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Original Study

Analytic solution of fractional order Pseudo-Hyperbolic Telegraph equation using modified double Laplace transform method

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Abstract

The Pseudo-Hyperbolic Telegraph partial differential equation (PHTPDE) based on the Caputo fractional derivative is investigated in this paper. The modified double Laplace transform method (MDLTM) is constructed for the proposed model. The MDLTM was used to obtain the analytic solution for the pseudo-hyperbolic telegraph equation of fractional order defined by the Caputo derivative. The proposed method is a highly effective analytical method for the fractional-order pseudo-hyperbolic telegraph equation. A test problem was presented as an example. Based on the results, it is clear that this method is more convenient and produces an analytic solution in fewer steps than other methods that require more steps to have an identical analytical solution. This paper claims to provide an analytic solution to the fractional order pseudohyperbolic telegraph equation order using the MDLTM. An analytical solution leads to an exact, closed-form solution that can be expressed in mathematical functions or known operations. Obtaining analytic solutions to PDEs is often challenging, especially for fractional order equations, making this achievement noteworthy.

Keywords: Pseudo-hyperbolic telegraph equation, caputo fractional derivative, modified double Laplace transform method, analytical solution.

AMS 2020 codes: 35R11; 35C10; 35J05; 35A22.

1 Introduction

The study of integrals and differentials of non-integer order, as well as their applications, is known as fractional calculus. Fractional models outperform classical models in terms of accuracy and efficiency. Furthermore, noninteger derivative operators have been discovered to be crucial in understanding physical occurrences [1–5]. The advantages of fractional calculus have been investigated in various fields, including engineering, mechanics, control theory, medicine, biology, and seismology [6–9]. The PHTPDE based on the Caputo derivative is a high-order partial differential equation with mixed partial derivative. The proposed model is well known as a mathematical physics equation, which has an exhaustive range of applications in areas like longitudinal vibrations, nerve conduction, plasma physics, and many other physical phenomena. The behaviour of the pseudo-hyperbolic equation has been covered in several articles, including those that discuss the existence and uniqueness of solution [10], analytical and approximate solutions [11], numerical computations and stability estimates [12], and the

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presence and nonexistence of solution for Cauchy type issues with pseudo-hyperbolic equation [13]. In recent years, many efficient techniques have been used to obtain approximate and analytical solutions for a fractional-order partial differential equation, including the double Laplace transform method [14], theta-method [15], reproducing kernel function and CrankNicholson difference method [16], finite element method [17], Laplace decomposition method [18], fractional sub equation method [19], MDLTM [20], Fibonacci wavelet method [21]. The authors in [22] explored the investigation of the finite time stability (FTS) of multi-state neutral fractional order systems when subjected to impulsive perturbations and state delays. The non-local fractional differential equation of the Sobolev type with impulsive conditions was addressed in [23], and the existence and uniqueness of mild solutions using the Banach fixed point technique and analytic semigroup were examined in every approximate solution. Then, for approximative solutions, the Faedo-Galerkin approximation was employed to establish specific convergence outcomes. The existence and uniqueness of solutions to the neutral functional sequential integro-differential equations with Caputo fractional derivative on the time scales T-Cauchy problem were examined by the researchers in [24]. The researchers in [25,26] utilized the q-homotopy analysis transform technique to derive some solutions for several important models arising in real world problems.

In contrast, academics have extensively employed the double Laplace transform method, a powerful method for solving the fractional-order partial differential equation, to address a wide range of issues, including the authors in [20] arriving at the analytical solution to the partial differential equation of fractional order using the double conformable Laplace transform method. The time-dependent (FSPPPD) Equation with Caputo Derivative was solved analytically and approximately using the (DLDM) and the (FDM), which were both suggested by the authors in [27]. And the time-fractional wave equation was analytically solved by the authors in [28] using the double Laplace transform technique. The method considered in this study, a MDLTM, differs from other approaches frequently used to solve fractional order differential equations, such as the Fourier transform, the Laplace transform, and other numerical methods, in a number of ways. The modified double Laplace transform method is a revolutionary strategy that has several advantages. It is more practical and takes fewer stages to provide an analytical solution than existing approaches that need more steps to achieve the same result. In this paper, we consider the following fractional order Pseudo-Hyperbolic Telegraph equation (PHTE) [29] based on the Caputo fractional derivative

