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Original Study

Wavelets approach for the solution of nonlinear variable delay differential equations

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Abstract

In this study, the Laguerre wavelet-oriented numerical scheme for nonlinear first and second-order delay differential equations (DDEs) is offered. The proposed technique is dependent on the truncated series of the Laguerre wavelets approximation of an unknown function. Here, we transform the different ordered DDEs into a system of non-linear algebraic equations with the help of limit points of a sequence of collocation points. Four nonlinear illustrations are involved to prove the efficiency of the planned technique. The obtained results are equated with the current results, indicating the proposed technique's accuracy and efficiency.

Keywords: Delay differential equations, laguerre wavelets, limit point, collocation process.
AMS 2020 codes: 34K28; 34K40.

1 Introduction

The differential equation (DE) is a mathematical model of numerous physical wonders. Applications of such equations can be perceived in many different areas such as biological, chemical, electronic, and transportation structures. The DEs with time delays are requested in the modeling of real-life problems. DEs with variable delay have plentiful applications in modeling [1], for instance, control systems on the human body, physiological and kinetics, electrical circuits, kinetics, ships and aircraft control systems, and transferrable diseases. In DDEs, it is known that the occurrence of the delay term generates difficulties in the analysis of DEs. Moreover, some research works about diffusion systems with delay term through finite-difference [2], parabolic type problems for delay partial DEs [3], Chebyshev function method [4], method of Bellman's [5], Runge-Kutta (R-K) method [6], Spline polynomial methods [7, 8], ADM [9], Radau method [10], Multiquadric approximation of Multiquadric outline [11], VIM [12], and HPM [13] have been introduced by many scientists.

The Wavelet theory is a recently emerging concept in the field of mathematics. Due to its numerous applications, many researchers are getting attention more and more interested in wavelets. Simply, wavelets are defined in a small domain instead of being defined in a large domain. That is a localized function. Wavelets

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are used in many fields of science such as engineering, signal analysis for waveform illustration and segmentation, time-frequency investigation, harmonic scrutiny, etc. Wavelets are seen as the precise representation of an assortment of operators and functions. Wavelets $\psi_{i,j}(x)$ form a basis function, that is, any continuous function can be expressed as a linear combination of basis elements. Mathematically, we represent any arbitrary function $f(x)$ in wavelet space as $f(x) = \Sigma_{i,j} a_{i,j} \psi_{i,j}(x)$. This wavelet basis originated from a single function called the mother wavelet $\psi(x)$, which is a small beat. In literature, wavelet methods such as the Euler wavelet scheme for volterra delay integral DEs [14], Hermite wavelet scheme for nonlinear singular initial value problems (IVPs) [15], Legendre wavelet method for nonlinear DDEs [16], continuous wavelet series method for Lane-Emden equations [17], B-spline method for Burgers-Huxley equation [18], Haar wavelet method for the Chen-Lee-Liu equation [19], DDEs based on Euler wavelets [20], R-K method for the DDEs [21] and A novel approach for Pantograph equations [22], and so on [23, 24] have been presented.

The rest of this article is organized as follows. Section 2 and 3 reflect the preliminaries of the Laguerre wavelets and the method of solution of DDEs respectively. Four illustrations were demonstrated to show the efficiency and accuracy of the current approach in section 4. In section 5 discussion and conclusion are introduced.

2 Laguerre wavelets

Let $(a,b) \subset \mathbb{R}$ be an interval and $y(x) : (a,b) \to \mathbb{R}$ be continuous real-valued functions. In this study, we employed the following DDE having the form:

$$\frac{d^n y(x)}{dx^n} = f(y^{n-1}(x), \cdots, y'(x), y(x), x, y(g(x))), a < x < b. \tag{1}$$

