\mathrm{Field}_{\mathrm{Copy}}(f, S)$, there exists an element t of S such that $s = f^{\circ}t$ and $M_1(s) = M(t)$. For every elements A, B of $\mathrm{Field}_{\mathrm{Copy}}(f, S)$ such that A misses B and $A \cup B \in \mathrm{Field}_{\mathrm{Copy}}(f, S)$ holds $M_1(A \cup B) = M_1(A) + M_1(B)$. $\square$

Let X, Y be non empty sets, f be a function from X into Y, S be a field of subsets of X, and M be a measure on S. Assume f is bijective. The functor $\mathrm{Measure}_{\mathrm{Copy}}(f, M)$ yielding a measure on $\mathrm{Field}_{\mathrm{Copy}}(f, S)$ is defined by

(Def. 3) $\mathit{it} = M \cdot (({}^{\circ}f \restriction S)^{-1})$ and for every element s of $\mathrm{Field}_{\mathrm{Copy}}(f, S)$, there exists an element t of S such that $s = f^{\circ}t$ and $\mathit{it}(s) = M(t)$.

Now we state the proposition:

(6) Let us consider non empty sets X, Y, a function f from X into Y, a σ-field S of subsets of X, and a σ-measure M on S. Suppose f is bijective. Then there exists a σ-measure M_1 on $\mathrm{Field}_{\mathrm{Copy}}(f, S)$ such that

(i) $M_1 = M \cdot (({}^{\circ}f \restriction S)^{-1})$, and

(ii) for every element s of $\mathrm{Field}_{\mathrm{Copy}}(f,S)$, there exists an element t of S such that $s = f^{\circ}t$ and $M_1(s) = M(t)$.

Proof: Reconsider $S_0 = S$ as a field of subsets of X. Consider F being a function from $\mathrm{Field}_{\mathrm{Copy}}(f,S_0)$ into S_0 such that $F = ({}^{\circ}f{\restriction}S_0)^{-1}$ and $\mathrm{rng}\,F = S_0$ and $\mathrm{dom}\,F = \mathrm{Field}_{\mathrm{Copy}}(f,S_0)$ and F is bijective. Consider G being a function from S_0 into $\mathrm{Field}_{\mathrm{Copy}}(f,S_0)$ such that $G = {}^{\circ}f{\restriction}S_0$ and $\mathrm{dom}\,G = S_0$ and $\mathrm{rng}\,G = \mathrm{Field}_{\mathrm{Copy}}(f,S_0)$ and G is bijective. Consider M_1 being a measure on $\mathrm{Field}_{\mathrm{Copy}}(f,S_0)$ such that $M_1 = M \cdot (({}^{\circ}f{\restriction}S_0)^{-1})$ and for every element s of $\mathrm{Field}_{\mathrm{Copy}}(f,S_0)$, there exists an element t of S_0 such that $s = f^{\circ}t$ and $M_1(s) = M(t)$. For every sequence s of separated subsets of $\mathrm{Field}_{\mathrm{Copy}}(f,S)$, $\overline{\sum} M_1 \cdot s = M_1(\bigcup \mathrm{rng}\, s)$. □

Let X, Y be non empty sets, f be a function from X into Y, S be a σ-field of subsets of X, and M be a σ-measure on S. Assume f is bijective. The functor $\mathrm{Measure}_{\mathrm{Copy}}(f,M)$ yielding a σ-measure on $\mathrm{Field}_{\mathrm{Copy}}(f,S)$ is defined by

(Def. 4) $it = M \cdot (({}^{\circ}f{\restriction}S)^{-1})$ and for every element s of $\mathrm{Field}_{\mathrm{Copy}}(f,S)$, there exists an element t of S such that $s = f^{\circ}t$ and $it(s) = M(t)$.

2. Correspondence between Types

Let m be a non zero natural number and X be a non-empty, m-element finite sequence. The functor Pt2FinSeq(X) yielding an m-element finite sequence is defined by

(Def. 5) there exists a function i_1 from $\prod_{\mathrm{FS}} \mathrm{SubFin}(X,1)$ into $\prod \mathrm{SubFin}(X,1)$ such that $it(1) = i_1$ and i_1 is bijective and for every object x such that $x \in \prod_{\mathrm{FS}} \mathrm{SubFin}(X,1)$ holds $i_1(x) = \langle x \rangle$ and for every non zero natural number i such that $i < m$ there exists a function F_2 from $\prod_{\mathrm{FS}} \mathrm{SubFin}(X,i)$ into $\prod \mathrm{SubFin}(X,i)$ and there exists a function I_3 from $\prod_{\mathrm{FS}} \mathrm{SubFin}(X,i) \times \mathrm{ElmFin}(X,i+1)$ into $\prod \mathrm{SubFin}(X,i+1)$ such that $F_2 = it(i)$ and $I_3 = it(i+1)$ and F_2 is bijective and I_3 is bijective and for every objects x, y such that $x \in \prod_{\mathrm{FS}} \mathrm{SubFin}(X,i)$ and $y \in \mathrm{ElmFin}(X,i+1)$ there exists a finite sequence s such that $F_2(x) = s$ and $I_3(x,y) = s \frown \langle y \rangle$.

Now we state the proposition:

(7) Let us consider non zero natural numbers m, n, and a non-empty, m-element finite sequence X. Suppose $n \leqslant m$. Then $(\mathrm{Pt2FinSeq}(X))(n)$ is a function from $\prod_{\mathrm{FS}} \mathrm{SubFin}(X,n)$ into $\prod \mathrm{SubFin}(X,n)$.