$$\begin{aligned} u_{tt}(x,t)+D_t^{\alpha}u(x,t)+u(x,t)-\lambda u_{txx}(x,t)-u_{xx}(x,t)&=f(x,t), \\ 0<x<L, t>0,\ \lambda>0,\ 0<\alpha\leq 1&, \\ u(x,0)=h(x),\ u_t(x,0)=g(x),\ 0\leq x\leq L&, \\ u(0,t)=u(L,t)=0,\ t\geq 0&, \end{aligned} \tag{1}$$

where $D_t^{\alpha}u(x,t)$ is Caputo fractional derivative [30], that is defined as follow

$$\frac{D^{\alpha}u(x,t)}{\partial t^{\alpha}}=D_t^{\alpha}u(x,t)=\frac{1}{\Gamma(n-\alpha)}\int_0^t\frac{1}{(t-p)^{n-\alpha-1}}\frac{\partial^n u(p,x)}{\partial p^{\alpha}}dp,(n-1<\alpha\leq n), \tag{2}$$

if $\alpha=n\in N$, then we can write as:

$$D_t^{\alpha}u(x,t)=\frac{D^{\alpha}u(x,t)}{\partial t^{\alpha}}=\frac{D^n u(x,t)}{\partial t^n}.$$

The numerical solutions for the suggested model (1) were derived by [31] using the Dufort Frankel difference scheme approach. Since the Caputo derivative [30] is the most frequently utilized derivative characteristic in the literature on fractional analysis, we adopted it in our model. The authors in [29] employed the explicit finite difference technique to solve the model (1). Additionally, they developed a first-order difference technique to investigate the stability analysis and obtained the approximate solutions of the given issue. To obtain the analytical solution to PHTPDE based on the Caputo fractional derivative of α order and $0<\alpha\leq 1$, this research article implements an efficient analytical technique called MDLTM to solve the proposed model analytically, and this method is incredibly effective because it generates an accurate solution in relatively few steps than other approaches that need more steps.

This research paper is organized as follows: section 2 gives background information. Section 3 provides the general solution of fractional order PHTPDE defined by the Caputo derivative by the MDLTM. Section 4 presents the applications of MDLTM. Via section 5, the conclusion is given.

2 Basic concept

In this section, we define a few terms and give some fundamental details about the double Laplace Decomposition Method.

Definition 1. In the positive quadrant of the xt-plane, the function $u(x,t)$ as a function of two variables is defined. In [32], the double Laplace transform for the function $u(x,t)$ is specified as follows:

$$L_xL_t\{u(x,t)\} = \bar{u}(s,p) = \int_0^{\infty}\int_0^{\infty} e^{-px-st}u(x,t)dxdt \tag{3}$$

whenever the integral is present the numbers p and s are both complex numbers.

From this definition, we can thus deduce:

$$L_xL_t\{u(x)g(t)\} = \bar{u}(p)\bar{g}(s) = L_x\{u(x)\}L_t\{g(t)\} \tag{4}$$

Definition 2. The complex double integral formula is used to define the inverse double Laplace transform $L_x^{-1}L_t^{-1}\{\bar{u}(s,p)\} = u(x,t)$, as in [33]:

$$L_x^{-1}L_t^{-1}\{\bar{u}(s,p)\} = u(x,t) = \frac{1}{2i\pi}\left[\int_{c-i\infty}^{c+i\infty} e^{px}dp\int_{d-i\infty}^{d+i\infty} e^{st}\bar{u}(s,p)ds\right] \tag{5}$$

where c and d are appropriate real constants, and $\bar{u}(s,p)$ must be an analytical function for each p and s in the area denoted by the inequalities $\mathrm{Re}(p) \geq c$ and $\mathrm{Re}(s) \geq d$.