Subject to the physical conditions:

$$y^i(0) = \alpha_i, i = 0,1,2,\cdots,N-1,$$

where $\alpha_i, i = 0,1,2,\cdots$ are known real constants, and $g(x)$ is the variable delay term. As far as our paper is concerned, no literatureis available on the above problem by the Laguerre wavelets series method which impels us to consider this problem according to the projected technique. Details of preliminaries of the wavelets can be seen in [17]. Definition of the Laguerre wavelets is defined as:

$$\psi_{n,m}(u) = \begin{cases} \frac{2^{\frac{k}{2}}}{m!} L_m(2^k u - 2n + 1), & \frac{n-1}{2^{k-1}} \leq u < \frac{n}{2^{k-1}} \\ 0, & otherwise \end{cases} \tag{2}$$

where $m = 0,1,2,\cdots,M-1$ and $n = 0,1,2,\cdots,2^{k-1}$ where $k \in N$. Here $L_m(u)$ are Laguerre polynomials of degree m concerning weight function $W(u) = 1$ on the interval $[0,\infty)$ and satisfy the following recurrence formula

$$\mathrm{L}_0(u) = 1, L_1(u) = 1-u,$$

$$L_{m+2}(u) = \frac{(2m+3-u)L_{m+1}(u) - (m+1)L_m(u)}{m+2},$$

where $m = 0,1,2,\cdots$. Some theorems on convergence analysis are discussed in [17].

3 Laguerre wavelets method

In this section of the paper, we introduce the general properties of Laguerre wavelets method (LWM). We want to convey a solution $y(x)$ of DDEs under Laguerre wavelet space by resembling the $y(x)$ by Laguerre wavelet basis as

$$y(x)=\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}G_{n,m}\psi_{n,m}(x), \tag{3}$$

where $\psi_{n,m}(x)$ is given in (2). Approximate $y(x)$ by shortening the series in the equation (3) as

$$y(x)\approx\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}G_{n,m}\psi_{n,m}(x)=G^T\psi(x), \tag{4}$$

where G and $\psi(x)$ are $2^{k-1}M\times 1$ matrix defined as,

$$G^T=[G_{1,0},\cdots,G_{1,M-1},G_{2,0},\cdots,G_{2,M-1},G_{2^{k-1},0},\cdots,G_{2^{k-1},M-1}], \tag{5}$$

$$\psi(x)=[\psi_{1,0},\cdots,\psi_{1,M-1},\psi_{2,0},\cdots,\psi_{2,M-1},\psi_{2^{k-1},0},\cdots,\psi_{2^{k-1},M-1}]. \tag{6}$$

Then $2^{k-1}M$ number of conditions required to determine $2^{k-1}M$ number of coefficients such as

$$G_{1,0},\cdots,G_{1,M-1},G_{2,0},\cdots,G_{2,M-1},G_{2^{k-1},0},\cdots,G_{2^{k-1},M-1}.$$

Case 1. Suppose DDE is of order one, then there is an initial constraint, namely

$$y(0)=\alpha_1.$$

Then there should be $2^{k-1}M-1$ extra constraints required to recuperate the unknown coefficients $G_{n,m}$. These conditions can be obtained by substituting (4) in (1) we get,

$$\frac{d(\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}G_{n,m}\psi_{n,m}(x))}{dx}=f\big(x,\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}G_{n,m}\psi_{n,m}(x),\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}G_{n,m}\psi_{n,m}(g(x))\big). \tag{7}$$

Assume (7) is precise at $2^{k-1}M-1$ limit points of the following form

$$x_i=\frac{1}{2}\big(1+\cos(\frac{(i-1)\pi}{2^{k-1}M})\big), i=2,3,\cdots. \tag{8}$$

Then (7) will become

$$\frac{d(\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}G_{n,m}\psi_{n,m}(x_i))}{dx}=f\big(x_i,\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}G_{n,m}\psi_{n,m}(x_i),\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}G_{n,m}\psi_{n,m}(g(x_i))\big). \tag{9}$$

The above equation contributes $2^{k-1}M-1$ algebraic equations and one more constraint will appear from the initial condition. Therefore, we obtain a system containing $2^{k-1}M$ number of linear/nonlinear algebraic equations with $2^{k-1}M$ unknown. By solving this system using the Newtons Raphson method, we get $2^{k-1}M$ unknown coefficient values and substitute these coefficients in (4) which yields the solution of (1).