Proof: Define $\mathcal{P}$[natural number] $\equiv$ if $1 \leqslant \$_1 \leqslant n$, then there exists a non zero natural number i such that $\$_1 = i$ and $(\mathrm{Pt2FinSeq}(X))(i)$ is a function from $\prod_{\mathrm{FS}} \mathrm{SubFin}(X,i)$ into $\prod \mathrm{SubFin}(X,i)$. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$. For every natural number k, $\mathcal{P}[k]$. Consider

i being a non zero natural number such that $i = n$ and $(\mathrm{Pt2FinSeq}(X))(i)$ is a function from $\prod_{\mathrm{FS}} \mathrm{SubFin}(X, i)$ into $\prod \mathrm{SubFin}(X, i)$. □

Let us consider non zero natural numbers m, n_1, n_2, k and a non-empty, m-element finite sequence X. Now we state the propositions:

(8) Suppose $k \leqslant n_1 \leqslant n_2 \leqslant m$. Then

(i) $\mathrm{SubFin}(\mathrm{SubFin}(X, n_1), k) = \mathrm{SubFin}(\mathrm{SubFin}(X, n_2), k)$, and

(ii) $\mathrm{ElmFin}(\mathrm{SubFin}(X, n_1), k) = \mathrm{ElmFin}(\mathrm{SubFin}(X, n_2), k)$.

(9) If $k \leqslant n_1 \leqslant n_2 \leqslant m$, then $(\mathrm{Pt2FinSeq}(\mathrm{SubFin}(X, n_1)))(k) = (\mathrm{Pt2FinSeq}(\mathrm{SubFin}(X, n_2)))(k)$.

Proof: Set $X_1 = \mathrm{SubFin}(X, n_1)$. Set $X_2 = \mathrm{SubFin}(X, n_2)$. Define $\mathcal{P}$[natural number] $\equiv$ if $1 \leqslant \$_1 \leqslant n_1$, then there exists a non zero natural number i such that $i = \$_1$ and $(\mathrm{Pt2FinSeq}(X_1))(i) = (\mathrm{Pt2FinSeq}(X_2))(i)$. For every natural number i such that $\mathcal{P}[i]$ holds $\mathcal{P}[i + 1]$. For every natural number i, $\mathcal{P}[i]$. □

(10) Let us consider non zero natural numbers m, n, k, and a non-empty, m-element finite sequence X. Suppose $k \leqslant n \leqslant m$. Then $(\mathrm{Pt2FinSeq}(X))(k) = (\mathrm{Pt2FinSeq}(\mathrm{SubFin}(X, n)))(k)$. The theorem is a consequence of (9).

(11) Let us consider non zero natural numbers m, n, a non-empty, m-element finite sequence X, and a function P from $\prod_{\mathrm{FS}} \mathrm{SubFin}(X, n)$ into $\prod \mathrm{SubFin}(X, n)$. If $n \leqslant m$ and $P = (\mathrm{Pt2FinSeq}(X))(n)$, then P is bijective.

Proof: Define $\mathcal{P}$[natural number] $\equiv$ if $1 \leqslant \$_1 \leqslant n$, then there exists a non zero natural number i and there exists a function F from $\prod_{\mathrm{FS}} \mathrm{SubFin}(X, i)$ into $\prod \mathrm{SubFin}(X, i)$ such that $\$_1 = i$ and $F = (\mathrm{Pt2FinSeq}(X))(i)$ and F is bijective. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$. For every natural number k, $\mathcal{P}[k]$. □

Let m be a non zero natural number and X be a non-empty, m-element finite sequence. The functor $\mathrm{CarProd}(X)$ yielding a function from $\prod_{\mathrm{FS}} X$ into $\prod X$ is defined by the term

(Def. 6) $(\mathrm{Pt2FinSeq}(X))(m)$.

Now we state the propositions:

(12) Let us consider a non zero natural number m, and a non-empty, m-element finite sequence X. Then $\mathrm{CarProd}(X)$ is bijective. The theorem is a consequence of (11).

(13) Let us consider a non zero natural number n, a non-empty, $(n + 1)$-element finite sequence X, and objects x, y. Suppose $x \in \prod_{\mathrm{FS}} \mathrm{SubFin}(X, n)$ and $y \in \mathrm{ElmFin}(X, n+1)$. Then there exist finite sequences s, t such that

(i) $(\mathrm{CarProd}(\mathrm{SubFin}(X, n)))(x) = s$, and

(ii) $\langle y \rangle = t$, and

(iii) $(\mathrm{CarProd}(X))(x, y) = s \frown t$.

The theorem is a consequence of (10).

Let n be a non zero natural number, X be a non-empty, n-element finite sequence, and S be a family of σ-fields of X. The functor XProd-Field(S) yielding a σ-field of subsets of $\prod X$ is defined by the term

(Def. 7) $\mathrm{Field}_{\mathrm{Copy}}(\mathrm{CarProd}(X), \prod_{\mathrm{Field}} S)$.

Let m be a family of σ-measures of S. The functor XProd-Measure(m) yielding a σ-measure on XProd-Field(S) is defined by the term

(Def. 8) $\mathrm{Measure}_{\mathrm{Copy}}(\mathrm{CarProd}(X), \mathrm{Measure}_{\mathrm{Prod}}(m))$.