The double Laplace transform formula in [27] is used to transform the partial derivatives of any integer order.
$L_xL_t\left\{\frac{\partial^n u(x,t)}{\partial x^n}\right\} = p^n\bar{u}(s,p) - \sum_{i=0}^{n-1} p^{n-1-i}L_t\left\{\frac{\partial^i u(0,t)}{\partial x^i}\right\}$
$L_xL_t\left\{\frac{\partial^k u(x,t)}{\partial t^k}\right\} = s^k\bar{u}(s,p) - \sum_{j=0}^{k-1} s^{n-1-i}L_x\left\{\frac{\partial^j u(x,0)}{\partial x^j}\right\}$
$L_xL_t\left\{\frac{\partial^{k+n} u(x,t)}{\partial x^n\partial t^k}\right\} =$
$p^ns^k\left[\bar{u}(s,p) - \sum_{i=0}^{n-1} p^{n-1-i}L_t\left\{\frac{\partial^i u(0,t)}{\partial x^i}\right\} - \sum_{j=0}^{k-1} s^{n-1-j}L_x\left\{\frac{\partial^j u(x,0)}{\partial x^j}\right\} + \sum_{i=0}^{n-1}\sum_{j=0}^{k-1} p^{-1-i}s^{-1-j}L_x\left\{\frac{\partial^{i+j} u(0,0)}{\partial x^{i+j}}\right\}\right].$

Definition 3. According to [27], the double Laplace transform formula for Caputo fractional derivative (2) is defined as

$$L_xL_t\left\{\frac{D^{\alpha}u(x,t)}{\partial t^{\alpha}}\right\} = s^{\alpha}\bar{u}(s,p) - \sum_{k=0}^{n-1} s^{\alpha-1-k}L_x\left\{\frac{D^k u(0,x)}{\partial t^k}\right\}. \tag{6}$$

Definition 4. In [30], the gamma function is defined as

$$\Gamma(z) = \int_0^{\infty} e^{-t}\mathrm{t}^{z-1}dt,\ \mathrm{Re}(z) > 0. \tag{7}$$

3 The projected scheme

In order to solve fractional-order PHTE in the Caputo sense, this section develops the key concept of the double Laplace decomposition approach for the proposed problem.

For this purpose, we examine the fractional-order PHTE (1) in the following form

$$u_{tt}(x,t) = \lambda u_{txx}(x,t) - D_t^{\alpha} u(x,t) + u_{xx}(x,t) - u(x,t) + f(x,t), \tag{8}$$

which is $0 < x < L, t > 0$ and also subjected to the initial and boundary conditions

$$u(0,x) = h(x),\ u_t(x,0) = g(x), \tag{9}$$

where $0 \leq x \leq L$ and also

$$u(0,t) = u(L,t) = 0, \tag{10}$$

where $t \geq 0$. Here $\lambda > 0$ and $D_t^{\alpha} u(x,t)$ is Caputo fractional derivative of order $0 < \alpha \leq 1$, while $f(x,t)$ is the given function. To get the general solution of equation (8), firstly we utilize double Laplace transform to equation (8), we obtain

$$s^2 \bar{u}(s,p) - s\bar{u}(0,p) - \bar{u}_t(0,p) = L_x L_t \{\lambda u_{txx}(x,t) - D_t^{\alpha} u(x,t) + u_{xx}(x,t) - u(x,t)\} + \bar{f}(s,p), \tag{11}$$

where $\bar{f}(s,p) = L_x L_t\{f(x,t)\}$. After that, by applying a single Laplace transform to equations equation (9) and equation (10), we get

$$\begin{aligned} &\bar{u}(0,p) = \bar{h}(p), \\ &\bar{u}_t(0,p) = \bar{g}(p), \\ &\bar{u}(s,0) = \bar{u}(s,L) = 0,\ 0 \leq x \leq L,\ t \geq 0. \end{aligned} \tag{12}$$

by substituting formula (12) into formula (11) and simplifying, we obtain the following formula

$$\bar{u}(s,p) = \frac{1}{s^2}\left[L_x L_t \{\lambda u_{txx}(x,t) - D_t^{\alpha} u(x,t) + u_{xx}(x,t) - u(x,t)\} + \bar{f}(s,p) + s\bar{h}(p) + \bar{g}(p)\right]. \tag{13}$$

When we apply the inverse double Laplace transform for equation (13), yields

$$\begin{aligned} u(x,t) = &L_x^{-1} L_t^{-1} \frac{1}{s^2} \{L_x L_t \{\lambda u_{txx}(x,t) - D_t^{\alpha} u(x,t) + u_{xx}(x,t) - u(x,t)\}\} \\ &+ L_x^{-1} L_t^{-1} \left\{\frac{1}{s^2}\bar{f}(s,p) + \frac{1}{s}\bar{h}(p) + \frac{1}{s^2}\bar{g}(p)\right\}. \end{aligned} \tag{14}$$

