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Original Study

Approximation by nonlinear Meyer-König and Zeller operators based on q-integers

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Abstract

In this paper, we introduce the nonlinear Meyer-König and Zeller operators based on q-integers. Firstly, we describe the q-Meyer-König and Zeller operators of max-product type. Then, we give an error estimation for the q-Meyer-König and Zeller operators of max-product kind by using a modulus of continuity.

Keywords: q-integers, nonlinear operators, Meyer-König and Zeller operators, modulus of continuity.
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1 Introduction

The qualification of approximation for linear positive operators has a significant impact on the approximation theory. Many researchers have studied in this field [1–9]. However, Bernstein operators and its generalizations have a significant status in Computer-Aided Geometric Design (CAGD) to introduce surfaces and curves and have been investigated in many papers [10–12]. Some application areas include the numerical solutions of partial differential equations, CAGD, 3D modeling.

In recent years, many articles have centered on the subject of approximating continuous functions with q-Calculus [3–9]. Initially, Lupas [3] and Philips [4] introduced the q-Bernstein operators generalization and examined approximation qualifications of these operators. Then, Derriennic introduced many qualifications of the q-Durrmeyer operators in [8].

In [13], using discrete linear approximation operators to approximate the nonlinear positive operators were introduced. The operators of max-product type were first used to describe linear operators that used maximum as the name of the sum and provided a Jackson-type error estimate with regard to the continuity modulus [14–22]. Since the max-product kind of approximation theory is a very rich and useful phenomena of approximating continuous functions, researchers have turned to this new field in recent years. For another approximation theory studies including univariate and bivariate type of operators can be seen via [23–32].

In [2], the nonlinear Meyer-König and Zeller operators of max-product type were first described from the

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linear counterpart by using the maximum operator in place of the sum operator, as below

$$Z_n^{(M)}(f)(v) = \frac{\bigvee_{i=0}^{\infty} s_{n,i}(v) f\left(\frac{i}{n+i}\right)}{\bigvee_{i=0}^{\infty} s_{n,i}(v)},$$

where $s_{n,i}(v) = \binom{n+i}{i}(v)^i$ and $\bigvee_{i=0}^{\infty} s_{n,i}(v) = \sup_{i\in\mathbb{N}}\{s_{n,i}(v)\}$, $v \in [0,1)$.

In [17], Bede et al. proved that $Z_n^{(M)}$ operators are a well defined nonlinear operator for all $x \in [0,1]$ and $Z_n^{(M)}$ operators have approximation conclusions and shape preserving properties. According to the usual Meyer-König and Zeller operators, the max-product kind of these operators also preserve approximation properties over the class of continuous functions. Additionally, $Z_n^{(M)}$ operators are continuous for any $f > 0$ and preserve the quasi-convexity of f on $[0,1]$ and the monocity in [17].

2 Preliminaries

In this study, we define nonlinear q-Meyer-König and Zeller operators of max-product type and give the approximation qualifications of these operators. Firstly, we indicate some basic definition and general notations. Now, let's consider the operations "$\bigvee$" (maximum) and "." (product) over the max-product algebra $(\mathbb{R}_+, \vee, \cdot)$. Assume $I \subset \mathbb{R}$ is a finite or infinite interval, and set

$$\mathbb{A} := \{u : I \longrightarrow \mathbb{R}_+; u \textit{ continous and bounded on } I\}.$$

The max-product type of discrete approximation operators' standard form is defined as

$$L_n(u)(t) = \bigvee_{r=0}^{n} K_n(t,t_r)u(t_r), \quad L_n(u)(t) = \bigvee_{r=0}^{\infty} K_n(t,t_r)u(t_r),$$

where $n \in \mathbb{N}$, $u \in \mathbb{A}$, $K_n(.,t_r) \in \mathbb{A}$ and $t_r \in I$, for all r. The pseudo-linearity property is verified by these nonlinear positive operators as below; for $u,w : I \to \mathbb{R}_+$

$$L_n(\zeta.u \vee \gamma.w)(t) = \zeta.L_n(u)(t) \vee \gamma.L_n(w)(t),$$

which $\zeta, \gamma \in \mathbb{R}_+$. Also, the operators of max-product kind are positive homogenous, i.e. $\forall\lambda \geq 0$, $L_n(\lambda u) = \lambda L_n(u)$ (for the other details, one can see [17]).

Lemma 1. *For $n \in \mathbb{N}$, let take $L_n : \mathbb{A} \to \mathbb{A}$ be a sequence of operators verifying the below circumstances:*

i. For all $u,w \in \mathbb{A}$, $L_n(u+w) \leq L_n(u) + L_n(w)$,

ii. For all $u,w \in \mathbb{A}$ and $t \in I$, $|L_n(u)(t) - L_n(w)(t)| \leq L_n(|u-w|)(t)$.

[18] provides proof for the Lemma.