Now we state the propositions:

(14) Let us consider non empty sets X, Y, and a function f from X into Y. Suppose f is bijective. Then there exists a function g from Y into X such that

(i) g is bijective, and

(ii) $g = f^{-1}$, and

(iii) $^{\circ}g = (^{\circ}f)^{-1}$.

PROOF: Reconsider $g = f^{-1}$ as a function from Y into X. $^{\circ}f$ is bijective. $^{\circ}g$ is bijective. For every objects x, y such that $x \in \mathrm{dom}(^{\circ}f)$ and $y \in \mathrm{dom}(^{\circ}g)$ holds $(^{\circ}f)(x) = y$ iff $(^{\circ}g)(y) = x$. □

(15) Let us consider non empty sets X, Y, a function T from X into Y, a partial function f from X to $\overline{\mathbb{R}}$, and a partial function g from Y to $\overline{\mathbb{R}}$. Suppose T is bijective and $g = f \cdot (T^{-1})$. Then

(i) $\mathrm{dom}\, g = T^{\circ}\, \mathrm{dom}\, f$, and

(ii) $\mathrm{dom}\, g = (^{\circ}T)(\mathrm{dom}\, f)$.

The theorem is a consequence of (1).

(16) Let us consider non empty sets X, Y, a σ-field S of subsets of X, a function T from X into Y, a partial function f from X to $\overline{\mathbb{R}}$, a partial function g from Y to $\overline{\mathbb{R}}$, an element A of S, and an element B of $\mathrm{Field}_{\mathrm{Copy}}(T, S)$. Suppose T is bijective and $g = f \cdot (T^{-1})$. Let us consider a real number r. Then $T^{\circ}(\text{LE-dom}(f, r)) = \text{LE-dom}(g, r)$.

PROOF: For every object x, $x \in T^{\circ}(\text{LE-dom}(f, r))$ iff $x \in \text{LE-dom}(g, r)$. □

(17) Let us consider non empty sets X, Y, a σ-field S of subsets of X, and a function T from X into Y. Suppose T is bijective. Then there exists a function H from Y into X such that

(i) H is bijective, and

(ii) $H = T^{-1}$, and

(iii) $H^{-1} = T$, and

(iv) $^{\circ}H = (^{\circ}T)^{-1}$, and

(v) $(^{\circ}H)^{\circ}(\mathrm{Field}_{\mathrm{Copy}}(T, S)) = S$, and

(vi) $\mathrm{Field}_{\mathrm{Copy}}(H, \mathrm{Field}_{\mathrm{Copy}}(T, S)) = S$.

The theorem is a consequence of (1) and (14).

(18) Let us consider non empty sets X, Y, a σ-field S of subsets of X, a function T from X into Y, and a subset A of X. Suppose T is bijective. Then $A \in S$ if and only if $T^{\circ}A \in \mathrm{Field}_{\mathrm{Copy}}(T, S)$. The theorem is a consequence of (17) and (1).

(19) Let us consider non empty sets X, Y, a σ-field S of subsets of X, a function T from X into Y, and a subset B of Y. Suppose T is bijective. Then $T^{-1}(B) \in S$ if and only if $B \in \mathrm{Field}_{\mathrm{Copy}}(T, S)$. The theorem is a consequence of (17) and (18).

3. Integral on a Tuple Type Set (one-dimensional)

Now we state the propositions:

(20) Let us consider non empty sets X, Y, a σ-field S of subsets of X, a function T from X into Y, a partial function f from X to $\overline{\mathbb{R}}$, a partial function g from Y to $\overline{\mathbb{R}}$, an element A of S, and an element B of $\mathrm{Field}_{\mathrm{Copy}}(T, S)$. Suppose T is bijective and $B = T^{\circ}A$ and $g = f \cdot (T^{-1})$. Then f is A-measurable if and only if g is B-measurable. The theorem is a consequence of (17), (1), and (16).

(21) Let us consider non empty sets X, Y, a σ-field S of subsets of X, a function T from X into Y, and a finite sequence F of separated subsets of S. Suppose T is bijective. Then $(^{\circ}T \restriction S) \cdot F$ is a finite sequence of separated subsets of $\mathrm{Field}_{\mathrm{Copy}}(T, S)$.

Proof: Set $H = {}^{\circ}T \restriction S$. Reconsider $G = H \cdot F$ as a finite sequence of elements of $\mathrm{Field}_{\mathrm{Copy}}(T, S)$. For every objects m, n such that $m \neq n$ holds $G(m)$ misses $G(n)$. □

(22) Let us consider non empty sets X, Y, a σ-field S of subsets of X, a function T from X into Y, a partial function f from X to $\overline{\mathbb{R}}$, and a partial function g from Y to $\overline{\mathbb{R}}$. Suppose T is bijective and $g = f \cdot (T^{-1})$. Then f is simple function in S if and only if g is simple function in $\mathrm{Field}_{\mathrm{Copy}}(T, S)$. The theorem is a consequence of (17).

(23) Let us consider non empty sets X, Y, a function T from X into Y, a partial function f from X to $\overline{\mathbb{R}}$, and a partial function g from Y to $\overline{\mathbb{R}}$. Suppose T is bijective and $g = f \cdot (T^{-1})$. Then f is non-negative if and only if g is non-negative.