The general solutions to equation (13), can be expressed as an infinite series, as shown below

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t). \tag{15}$$

Consequently, equation (13) becomes

$$\begin{aligned} \sum_{n=0}^{\infty} u_n(x,t) = &L_x^{-1} L_t^{-1} \left\{\frac{1}{s^2} L_x L_t \left\{\lambda \frac{\partial^3}{\partial x^2 \partial t} \sum_{n=0}^{\infty} u_n(x,t) + \frac{\partial^2}{\partial x^2} \sum_{n=0}^{\infty} u_n(x,t) - \frac{D^{\alpha} u(x,t)}{\partial t^{\alpha}} \sum_{n=0}^{\infty} u_n(x,t) - \sum_{n=0}^{\infty} u_n(x,t)\right\}\right\} \\ &+ L_x^{-1} L_t^{-1} \left\{\frac{1}{s^2}\bar{f}(s,p) + \frac{1}{s}\bar{h}(p) + \frac{1}{s^2}\bar{g}(p)\right\}. \end{aligned} \tag{16}$$

The general solution of equation (8) is described as the following formulas:

$$u_0(x,t) = L_x^{-1}L_t^{-1}\left\{\frac{1}{s^2}\bar{f}(s,p) + \frac{1}{s}\bar{h}(p) + \frac{1}{s^2}\bar{g}(p)\right\}, \tag{17}$$

and

$$u_{n+1}(x,t) = L_x^{-1}L_t^{-1}\left\{\frac{1}{s^2}L_xL_t\left\{\lambda\frac{\partial^3}{\partial x^2\partial t}\sum_{n=0}^{\infty}u_n(x,t) + \frac{\partial^2}{\partial x^2}\sum_{n=0}^{\infty}u_n(x,t) - \frac{D^\alpha u(x,t)}{\partial t^\alpha}\sum_{n=0}^{\infty}u_n(x,t) - \sum_{n=0}^{\infty}u_n(x,t)\right\}\right\}, n \geq 0. \tag{18}$$

We suppose that each part on the right side of (18) has an inverse double Laplace transform.

4 Applications of MDLTM

In this section, we provide an illustration and applicability of how to solve the fractional-order PHTE using the provided approach. We consider the following fractional-order PHTE [29], as

$$u_{tt}(x,t) = \lambda u_{txx}(x,t) - D_t^\alpha u(x,t) + u_{xx}(x,t) - u(x,t) + \left(2 + \frac{2t^{2-\alpha}}{\Gamma(3-\alpha)} + 2\lambda t + 2\left(t^2+1\right)\right)\sin(x), \tag{19}$$

where $0 \leq x \leq \pi,\ 0 \leq t \leq 1, \lambda > 0,\ 0 < \alpha \leq 1$ subject to the initial and boundary conditions

$$u(x,0) = \sin(x),\ u_t(x,0) = 0,\ 0 \leq x \leq \pi \tag{20}$$

and also

$$u(0,t) = u(\pi,t) = 0,\ t \geq 0. \tag{21}$$

Taking the MDLTM to both sides of (19) and by using the initial conditions (20) and simplifying, we get

$$\begin{aligned} \bar{u}(s,p) &= L_xL_t\{u(x,t)\} \\ &= \left[\frac{1}{s} + \frac{2}{s^3} + \frac{2}{s^{5-\alpha}} + \frac{2\lambda}{s^4} + \frac{4}{s^5} + \frac{2}{s^3}\right]\frac{1}{p^2+1} \\ &+ \frac{1}{s^2}L_xL_t\left\{\lambda u_{txx}(x,t) - D_t^\alpha u(x,t) + u_{xx}(x,t) - u(x,t)\right\}. \end{aligned} \tag{22}$$