Corollary 2. *Let's suppose that the sequence L_n provides $L_n(e_0) = e_0$ for all $n \in \mathbb{N}$ in addition the conditions given Lemma 1 [18]. Then for all $u \in \mathbb{A}$ and $t \in I$, we get*

$$|u(t) - L_n(u)(t)| \leq \left[\frac{1}{\delta} L_n(\eta_t)(t) + 1\right] \omega_1(u;\delta),$$

where $\delta > 0$, $\eta_t(a) = |a-t|$ for all $a,t \in I$ and $\omega_1(u;\delta) = \max\{|u(t)-u(s)|; t,s \in I, |t-s| \leq \delta\}$.

Let's give some basic description of the q-calculus. For the parameter $q>0$ and $m\in\mathbb{N}$, one gives the q-integer $[m]_q$ as below

$$[m]_q=\begin{cases}\frac{1-q^m}{1-q} & \text{if } q\neq 1\\ m & \text{if } q=1\end{cases},\quad [0]_q=0,$$

and q-factorial $[m]_q!$ as

$$[m]_q!=[1]_q[2]_q\cdots[m]_q \quad \text{for} \quad m\in\mathbb{N} \quad \text{and} \quad [0]_q!=1.$$

For integers $0\le l\le m$, q-binomial is introduced as:

$$\begin{bmatrix} m\\ l\end{bmatrix}_q=\frac{[m]_q!}{[l]_q![m-l]_q!}.$$

Finally, let q-binomial coefficient and $1\le i\le m-1$, one get q-Pascal Rules as follows

$$\begin{bmatrix} m\\ i\end{bmatrix}_q=\begin{bmatrix} m-1\\ i-1\end{bmatrix}_q+q^i\begin{bmatrix} m-1\\ i\end{bmatrix}_q,\quad \begin{bmatrix} m\\ i\end{bmatrix}_q=q^{m-i}\begin{bmatrix} m-1\\ i-1\end{bmatrix}_q+\begin{bmatrix} m-1\\ i\end{bmatrix}_q. \tag{1}$$

3 Construction of the operators

In this section, we define nonlinear q-Meyer-König and Zeller operators of max-product type as below:

$$Z_n^{(M)}(f)(\varsigma;q)=\frac{\bigvee_{\zeta=0}^{\infty}t_{n,\zeta}(\varsigma,q)f\left(\frac{[\zeta]_q}{[n+\zeta]_q}\right)}{\bigvee_{\zeta=0}^{\infty}t_{n,\zeta}(\varsigma,q)},\ \varsigma\in[0,1), \tag{2}$$

which $t_{n,\zeta}(\varsigma,q)=\begin{bmatrix} n+\zeta\\ \zeta\end{bmatrix}_q(\varsigma)^{\zeta}$. Here, the function $f:[0,1]\to\mathbb{R}^+$ is continuous.

The operators given in (2) are positive and continuous on the interval $[0,1]$ for a continuous function $f:[0,1]\to\mathbb{R}^+$. Indeed, $f\in C_+([0,1])$ and $t_{n,\zeta}(\varsigma,q)$ is positive for all $[0,1]$, we have our operator being positive. For the nonlinearity of $Z_n^{(M)}(f)(\varsigma;q)$ for any $f,h\in C_+([0,1])$, we obtain $Z_n^{(M)}(f+h)(\varsigma;q)\le Z_n^{(M)}(f)(\varsigma;q)+Z_n^{(M)}(h)(\varsigma;q)$. Also, the pseudo-linearity property is provided by these operators, and these operators are positive homogenous. Also, we handily show that $Z_n^{(M)}(f;q)(0)-f(0)=Z_n^{(M)}(f;q)(1)-f(1)=0$ for any n consider that in the indications, proofs and expressions of all approximation conclusions in fact we may assume that $0<\varsigma<1$. Additionally, we provide an error estimate for the operators $Z_n^{(M)}(f)(\varsigma;q)$ described by (2) with regard to the modulus of continuity.

For each $\zeta,\gamma\in\{0,1,2,\cdots\}$ and $x\in\left[\frac{[\gamma]_q}{[n+\gamma]_q},\frac{[\gamma+1]_q}{[n+\gamma+1]_q}\right]$, we obtained in the following structure

$$P_{\zeta,n,\gamma}(\varsigma,q)=\frac{t_{n,\zeta}(\varsigma,q)\left|\frac{[\zeta]_q}{[n+\zeta]_q}-\varsigma\right|}{t_{n,\gamma}(\varsigma,q)},$$

$$p_{\zeta,n,\gamma}(\varsigma,q)=\frac{t_{n,\zeta}(\varsigma,q)}{t_{n,\gamma}(\varsigma,q)}. \tag{3}$$

It follows that if $\zeta\ge j+1$, then

$$P_{\zeta,n,\gamma}(\varsigma,q)=\frac{t_{n,\zeta}(\varsigma,q)\left(\frac{[\zeta]_q}{[n+\zeta]_q}-\varsigma\right)}{t_{n,\gamma}(\varsigma,q)}, \tag{4}$$

and if $\zeta\le\gamma$, then

$$P_{\zeta,n,\gamma}(\varsigma,q)=\frac{t_{n,\zeta}(\varsigma,q)\left(\varsigma-\frac{[\zeta]_q}{[n+\zeta]_q}\right)}{t_{n,\gamma}(\varsigma)}. \tag{5}$$