(24) Let us consider non empty sets X, Y, a σ-field S of subsets of X, a function T from X into Y, a finite sequence F of separated subsets of S, a finite sequence a of elements of $\overline{\mathbb{R}}$, a partial function f from X to $\overline{\mathbb{R}}$, and a partial function g from Y to $\overline{\mathbb{R}}$. Suppose T is bijective and $g = f \cdot (T^{-1})$ and F and a are representation of f. Then there exists a finite sequence G of separated subsets of $\mathrm{Field}_{\mathrm{Copy}}(T, S)$ such that

(i) $G = ({}^{\circ}T{\restriction}S) \cdot F$, and

(ii) G and a are representation of g.

Proof: Set $H = {}^{\circ}T{\restriction}S$. Reconsider $G = H \cdot F$ as a finite sequence of separated subsets of $\mathrm{Field}_{\mathrm{Copy}}(T, S)$. For every object x, $x \in \mathrm{dom}\, g$ iff $x \in \bigcup \mathrm{rng}\, G$. For every natural number n such that $n \in \mathrm{dom}\, G$ for every object x such that $x \in G(n)$ holds $g(x) = a(n)$. $\square$

Let us consider non empty sets X, Y, a σ-field S of subsets of X, a function T from X into Y, a σ-measure M on S, a partial function f from X to $\overline{\mathbb{R}}$, and a partial function g from Y to $\overline{\mathbb{R}}$. Now we state the propositions:

(25) Suppose T is bijective and $g = f \cdot (T^{-1})$ and f is simple function in S and f is non-negative. Then $\int\limits_{g} (\mathrm{Measure}_{\mathrm{Copy}}(T, M))(x) dx = \int\limits_{f} M(x) dx$.

Proof: g is simple function in $\mathrm{Field}_{\mathrm{Copy}}(T, S)$ and g is non-negative. Consider F being a finite sequence of separated subsets of S, a, x being finite sequences of elements of $\overline{\mathbb{R}}$ such that F and a are representation of f and $a(1) = 0_{\overline{\mathbb{R}}}$ and for every natural number n such that $2 \leqslant n$ and $n \in \mathrm{dom}\, a$ holds $0_{\overline{\mathbb{R}}} < a(n) < +\infty$ and $\mathrm{dom}\, x = \mathrm{dom}\, F$ and for every natural number n such that $n \in \mathrm{dom}\, x$ holds $x(n) = a(n) \cdot (M \cdot F)(n)$ and $\int\limits_{f} M(x) dx = \sum x$. Consider G being a finite sequence of separated subsets of $\mathrm{Field}_{\mathrm{Copy}}(T, S)$ such that $G = ({}^{\circ}T{\restriction}S) \cdot F$ and G and a are representation of g. Set $L = \mathrm{Measure}_{\mathrm{Copy}}(T, M)$. For every natural number n such that $n \in \mathrm{dom}\, x$ holds $x(n) = a(n) \cdot (L \cdot G)(n)$. $\square$

(26) Suppose T is bijective and $g = f \cdot (T^{-1})$ and f is simple function in S and f is non-negative. Then $\int' g \, \mathrm{d}\, \mathrm{Measure}_{\mathrm{Copy}}(T, M) = \int' f \, \mathrm{d}M$. The theorem is a consequence of (25).

(27) Let us consider non empty sets X, Y, a function T from X into Y, a partial function f from X to $\overline{\mathbb{R}}$, and a partial function g from Y to $\overline{\mathbb{R}}$.

Suppose T is bijective and $g = f \cdot (T^{-1})$. Then

(i) $\max_+(g) = (\max_+(f)) \cdot (T^{-1})$, and

(ii) $\max_-(g) = (\max_-(f)) \cdot (T^{-1})$.

PROOF: Reconsider $H = T^{-1}$ as a function from Y into X. Reconsider $g_1 = (\max_+(f)) \cdot H$ as a partial function from Y to $\overline{\mathbb{R}}$. For every object x, $x \in \operatorname{dom} g_1$ iff $x \in \operatorname{dom} g$. For every element y of Y such that $y \in \operatorname{dom} g_1$ holds $g_1(y) = \max(g(y), 0_{\overline{\mathbb{R}}})$. Reconsider $g_1 = (\max_-(f)) \cdot H$ as a partial function from Y to $\overline{\mathbb{R}}$. For every object x, $x \in \operatorname{dom} g_1$ iff $x \in \operatorname{dom} g$. For every element y of Y such that $y \in \operatorname{dom} g_1$ holds $g_1(y) = \max(-g(y), 0_{\overline{\mathbb{R}}})$. □

(28) Let us consider non empty sets X, Y, a σ-field S of subsets of X, a function T from X into Y, a σ-measure M on S, a partial function f from X to $\overline{\mathbb{R}}$, a partial function g from Y to $\overline{\mathbb{R}}$, and an element A of S. Suppose T is bijective and $g = f \cdot (T^{-1})$ and $A = \operatorname{dom} f$ and f is A-measurable. Then there exists an element B of $\mathrm{Field}_{\mathrm{Copy}}(T, S)$ such that

(i) $B = T^{\circ}A$, and

(ii) $B = \operatorname{dom} g$, and

(iii) g is B-measurable.

The theorem is a consequence of (1) and (20).

(29) Let us consider non empty sets X, Y, a σ-field S of subsets of X, a function T from X into Y, a σ-measure M on S, a partial function f from X to $\overline{\mathbb{R}}$, an element A of S, and a partial function g from Y to $\overline{\mathbb{R}}$. Suppose T is bijective and $g = f \cdot (T^{-1})$ and f is non-negative and $A = \operatorname{dom} f$ and f is A-measurable. Then $\int^+ g \, \mathrm{d}\,\mathrm{Measure}_{\mathrm{Copy}}(T, M) = \int^+ f \, \mathrm{d}M$.