Equation (22), when transformed using the inverse double Laplace method, yields

$$\begin{aligned} u(x,t) = &\left(1 + 2t^2 + \frac{2}{\Gamma(5-\alpha)}t^{4-\alpha} + \frac{\lambda}{3}t^3 + \frac{1}{6}t^4\right)\sin(x) \\ &+ L_x^{-1}L_t^{-1}\left\{\frac{1}{s^2}L_xL_t\left\{\lambda u_{txx}(x,t) - D_t^\alpha u(x,t) + u_{xx}(x,t) - u(x,t)\right\}\right\}. \end{aligned} \tag{23}$$

When the decomposition series is applied to equation (23) for $u(x,t)$, then we have

$$\begin{aligned} \sum_{n=0}^{\infty}u_n(x,t) = &\left(+2t^2 + \frac{2}{\Gamma(5-\alpha)}t^{4-\alpha} + \frac{\lambda}{3}t^3 + \frac{1}{6}t^4\right)\sin(x) \\ &+ L_x^{-1}L_t^{-1}\left\{\frac{1}{s^2}L_xL_t\left\{\lambda\frac{\partial^3}{\partial x^2\partial t}\sum_{n=0}^{\infty}u_n(x,t) + \frac{\partial^2}{\partial x^2}\sum_{n=0}^{\infty}u_n(x,t) - \frac{D^\alpha u(x,t)}{\partial t^\alpha}\sum_{n=0}^{\infty}u_n(x,t)\right.\right. \\ &\left.\left. - \sum_{n=0}^{\infty}u_n(x,t)\right\}\right\}. \end{aligned} \tag{24}$$

The following recursive relationship can be deduced from equation (24)

$$u_0(x,t) = \left(1 + 2t^2 + \frac{2}{\Gamma(5-\alpha)}t^{4-\alpha} + \frac{\lambda}{3}t^3 + \frac{1}{6}t^4\right)\sin(x) \tag{25}$$

$$\begin{aligned} u_{n+1}(x,t) = L_x^{-1}L_t^{-1}\left\{\frac{1}{s^2}L_xL_t\left\{\lambda\frac{\partial^3}{\partial x^2\partial t}\sum_{n=0}^{\infty}u_n(x,t) + \frac{\partial^2}{\partial x^2}\sum_{n=0}^{\infty}u_n(x,t) - \frac{D^\alpha u(x,t)}{\partial t^\alpha}\sum_{n=0}^{\infty}u_n(x,t)\right.\right. \\ \left.\left. - \sum_{n=0}^{\infty}u_n(x,t)\right\}\right\}, n \geq 0. \end{aligned} \tag{26}$$

From equation (26), we may get the following formula

$$\begin{aligned} u_1(x,t) = -\left(\frac{2}{\Gamma(5-\alpha)}t^{4-\alpha} + \frac{2}{\Gamma(7-2\alpha)}t^{6-2\alpha} + \frac{4\lambda}{\Gamma(6-\alpha)}t^{5-\alpha} + \frac{8}{\Gamma(7-\alpha)}t^{6-\alpha} + t^2 + \frac{1}{3}t^4 + \frac{\lambda}{30}t^5 + \frac{1}{90}t^6\right. \\ \left. + \frac{2\lambda}{3}t^3 + \frac{\lambda^2}{12}t^4 + \frac{\lambda}{10}t^5\right)\sin(x) \\ u_2(x,t) = \left(\frac{6\lambda}{\Gamma(6-\alpha)}t^{5-\alpha} + \frac{6\lambda}{\Gamma(8-2\alpha)}t^{7-2\alpha} + \frac{6\lambda^2}{\Gamma(7-\alpha)}t^{6-\alpha} + \frac{32\lambda}{\Gamma(8-\alpha)}t^{7-\alpha} + \frac{\lambda}{3}t^3 + \frac{16\lambda}{120}t^5 + \frac{\lambda^2}{36}t^6\right. \\ + \frac{56\lambda}{\Gamma(8)}t^7 + \frac{\lambda^2}{6}t^4 + \frac{\lambda^3}{60}t^5 + \frac{12}{\Gamma(7-\alpha)}t^{6-\alpha} + \frac{12}{\Gamma(9-2\alpha)}t^{8-2\alpha} + \frac{24}{\Gamma(9-\alpha)}t^{8-\alpha} + \frac{1}{6}t^4 + \frac{16}{\Gamma(7)}t^6 \\ \left. + \frac{16}{\Gamma(9)}t^8 + \frac{2}{\Gamma(7-2\alpha)}t^{6-2\alpha} + \frac{2}{\Gamma(9-3\alpha)}t^{8-3\alpha}\right)\sin(x), \end{aligned}$$

