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Variance and information potential of some random variables [1]

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Abstract

We investigate random variables for which the variance and the information potential satisfy a preservation law.

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1 Introduction

The theory of information potential and its applications is extensively presented in [3]. Recent results and applications can be found in [1], [2], [4]. These papers are concerned, in particular, with a preservation law involving information potential and variance. More precisely, let Y_x be a random variable with probability density function $p(t,x)$ depending on a parameter x. Let $V(x)$ be the corresponding variance of Y_x and $S(x)$ the associated information potential

$$S(x) := \int_{\mathbb{R}} p^2(t,x)dt.$$

For certain random variables Y_x the following result holds:

(1) $$V(x)S^2(x) = \text{ constant with respect to } x.$$

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This shows, in particular, that $V(x)$ and $S(x)$ are asynchronous functions. Some examples are presented in [1] and [2]. See also [4, Remark 10].

In Section 2 we present a general method for constructing random variables which satisfy (1). Section 3 is devoted to an example where (1) is not satisfied, but $V(x)$ and $S(x)$ are asynchronous.

2 Random variables obeying the preservation law

Let X be a continuous random variable having the probability density function $\varphi(s)$, $s \in \mathbb{R}$. For $x > 0$ let $Y_x := \frac{1}{x} X$.

Theorem 1 *The associated variances and information potentials satisfy*

$$V[Y_x]S^2[Y_x] = V[X]S^2[X],\ x > 0. \tag{2}$$

In particular, $V[Y_x]S^2[Y_x]$ does not depend on x.

Proof. Let $p(t,x) := x\varphi(xt)$, $t \in \mathbb{R}$, $x > 0$. Then, for $y \in \mathbb{R}$ we have

$$\begin{aligned}\int_{-\infty}^{y} p(t,x)dt &= \int_{-\infty}^{y} x\varphi(xt)dt = \int_{-\infty}^{xy} x\varphi(s)\frac{ds}{x} = \int_{-\infty}^{xy} \varphi(s)ds \\ &= P(X < xy) = P(Y_x < y).\end{aligned}$$

It follows that the probability density function of Y_x is $p(t,x)$.

Now

$$\begin{aligned}S[Y_x] &= \int_{\mathbb{R}} p^2(t,x)dt = \int_{\mathbb{R}} x^2\varphi^2(xt)dt \\ &= \int_{\mathbb{R}} x^2\varphi^2(s)\frac{ds}{x} = x\int_{\mathbb{R}} \varphi^2(s)ds = xS[X].\end{aligned}$$

Moreover, $V[Y_x] = \frac{1}{x^2}V[X]$, and so $V[Y_x]S^2[Y_x] = \frac{1}{x^2}V[X]x^2S^2[X] = V[X]S^2[X]$ and the proof of (2) is complete.

Example 1 *Let $\alpha > 0$, $\beta > -1$, $\lambda > 0$,*

$$\varphi(s) = \begin{cases} \alpha s^\beta e^{-\lambda s^\alpha}\left(\Gamma\left(\frac{\beta+1}{\alpha}\right)\right)^{-1}\lambda^{\frac{\beta+1}{\alpha}},\ s > 0, \\ 0,\ s \le 0. \end{cases} \tag{3}$$

If φ is the probability density function of X, and $x > 0$, then

$$p(t,x) = \begin{cases} \alpha x^{\beta+1}\left(\Gamma\left(\frac{\beta+1}{\alpha}\right)\right)^{-1} t^\beta e^{-\lambda(xt)^\alpha}\lambda^{\frac{\beta+1}{\alpha}},\ t > 0, \\ 0,\ t \le 0 \end{cases}$$

is the probability density function of $Y_x = \frac{1}{x}X$. *So, according to Theorem 1 we have* $V[Y_x]S^2[Y_x] = V[X]S^2[X]$. *This function* $p(t,x)$ *can be obtained from [1, (2.2)] if we take there* $a(x) := \lambda x^\alpha$. *So, by a direct calculation or by using [1, (2.3)] we get*

$$V[Y_x]S^2[Y_x] = \left(\alpha 2^{-\frac{2\beta+1}{\alpha}}\right)^2 \Gamma^2\left(\frac{2\beta+1}{\alpha}\right)\Gamma^{-4}\left(\frac{\beta+1}{\alpha}\right) \cdot \left[\Gamma\left(\frac{\beta+1}{\alpha}\right)\Gamma\left(\frac{\beta+3}{\alpha}\right) - \Gamma^2\left(\frac{\beta+2}{\alpha}\right)\right],$$