PROOF: Reconsider $B = T^{\circ}A$ as an element of $\mathrm{Field}_{\mathrm{Copy}}(T, S)$. g is B-measurable. g is non-negative. Consider F being a sequence of partial functions from X into $\overline{\mathbb{R}}$, K being a sequence of extended reals such that for every natural number n, $F(n)$ is simple function in S and $\operatorname{dom}(F(n)) = \operatorname{dom} f$ and for every natural number n, $F(n)$ is non-negative and for every natural numbers n, m such that $n \leqslant m$ for every element x of X such that $x \in \operatorname{dom} f$ holds $F(n)(x) \leqslant F(m)(x)$ and for every element x of X such that $x \in \operatorname{dom} f$ holds $F\#x$ is convergent and $\lim(F\#x) = f(x)$ and for every natural number n, $K(n) = \int' F(n) \, \mathrm{d}M$ and K is convergent and $\int^+ f \, \mathrm{d}M = \lim K$. Reconsider $H = T^{-1}$ as a function from Y into X. Consider H being a function from Y into X such that H is bijective and $H = T^{-1}$ and $H^{-1} = T$ and ${}^{\circ}H = ({}^{\circ}T)^{-1}$ and $({}^{\circ}H)^{\circ}(\mathrm{Field}_{\mathrm{Copy}}(T, S)) = S$ and $\mathrm{Field}_{\mathrm{Copy}}(H, \mathrm{Field}_{\mathrm{Copy}}(T, S)) = S$. For every object x, $x \in T^{\circ} \operatorname{dom} f$ iff $x \in \operatorname{dom} g$. For every natural number n, $\operatorname{dom}(F(n) \cdot H) = T^{\circ} \operatorname{dom}(F(n))$.

Define $\mathcal{N}$(natural number) $= F(\$_1) \cdot H$. Consider G being a sequence of partial functions from Y into $\overline{\mathbb{R}}$ such that for every natural number n, $G(n) = \mathcal{N}(n)$ from [11, (Sch. 1)]. Set $L = \mathrm{Measure}_{\mathrm{Copy}}(T, M)$. For every natural number n, $G(n)$ is simple function in $\mathrm{Field}_{\mathrm{Copy}}(T, S)$ and $\mathrm{dom}(G(n)) = \mathrm{dom}\, g$. For every natural number n, $G(n)$ is non-negative. For every natural numbers n, m such that $n \leqslant m$ for every element y of Y such that $y \in \mathrm{dom}\, g$ holds $G(n)(y) \leqslant G(m)(y)$. For every element y of Y such that $y \in \mathrm{dom}\, g$ holds $G\#y$ is convergent and $\lim(G\#y) = g(y)$. For every natural number n, $K(n) = \int' G(n)\, \mathrm{d}L$. □

(30) Let us consider non empty sets X, Y, a σ-field S of subsets of X, a function T from X into Y, a σ-measure M on S, a partial function f from X to $\overline{\mathbb{R}}$, a partial function g from Y to $\overline{\mathbb{R}}$, and an element B of $\mathrm{Field}_{\mathrm{Copy}}(T, S)$. Suppose T is bijective and $g = f \cdot (T^{-1})$ and $B = \mathrm{dom}\, g$ and g is B-measurable. Then there exists an element A of S such that

(i) $B = T^{\circ}A$, and

(ii) $A = \mathrm{dom}\, f$, and

(iii) f is A-measurable.

The theorem is a consequence of (17), (19), (1), (15), and (20).

Let us consider non empty sets X, Y, a σ-field S of subsets of X, a function T from X into Y, a σ-measure M on S, a partial function f from X to $\overline{\mathbb{R}}$, a partial function g from Y to $\overline{\mathbb{R}}$, and an element A of S. Now we state the propositions:

(31) Suppose T is bijective and $g = f \cdot (T^{-1})$ and $A = \mathrm{dom}\, f$ and f is A-measurable. Then

(i) $\int^+ \max_+(f)\, \mathrm{d}M = \int^+ \max_+(g)\, \mathrm{d}\, \mathrm{Measure}_{\mathrm{Copy}}(T, M)$, and

(ii) $\int^+ \max_-(f)\, \mathrm{d}M = \int^+ \max_-(g)\, \mathrm{d}\, \mathrm{Measure}_{\mathrm{Copy}}(T, M)$.

The theorem is a consequence of (27) and (29).

(32) Suppose T is bijective and $g = f \cdot (T^{-1})$ and $A = \mathrm{dom}\, f$ and f is A-measurable. Then $\int g\, \mathrm{d}\, \mathrm{Measure}_{\mathrm{Copy}}(T, M) = \int f\, \mathrm{d}M$. The theorem is a consequence of (31).

(33) Let us consider non empty sets X, Y, a σ-field S of subsets of X, a function T from X into Y, a σ-measure M on S, a partial function f from X to $\overline{\mathbb{R}}$, and a partial function g from Y to $\overline{\mathbb{R}}$. Suppose T is bijective and $g = f \cdot (T^{-1})$. Then f is integrable on M if and only if g is integrable on $\mathrm{Measure}_{\mathrm{Copy}}(T, M)$. The theorem is a consequence of (28), (31), (17), (1), and (20).