and so forth. The general solution of equation (26) can be expressed as an infinite series. As a result, the desired analytical solution is

$$u(x,t) = \left(t^2 + 1\right)\sin(x). \tag{27}$$

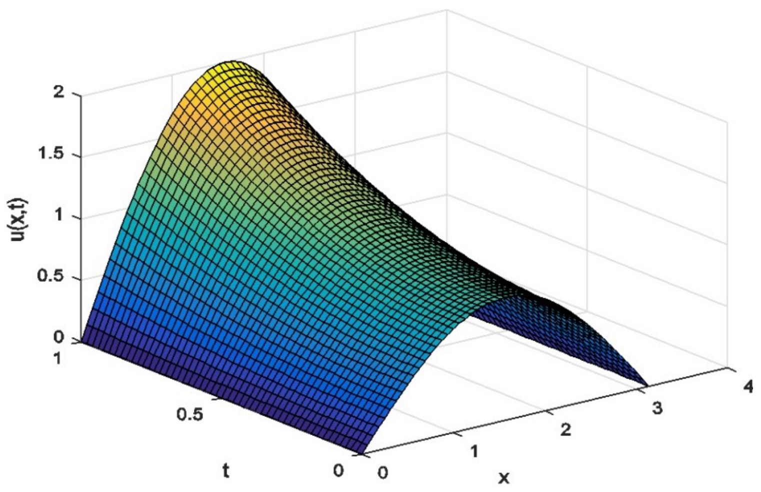

Fig. 1 3D simulation of the exact solution of equation (19) for $0 \leq x \leq \pi, 0 \leq t \leq 1, \lambda > 0$ and $0 \leq \alpha \leq 1$.

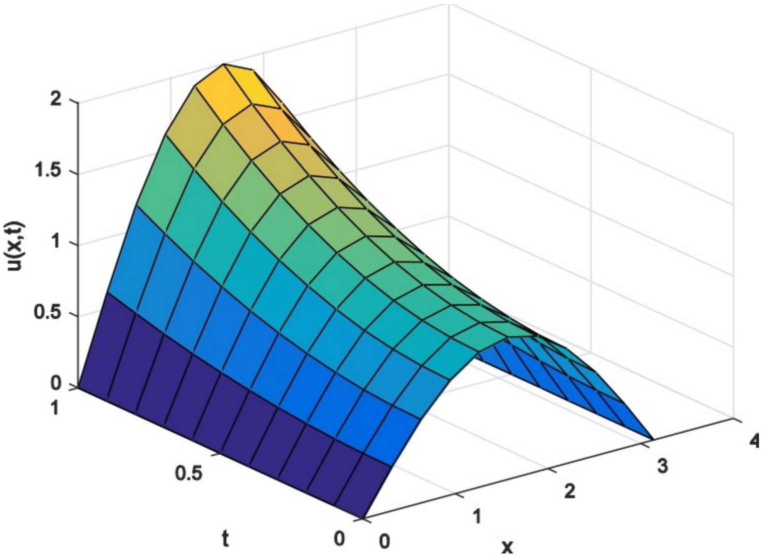

Fig. 2 3D simulation of the analytic solution of equation (19) for $0 \leq x \leq \pi, 0 \leq t \leq 1, \lambda > 0$ and $\alpha = 0.01$.