Case 2. Suppose DDE is second order, then there will be two constraints,

$$y(0)=\alpha_1, y'(0)=\alpha_2.$$

Then, there must be $2^{k-1}M-2$ constraints required to recuperate the unknown coefficients $G_{n,m}$. These constraints are obtained by replacing equation (4) in (1), we get

$$\frac{d^2(\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}G_{n,m}\psi_{n,m}(x))}{dx^2}=f\big(x,\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}G_{n,m}\psi_{n,m}(x),\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}G_{n,m}\psi_{n,m}(g(x))\big). \tag{10}$$

Assume (10) is precise at $2^{k-1}M-2$ limit points of the following sequence

$$x_i = \frac{1}{2}\big(1+\cos(\frac{(i-1)\pi}{2^{k-1}M-1})\big), i = 2,3,\cdots. \tag{11}$$

Then (10) may be transformed as

$$\frac{d^2(\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} G_{n,m}\psi_{n,m}(x_i))}{dx^2} = f\big(x_i, \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} G_{n,m}\psi_{n,m}(x), \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} G_{n,m}\psi_{n,m}(g(x_i))\big). \tag{12}$$

Since two equations are provided by the given constraints and the remaining system of equations is obtained by (12). So, we get a system with $2^{k-1}M$ algebraic equations with $2^{k-1}M$ unknown. By solving this system, we get $2^{k-1}M$ coefficient values. Replacing these coefficient values in (4), we get the solution of (1). The same practice is repeated for higher-order DDEs also.

4 Numerical illustrations

In this section of the current work, we present some applications of the handled scheme .

Application 1 Consider the following multi Pantograph equation [16],

$$y'(x) + xy(x-x^2) + xy^2(x) = x^2+1, 0 \le x \le 1, \tag{13}$$

given initial condition

$$y(0) = 0. \tag{14}$$

The precise solution of (13) is read as $y(x) = x$. By applying LWM with $k=1$ and $M=2$, we solve (13). Let's assume the solution $y(x)$ as

$$y(x) \approx \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} G_{n,m}\psi_{n,m}(x) = G^T\psi(x), \tag{15}$$

substitute (15) in (13) and discrete it using limit points of sequence discussed in section 3. Then we obtain the following equation

$$\frac{d\big(\sum_{m=0}^{1} G_{1,m}\psi_{1,m}(x_i)\big)}{dx} + x\big(\sum_{m=0}^{1} G_{1,m}\psi_{1,m}(x-x^2)\big) + x\big(\sum_{m=0}^{1} G_{1,m}\psi_{1,m}(x)\big) = x^2+1. \tag{16}$$

From (16) and (14), we obtain a system with two nonlinear equations given as

$$\begin{cases} \sqrt{2}C_{1,0} + 2\sqrt{2}C_{1,1} = 0 \\ \frac{C_{1,0}}{\sqrt{2}} - \frac{5\sqrt{2}C_{1,1}}{4} + C_{1,1}^2 + \frac{1}{2}(\sqrt{2}C_{1,0}+2\sqrt{2}C_{1,1})^2 - \sqrt{2}C_{1,1}\big(\sqrt{2}C_{1,0}+2\sqrt{2}C_{1,1}\big) - \frac{5}{4} = 0. \end{cases}$$

In solving this system, we get the vector G as,

$$G^T = [\frac{1}{\sqrt{2}}, \frac{1}{2\sqrt{2}}].$$

Then the approximate solution is $y(x) = x$, which is the same as the precise solution. The proposed technique yields the exact solution for the differential equations having the finite degree of the polynomial as a solution.