4. Integral over Tuple Type Sets (n-dimensional)

Now we state the propositions:

(34) Let us consider a non zero natural number n, a non-empty, n-element finite sequence X, a family of σ-fields S of X, a family of σ-measures m of S, a partial function f from $\prod_{\rm FS} X$ to $\overline{\mathbb{R}}$, a partial function g from $\prod X$ to $\overline{\mathbb{R}}$, an element A of $\prod_{\rm Field} S$, and an element B of XProd-Field(S). Suppose $B = (\mathrm{CarProd}(X))^{\circ}A$ and $g = f\cdot((\mathrm{CarProd}(X))^{-1})$. Then f is A-measurable if and only if g is B-measurable. The theorem is a consequence of (12) and (20).

(35) Let us consider a non zero natural number n, a non-empty, n-element finite sequence X, a family of σ-fields S of X, a family of σ-measures m of S, a partial function f from $\prod_{\rm FS} X$ to $\overline{\mathbb{R}}$, a partial function g from $\prod X$ to $\overline{\mathbb{R}}$, and an element A of $\prod_{\rm Field} S$. Suppose $g = f \cdot ((\mathrm{CarProd}(X))^{-1})$ and $A = \mathrm{dom}\, f$ and f is A-measurable. Then $\int g \,\mathrm{d}\,\mathrm{XProd\text{-}Measure}(m) = \int f \,\mathrm{d}\,\mathrm{Measure}_{\rm Prod}(m)$.

(36) Let us consider a non zero natural number n, a non-empty, n-element finite sequence X, a family of σ-fields S of X, a family of σ-measures m of S, a partial function f from $\prod_{\rm FS} X$ to $\overline{\mathbb{R}}$, and a partial function g from $\prod X$ to $\overline{\mathbb{R}}$. Suppose $g = f \cdot ((\mathrm{CarProd}(X))^{-1})$. Then f is integrable on $\mathrm{Measure}_{\rm Prod}(m)$ if and only if g is integrable on XProd-Measure(m). The theorem is a consequence of (12) and (33).

5. Lebesgue Type Measure and Lebesgue Integral on `REAL n`

Let n be a non zero natural number. Observe that $\mathrm{Seg}\, n \longmapsto \mathbb{R}$ is non-empty and n-element as a finite sequence.

The functor L-Field(n) yielding a family of σ-fields of $\mathrm{Seg}\, n \longmapsto \mathbb{R}$ is defined by the term

(Def. 9) $\mathrm{Seg}\, n \longmapsto$ L-Field.

The functor L-Meas(n) yielding a family of σ-measures of L-Field(n) is defined by the term

(Def. 10) $\mathrm{Seg}\, n \longmapsto$ L-Meas.

The functor XL-Field(n) yielding a σ-field of subsets of $\mathcal{R}^n$ is defined by the term

(Def. 11) XProd-Field(L-Field(n)).

The functor XL-Meas(n) yielding a σ-measure on XL-Field(n) is defined by the term

(Def. 12) XProd-Measure(L-Meas(n)).

Now we state the propositions:

(37) (i) $\prod_{\rm FS} {\rm Seg}\,1 \longmapsto \mathbb{R} = \mathbb{R}$, and

(ii) ElmFin(Seg 1 $\longmapsto \mathbb{R}, 1) = \mathbb{R}$, and

(iii) $\prod_{\rm FS} {\rm Seg}\,2 \longmapsto \mathbb{R} = \mathbb{R} \times \mathbb{R}$, and

(iv) ElmFin(Seg 2 $\longmapsto \mathbb{R}, 2) = \mathbb{R}$, and

(v) $\prod_{\rm FS} {\rm Seg}\,3 \longmapsto \mathbb{R} = \mathbb{R} \times \mathbb{R} \times \mathbb{R}$.

(38) (i) CarProd(Seg 1 $\longmapsto \mathbb{R}$) is a function from $\mathbb{R}$ into $\mathcal{R}^1$, and

(ii) for every object s such that $s \in \mathbb{R}$ holds (CarProd(Seg 1 $\longmapsto \mathbb{R}$))(s) = $\langle s \rangle$.

The theorem is a consequence of (37).

(39) (i) CarProd(Seg 2 $\longmapsto \mathbb{R}$) is a function from $\mathbb{R} \times \mathbb{R}$ into $\mathcal{R}^2$, and

(ii) for every objects s, t such that s, $t \in \mathbb{R}$ holds (CarProd(Seg 2 $\longmapsto$ $\mathbb{R}$))($\langle s, t \rangle$) = $\langle s, t \rangle$.

Proof: Set F = CarProd(Seg 2 $\longmapsto \mathbb{R}$). For every objects s, t such that s, $t \in \mathbb{R}$ holds $F(\langle s, t \rangle) = \langle s, t \rangle$. □

(40) (i) CarProd(Seg 3 $\longmapsto \mathbb{R}$) is a function from $\mathbb{R} \times \mathbb{R} \times \mathbb{R}$ into $\mathcal{R}^3$, and

(ii) for every objects s, t, u such that $s, t, u \in \mathbb{R}$ holds (CarProd(Seg 3 $\longmapsto$ $\mathbb{R}$))($\langle \langle s, t \rangle, u \rangle$) = $\langle s, t, u \rangle$.