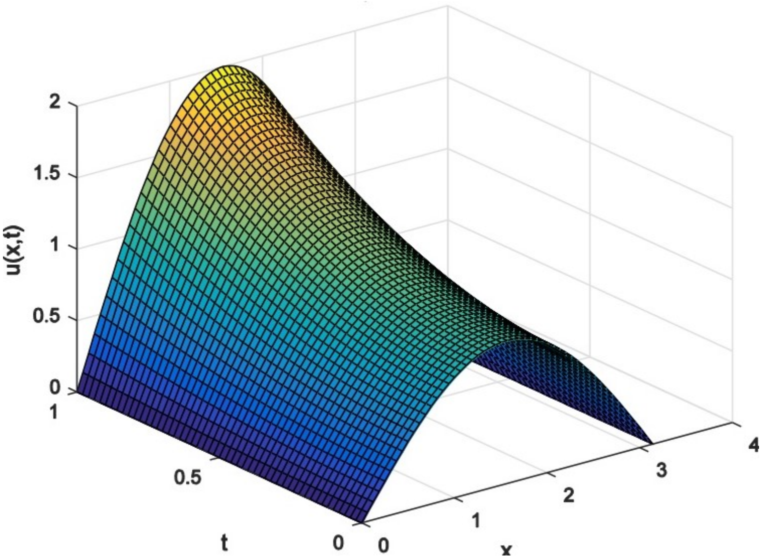

Fig. 3 3D simulation of the analytic solution of equation (19) for $0 \leq x \leq \pi, 0 \leq t \leq 1, \lambda > 0$ and $\alpha = 0.5$.

5 Conclusion

The fractional order PHTPDE based on the Caputo fractional derivative of order was considered in this study. The general solution to the proposed problem was examined utilizing the MDLT method. As an application, a test example was given. The MDLT method was applied to obtain the analytical solution. The acquired result showed that this method is incredibly efficient because we build an analytic solution in only a few steps, while other methods require more steps to achieve the same result. In conclusion, this paper presented an important contribution by proposing an analytic solution to the fractional order pseudo-hyperbolic telegraph equation using the modified double Laplace transform method. The obtained solution provides an exact, closed-form expression that can be expressed in mathematical functions or known operations, offering valuable insights into the behaviour of the equation. The significance of this achievement lies in the challenging nature of obtaining analytic solutions for fractional-order equations, making the proposed method noteworthy. By demonstrating the

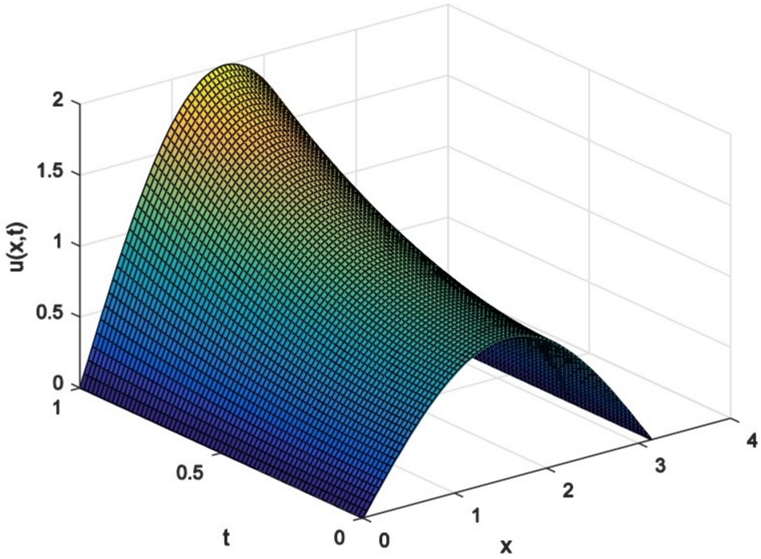

Fig. 4 3D simulation of the analytic solution of equation (19) for $0 \leq x \leq \pi, 0 \leq t \leq 1, \lambda > 0$ and $\alpha = 0.99$.

effectiveness of the MDLTM, the paper opens up new possibilities in order to solve similar types of equations in mathematical physics and engineering. By implementing the Modified Predictor-Corrector method for the fractional order pseudo-hyperbolic telegraph equation, researchers can obtain accurate numerical solutions that complement the analytic solutions obtained through the MDLTM. This would enable the exploration of complex phenomena, such as wave propagation, signal transmission, or diffusion processes, where numerical simulations are essential for capturing detailed behaviour and dynamics.

6 Declarations

6.1 Conflict of interest:

The authors hereby declare that there is no conflict of interests regarding the publication of this paper.

6.2 Funding:

Not applicable.

6.3 Authors Contribution:

S.T.A.-Conceptualization, Methodology, Software, Formal Analysis, Writing-Original Draft, Writing-Review and Editing. M.M.-Conceptualization, Validation, Formal Analysis, Writing-Original Draft. All authors read and approved the final submitted version of this manuscript.

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6.5 Data availability statement:

All data that support the findings of this study are included within the article (and any supplementary files).

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