Application 2 Consider the DDE with several delay terms such as x^2 and $\frac{x}{2}$ [16].

$$y''(x)+y'(x-x^2)-x^2y(x+\frac{x}{2})+(y')^2(x)-y'(x)y(x)=e^x+e^{x-x^2}-x^2e^{\frac{3x}{2}}, 0<x<1. \quad (17)$$

Subject to the initial conditions are

$$y(0)=y'(0)=1.$$

The exact solution of (17) is $y(x)=e^x$. We consider the above model by using LWM at $k=1$ and $M=10$. Table 1 compares the absolute error (AE) of the approximate solution with the exact solution produced by the scheme given in [16]. Graphical representation of the exact and approximation solution have been simulated by Figures 1 and 2.

Table 1 Comparison of AE for (17) by LWM and technique in [16].

x	Exact Sol.	AE by the method in [16]	AE by LWM
0.1	1.10517091807	3.75E-08	1.40E-10
0.2	1.22140275816	1.25E-07	1.10E-10
0.3	1.34985880757	8.14E-08	2.85E-10
0.4	1.49182469764	3.44E-07	4.64E-10
0.5	1.64872127070	4.76E-07	3.28E-10
0.6	1.82211880039	4.77E-07	2.28E-10
0.7	2.01375270747	1.10E-06	6.24E-10
0.8	2.22554092849	5.62E-06	2.00E-09
0.9	2.45960311115	3.90E-05	1.23E-08

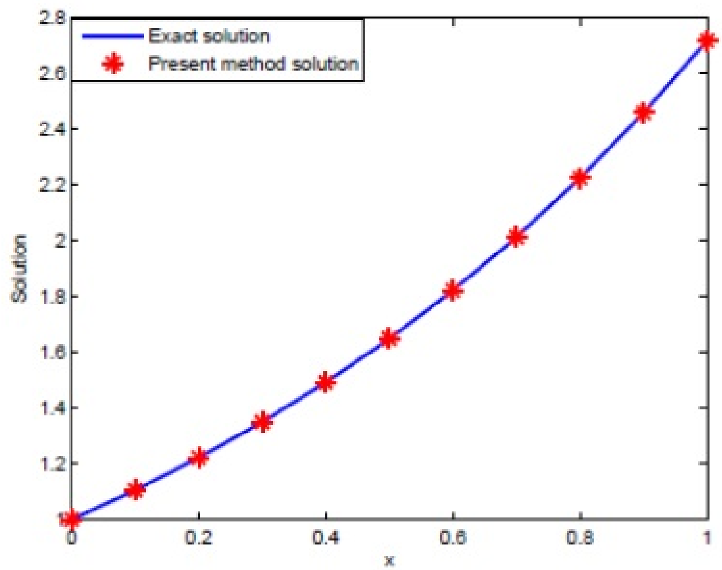

Fig. 1 Graphical interpretation of Exact solution with an approximate solution at $k=1$ and $M=10$ for (17).

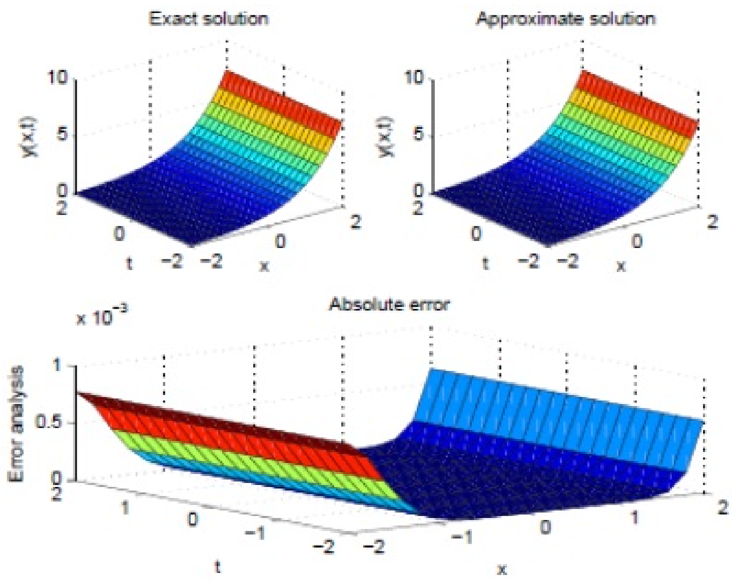

Fig. 2 Graphical interpretation of Exact solution with an approximate solution at $k = 1$ and $M = 10$ for (17) in the surface.