Remark 1 *If we choose* $\beta = \alpha - 1$, *(3) reduces to the Weibull probability density function.*

Example 2 *(see also [1, Example 2.2]) If* $n \in \mathbb{N}$ *and*

$$\varphi(s) = \begin{cases} \frac{s^n}{n!}e^{-s}, \ s > 0, \\ 0, s \le 0, \end{cases}$$

then

$$p(t,x) = \begin{cases} \frac{x^{n+1}}{n!}t^n e^{-xt}, \ t > 0, \\ 0, t \le 0, \end{cases}$$

and consequently

$$V[Y_x]S^2[Y_x] = \frac{n+1}{4^{2n+1}}\binom{2n}{n}^2.$$

Example 3 *For* $n \in \mathbb{N}$, $n > 2$, *let us consider the random variable* X *having the Student density of probability*

$$\varphi(s) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{n\pi}\,\Gamma\left(\frac{n}{2}\right)}\left(1 + \frac{s^2}{n}\right)^{-\frac{n+1}{2}}, \ s \in \mathbb{R}.$$

Then $V(X) = \frac{n}{n-2}$ *and* $S(X) = \dfrac{\Gamma\left(\frac{n+1}{2}\right)^2 \Gamma\left(n+\frac{1}{2}\right)}{\sqrt{n\pi}\,\Gamma\left(\frac{n}{2}\right)^2 \Gamma(n+1)}$.

The probability density function of $Y_x = \frac{1}{x}X$ *is*

$$p(t,x) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{n\pi}\,\Gamma\left(\frac{n}{2}\right)}x\left(1 + \frac{x^2t^2}{n}\right)^{-\frac{n+1}{2}}, \ t \in \mathbb{R}, \ x > 0.$$

According to Theorem 1 we have

$$V[Y_x]S^2[Y_x] = \frac{\Gamma\left(\frac{n+1}{2}\right)^4 \Gamma\left(n+\frac{1}{2}\right)^2}{\pi(n-2)\Gamma\left(\frac{n}{2}\right)^4 \Gamma(n+1)^2}.$$

3 Asynchronous variance and information potential

In this section we consider the vector $x = (a, \mu, \nu, \sigma)$ where $a \in [0, 1]$, $\mu \in \mathbb{R}$, $\nu \in \mathbb{R}$, $\sigma \in (0, \infty)$. Let Z_x be the random variable with probability density function

$$p(t, x) := \frac{1}{\sigma\sqrt{2\pi}} \left(a \exp\left(-\frac{(t-\mu)^2}{2\sigma^2}\right)\right) + (1-a) \exp\left(-\frac{(t-\nu)^2}{2\sigma^2}\right).$$

Theorem 2 *The variance $V[Z_x]$ is increasing with respect to $(\mu-\nu)^2$ and increasing with respect to σ. The information potential $S[Z_x]$ is decreasing in $(\mu-\nu)^2$ and decreasing in σ.*

Proof. By direct calculation we find that

$$\int_{\mathbb{R}} tp(t, x)dt = a\mu + (1-a)\nu,$$

$$\int_{\mathbb{R}} t^2 p(t, x)dt = \sigma^2 + a\mu^2 + (1-a)\nu^2,$$

$$V[Z_x] = Var[Z_x] = \sigma^2 + a(1-a)(\mu-\nu)^2. \tag{4}$$

Moreover,

$$\begin{aligned} S[Z_x] &= \int_{\mathbb{R}} p^2(t, x)dt \\ &= \frac{1}{2\pi\sigma^2} \int_{\mathbb{R}} \left[a^2 \exp\left(-\frac{(t-\mu)^2}{\sigma^2}\right) + (1-a)^2 \exp\left(-\frac{(t-\nu)^2}{\sigma^2}\right) \right. \\ &\left. + 2a(1-a) \exp\left(-\left(t - \frac{\mu+\nu}{2}\right)^2 - \frac{(\mu-\nu)^2}{4}\right)\right] dt. \end{aligned}$$

Therefore,

$$S[Z_x] = \frac{1}{2\sigma\sqrt{\pi}} \left[a^2 + (1-a)^2 + 2a(1-a) \exp\left(-\frac{(\mu-\nu)^2}{4}\right)\right]. \tag{5}$$

Using (4) and (5) we conclude the proof.

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