Proof: Set H = CarProd(Seg 3 $\longmapsto \mathbb{R}$). For every objects s, t, u such that s, t, $u \in \mathbb{R}$ holds $H(\langle \langle s, t \rangle, u \rangle) = \langle s, t, u \rangle$. □

(41) (i) $\prod_{\rm Field}$ L-Field(1) = L-Field, and

(ii) the Borel sets $\subseteq \prod_{\rm Field}$ L-Field(1), and

(iii) for every subset I of $\mathbb{R}$ such that I is an interval holds $I \in \prod_{\rm Field}$ L-Field(1).

(42) (i) $\prod_{\rm Field}$ L-Field(2) = σ(MeasRect(L-Field, L-Field)), and

(ii) MeasRect(L-Field, L-Field) $\subseteq \sigma$(MeasRect(L-Field, L-Field)), and

(iii) the set of all $A \times B$ where A is an element of the Borel sets, B is an element of the Borel sets $\subseteq$ MeasRect(L-Field, L-Field), and

(iv) $\{I \times J$, where I, J are subsets of $\mathbb{R}$: I is an interval and J is an interval$\}$ $\subseteq$ the set of all $A \times B$ where A is an element of the Borel sets, B is an element of the Borel sets.

(43) (i) $\prod_{\rm Field}$ L-Field(3) = σ(MeasRect(σ(MeasRect(L-Field, L-Field)), L-Field)), and

(ii) $\mathrm{MeasRect}(\sigma(\mathrm{MeasRect}(\text{L-Field}, \text{L-Field})), \text{L-Field})$
$\subseteq \sigma(\mathrm{MeasRect}(\sigma(\mathrm{MeasRect}(\text{L-Field}, \text{L-Field})), \text{L-Field}))$, and

(iii) the set of all $A \times B \times C$ where A is an element of the Borel sets, B is an element of the Borel sets, C is an element of the Borel sets $\subseteq \mathrm{MeasRect}(\sigma(\mathrm{MeasRect}(\text{L-Field}, \text{L-Field})), \text{L-Field})$, and

(iv) $\{I \times J \times K$, where I, J, K are subsets of $\mathbb{R}$: I is an interval and J is an interval and K is an interval$\} \subseteq$ the set of all $A \times B \times C$ where A is an element of the Borel sets, B is an element of the Borel sets, C is an element of the Borel sets.

(44) Let us consider a non zero natural number n. Then $\prod_{\mathrm{Field}} \text{L-Field}(n+1) = \sigma(\mathrm{MeasRect}(\prod_{\mathrm{Field}} \text{L-Field}(n), \text{L-Field}))$.

(45) (i) $\mathrm{Measure}_{\mathrm{Prod}}(\text{L-Meas}(1)) = \text{L-Meas}$, and

(ii) for every element E of L-Field, $E \in \prod_{\mathrm{Field}} \text{L-Field}(1)$.

The theorem is a consequence of (41).

(46) (i) $\mathrm{Measure}_{\mathrm{Prod}}(\text{L-Meas}(2)) = \mathrm{ProdMeas}(\text{L-Meas}, \text{L-Meas})$, and

(ii) for every elements E_1, E_2 of L-Field, $E_1 \times E_2 \in \mathrm{MeasRect}(\text{L-Field}, \text{L-Field})$ and $(\mathrm{Measure}_{\mathrm{Prod}}(\text{L-Meas}(2)))(E_1 \times E_2) = (\text{L-Meas})(E_1) \cdot (\text{L-Meas})(E_2)$.

Proof: For every elements E_1, E_2 of L-Field, $E_1 \times E_2 \in \mathrm{MeasRect}(\text{L-Field}, \text{L-Field})$ and $(\mathrm{Measure}_{\mathrm{Prod}}(\text{L-Meas}(2)))(E_1 \times E_2) = (\text{L-Meas})(E_1) \cdot (\text{L-Meas})(E_2)$ by [6, (16)], (37), (41), (45). □

(47) (i) $\mathrm{Measure}_{\mathrm{Prod}}(\text{L-Meas}(3)) = \mathrm{ProdMeas}(\mathrm{ProdMeas}(\text{L-Meas}, \text{L-Meas}), \text{L-Meas})$, and

(ii) for every elements E_1, E_2, E_3 of L-Field, $E_1 \times E_2 \times E_3 \in \mathrm{MeasRect}(\sigma(\mathrm{MeasRect}(\text{L-Field}, \text{L-Field})), \text{L-Field})$ and $(\mathrm{Measure}_{\mathrm{Prod}}(\text{L-Meas}(3)))(E_1 \times E_2 \times E_3) = (\text{L-Meas})(E_1) \cdot (\text{L-Meas})(E_2) \cdot (\text{L-Meas})(E_3)$.

Proof: For every elements E_1, E_2, E_3 of L-Field, $E_1 \times E_2 \times E_3 \in \mathrm{MeasRect}(\sigma(\mathrm{MeasRect}(\text{L-Field}, \text{L-Field})), \text{L-Field})$ and $(\mathrm{Measure}_{\mathrm{Prod}}(\text{L-Meas}(3)))(E_1 \times E_2 \times E_3) = (\text{L-Meas})(E_1) \cdot (\text{L-Meas})(E_2) \cdot (\text{L-Meas})(E_3)$. □

(48) Let us consider a non zero natural number n. Then $\mathrm{Measure}_{\mathrm{Prod}}(\text{L-Meas}(n+1)) = \mathrm{ProdMeas}(\mathrm{Measure}_{\mathrm{Prod}}(\text{L-Meas}(n)), \text{L-Meas})$.