Application 3 Let's choose the second-order DDE with variable delay term as $\frac{x^3}{8}$ [16]

$$y''(x)+2y(x)-y^2(x)+y(\frac{x^3}{8}) = \sin(x)-\sin^2(x)+\sin(\frac{x^3}{8}), 0 < x < 1. \tag{18}$$

In equation (18), initial conditions are given as

$$y(0) = y'(0) = 1.$$

The exact solution of (18) is $y(x) = \sin(x)$. Then, the proposed technique is applied to the problem at $k = 1$ and $M = 9$. Obtained outcomes are compared with other results in literature which can be seen in Table 2. Graphical comparisons of the solutions are shown in Figures 3 and 4.

Table 2 Comparison of AE for (18) by LWM and method in [16].

x	Exact Sol.	AE by the method in [16]	AE by LWM
0.1	0.09983341664	3.38E-10	1.57E-10
0.2	0.19866933079	3.61E-09	2.10E-10
0.3	0.29552020666	3.06E-09	2.00E-10
0.4	0.38941834230	7.99E-09	5.69E-10
0.5	0.47942553860	7.46E-09	3.67E-10
0.6	0.56464247339	1.88E-08	1.07E-10
0.7	0.64421768723	2.56E-08	1.20E-10
0.8	0.71735609089	3.42E-08	2.11E-09
0.9	0.78332690962	6.39E-08	1.29E-09

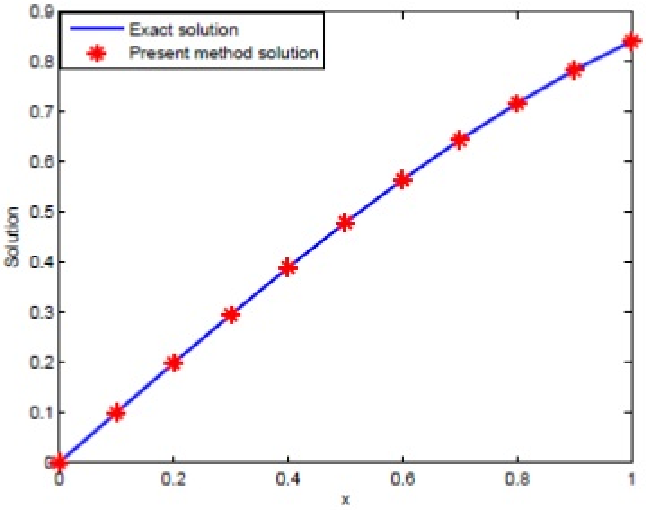

Fig. 3 Graphical interpretation of Exact solution with an approximate solution at $k = 1$ and $M = 9$ for (18).

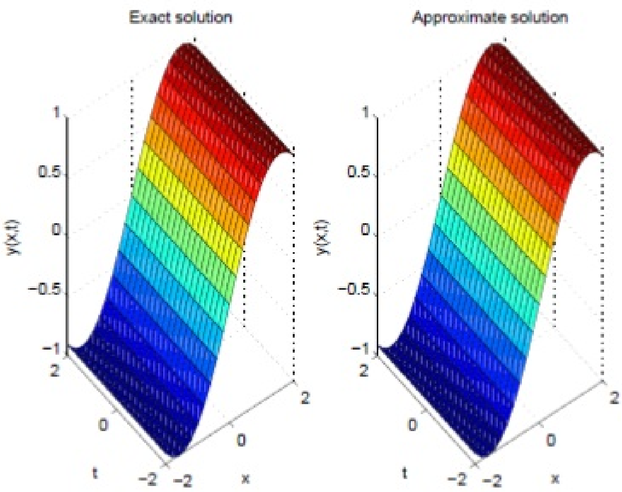

Fig. 4 Graphical interpretation of Exact solution with an approximate solution at $k = 1$ and $M = 9$ for (18) in the surface.

Application 4 Consider the Pantograph equation [24]

$$y'(x) = \frac{1}{2}y\Big(\frac{x}{2}\Big) - y(x) + \frac{1}{2}e^{\frac{-x}{2}}, 0 < x < 1, \tag{19}$$

which is the subject to the initial condition

$$y(0) = 1.$$

The exact solution of (19) is read as $y(x) = e^x$. Let's consider the above model by using LWM, Table 3 shows the existing method yields an improved solution than further approaches in literature. Figure 5 is the graphical simulation of the solutions. Also, Table 3 introduces that M is directly proportional to accuracy in the solution.

Table 3 Comparison of AE for (19) by LWM and other methods [16].

x	Exact Sol.	AE by LWM at M=10	AE by LWM at M=8	AE by the method [25] at M=8
0.0	1.0000000000	0	0	0
0.1	0.9048374180	8.1435E-10	8.6000E-08	2.9610E-06
0.2	0.8187307530	6.5498E-10	1.7200E-08	5.9220E-06
0.3	0.7408182206	5.1857E-10	2.5800E-08	8.8830E-06
0.4	0.6703200460	4.0219E-10	3.4400E-08	1.1844E-05
0.5	0.6065306597	3.0327E-10	4.3000E-08	1.4805E-05
0.6	0.5488116360	2.1952E-10	5.1600E-08	1.7766E-05
0.7	0.4965853037	1.4898E-10	6.0200E-08	2.0727E-05
0.8	0.4493289641	8.9866E-09	6.8800E-08	2.3688E-04
0.9	0.4065696597	4.0657E-09	7.7400E-07	2.6649E-04
1.0	0.3678794411	5.7683E-09	8.6000E-07	2.9610E-04

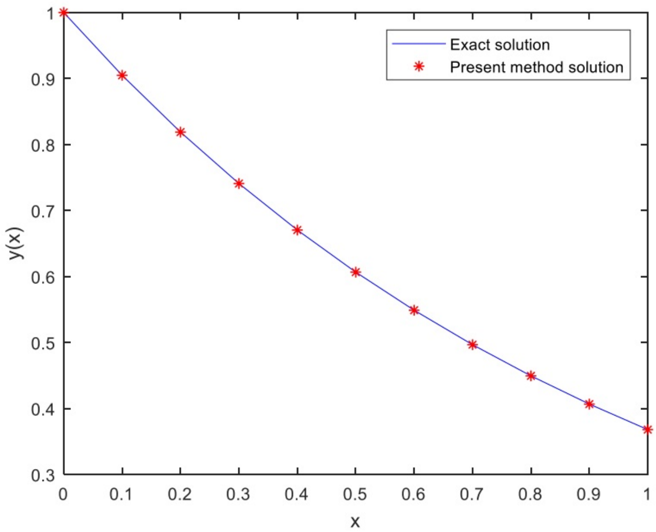

Fig. 5 Graphical interpretation of Exact solution with an approximate solution at $k = 1$ and $M = 9$ for (19).

5 Conclusion

We established the Laguerre wavelet-based mathematical method for DDEs and applied it to linear and nonlinear DDEs. The obtained results show that LWM effectively solves the variable DDEs with different initial constraints. A similar technique can be prolonged for higher-order also with slight modification in the planned method. The proposed technique is better than the method in [16, 25] as can be seen in the tables.

6 Declarations

6.1 Conflict of interest:

The authors have no competing interests to declare that are relevant to the content of this article.

6.2 Funding:

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6.3 Authors Contribution:

K.S.- Conceptualization, Methodology, Software, Writing-Review Editing. R.A.M.-Formal Analysis, Validation, Writing-Original Draft. All authors read and approved the final submitted version of this manuscript.

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6.5 Data availability statement:

All data that support the findings of this study are included within the article.

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