(49) Let us consider a non zero natural number n, a partial function f from $\prod_{\mathrm{FS}} \mathrm{Seg}\, n \longmapsto \mathbb{R}$ to $\overline{\mathbb{R}}$, a partial function g from $\mathcal{R}^n$ to $\overline{\mathbb{R}}$, an element A of $\prod_{\mathrm{Field}} \text{L-Field}(n)$, and an element B of XL-Field(n). Suppose $g = f \cdot ((\mathrm{CarProd}(\mathrm{Seg}\, n \longmapsto \mathbb{R}))^{-1})$ and $B = (\mathrm{CarProd}(\mathrm{Seg}\, n \longmapsto \mathbb{R}))^{\circ} A$.

Then f is A-measurable if and only if g is B-measurable. The theorem is a consequence of (34).

(50) Let us consider a partial function f_1 from $\mathbb{R}\times\mathbb{R}$ to $\overline{\mathbb{R}}$, a partial function f_2 from $\prod_{\mathrm{FS}} \mathrm{Seg}\, 2 \longmapsto \mathbb{R}$ to $\overline{\mathbb{R}}$, an element A_1 of $\sigma(\mathrm{MeasRect}(\text{L-Field}, \text{L-Field}))$, and an element A_2 of $\prod_{\mathrm{Field}} \text{L-Field}(2)$. Suppose $f_1 = f_2$ and $A_1 = A_2$. Then f_1 is A_1-measurable if and only if f_2 is A_2-measurable. The theorem is a consequence of (44), (37), and (41).

(51) Let us consider a partial function f from $\mathbb{R}\times\mathbb{R}$ to $\overline{\mathbb{R}}$, a partial function g from $\mathcal{R}^2$ to $\overline{\mathbb{R}}$, an element A of $\sigma(\mathrm{MeasRect}(\text{L-Field}, \text{L-Field}))$, and an element B of XL-Field(2). Suppose $g = f \cdot ((\mathrm{CarProd}(\mathrm{Seg}\, 2 \longmapsto \mathbb{R}))^{-1})$ and $B = (\mathrm{CarProd}(\mathrm{Seg}\, 2 \longmapsto \mathbb{R}))^{\circ} A$. Then f is A-measurable if and only if g is B-measurable. The theorem is a consequence of (44), (37), (41), (49), and (50).

(52) Let us consider a partial function f_1 from $(\mathbb{R}\times\mathbb{R})\times\mathbb{R}$ to $\overline{\mathbb{R}}$, a partial function f_2 from $\prod_{\mathrm{FS}} \mathrm{Seg}\, 3 \longmapsto \mathbb{R}$ to $\overline{\mathbb{R}}$, an element A_1 of $\sigma(\mathrm{MeasRect}(\sigma(\mathrm{MeasRect}(\text{L-Field}, \text{L-Field})), \text{L-Field}))$, and an element A_2 of $\prod_{\mathrm{Field}} \text{L-Field}(3)$. Suppose $f_1 = f_2$ and $A_1 = A_2$. Then f_1 is A_1-measurable if and only if f_2 is A_2-measurable. The theorem is a consequence of (44), (37), and (41).

(53) Let us consider a partial function f from $(\mathbb{R}\times\mathbb{R})\times\mathbb{R}$ to $\overline{\mathbb{R}}$, a partial function g from $\mathcal{R}^3$ to $\overline{\mathbb{R}}$, an element A of $\sigma(\mathrm{MeasRect}(\sigma(\mathrm{MeasRect}(\text{L-Field}, \text{L-Field})), \text{L-Field}))$, and an element B of XL-Field(3). Suppose $g = f \cdot ((\mathrm{CarProd}(\mathrm{Seg}\, 3 \longmapsto \mathbb{R}))^{-1})$ and $B = (\mathrm{CarProd}(\mathrm{Seg}\, 3 \longmapsto \mathbb{R}))^{\circ} A$. Then f is A-measurable if and only if g is B-measurable. The theorem is a consequence of (44), (37), (41), (49), and (52).

(54) Let us consider a non zero natural number n, a partial function f from $\prod_{\mathrm{FS}} \mathrm{Seg}\, n \longmapsto \mathbb{R}$ to $\overline{\mathbb{R}}$, a partial function g from $\mathcal{R}^n$ to $\overline{\mathbb{R}}$, and an element A of $\prod_{\mathrm{Field}} \text{L-Field}(n)$. Suppose $g = f \cdot ((\mathrm{CarProd}(\mathrm{Seg}\, n \longmapsto \mathbb{R}))^{-1})$ and $A = \mathrm{dom}\, f$ and f is A-measurable. Then $\int g \,\mathrm{d}\, \text{XL-Meas}(n) = \int f \,\mathrm{d}\, \mathrm{Measure}_{\mathrm{Prod}}(\text{L-Meas}(n))$. The theorem is a consequence of (12) and (32).

(55) Let us consider a non zero natural number n, a partial function f from $\prod_{\mathrm{FS}} \mathrm{Seg}\, n \longmapsto \mathbb{R}$ to $\overline{\mathbb{R}}$, and a partial function g from $\mathcal{R}^n$ to $\overline{\mathbb{R}}$. Suppose $g = f \cdot ((\mathrm{CarProd}(\mathrm{Seg}\, n \longmapsto \mathbb{R}))^{-1})$. Then f is integrable on $\mathrm{Measure}_{\mathrm{Prod}}(\text{L-Meas}(n))$ if and only if g is integrable on XL-Meas(n). The theorem is a consequence of (36).

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