Lemma 3. *For all* $\zeta,\gamma\in\{0,1,2,\cdots\}$ *and* $\varsigma\in\left[\frac{[\gamma]_q}{[n+\gamma]_q},\frac{[\gamma+1]_q}{[n+\gamma+1]_q}\right]$ *we obtain*

$$p_{\zeta,n,\gamma}(\varsigma,q)\leq 1.$$

Proof. We have two cases for the proof of the above lemma: 1) $\zeta\geq\gamma$, 2) $\zeta\leq\gamma$. *Case 1: Let* $\zeta\geq\gamma$. *From the definition* $p_{\zeta,n,\gamma}(\varsigma,q)$ *given (3), since the function* $\frac{1}{\varsigma}$ *is nonincreasing on* $\left[\frac{[\gamma]_q}{[n+\gamma]_q},\frac{[\gamma+1]_q}{[n+\gamma+1]_q}\right]$ *and* $[\gamma+1]_q\leq[\zeta+1]_q$, *we obtain*

$$\frac{p_{\zeta,n,\gamma}(\varsigma)}{p_{\zeta+1,n,\gamma}(\varsigma)}=\frac{[\zeta+1]_q}{[n+\zeta+1]_q}\cdot\frac{1}{\varsigma}\geq\frac{[\zeta+1]_q}{[n+\zeta+1]_q}\cdot\frac{[n+\gamma+1]_q}{[\gamma+1]_q}=\frac{1-q^{n+\gamma+1}}{1-q^{n+\zeta+1}}\geq 1.$$

which indicates

$$p_{\gamma,n,\gamma}(\varsigma,q)\geq p_{\gamma+1,n,\gamma}(\varsigma,q)\geq p_{\gamma+2,n,\gamma}(\varsigma,q)\geq\cdots.$$

Case 2: Let $\zeta\leq\gamma$.

$$\frac{p_{\zeta,n,\gamma}(\varsigma)}{p_{\zeta-1,n,\gamma}(\varsigma)}=\frac{[n+\zeta]_q}{[\zeta]_q}\cdot\varsigma\geq\frac{[n+\zeta]_q}{[\zeta]_q}\cdot\frac{[\gamma]_q}{[n+\gamma]_q}\geq 1.$$

which implies

$$p_{\gamma,n,\gamma}(\varsigma,q)\geq p_{\gamma-1,n,\gamma}(\varsigma,q)\geq p_{\gamma-2,n,\gamma}(\varsigma,q)\geq\cdots\geq p_{0,n,\gamma}(\varsigma,q).$$

Since $p_{\gamma,n,\gamma}(\varsigma,q)=1$, *the proof of lemma is finished.*

Lemma 4. *Let* $q\in(0,1)$, $\gamma\in\{1,2,\cdots\}$ *and* $\varsigma\in\left[\frac{[\gamma]_q}{[n+\gamma]_q},\frac{[\gamma+1]_q}{[n+\gamma+1]_q}\right]$.

(i) If $\zeta\in\{\gamma+1,\cdots\}$ *is such that* $[\gamma]_q\leq[\zeta]_q-\sqrt{[\zeta+1]_q+\frac{q^{\gamma}[\gamma+1]_q+[\gamma]_q[\gamma+1]_q}{[n]_q}}$, *then* $P_{\zeta,n,\gamma}(\varsigma,q)\geq P_{\zeta+1,n,\gamma}(\varsigma,q)$.

(ii) If $\zeta\in\{0,1\cdots,\gamma\}$ *is such that* $[\gamma]_q\leq[\zeta]_q+\sqrt{[\zeta]_q+\frac{[\gamma]_q^2}{[n]_q}}$, *then* $P_{\zeta,n,\gamma}(\varsigma)\geq P_{\zeta-1,n,\gamma}(\varsigma)$.

Proof. (i) Let $\zeta\in\{\gamma+1,\gamma+2,\cdots\}$ *with* $[\gamma]_q\leq[\zeta]_q-\sqrt{[\zeta+1]_q+\frac{q^{\gamma}[\gamma+1]_q+[\gamma]_q[\gamma+1]_q}{[n]_q}}$. *So, we obtain*

$$\frac{P_{\zeta,n,\gamma}(\varsigma,q)}{P_{\zeta+1,n,\gamma}(\varsigma,q)}=\frac{[\zeta+1]_q}{[n+\zeta+1]_q}\cdot\frac{1}{\varsigma}\cdot\frac{\frac{[\zeta]_q}{[n+\zeta]_q}-\varsigma}{\frac{[\zeta+1]_q}{[n+\zeta+1]_q}-x}.$$

Since the function $h(\varsigma)=\frac{1}{\varsigma}\cdot\frac{\frac{[\zeta]_q}{[n+\zeta]_q}-\varsigma}{\frac{[\zeta+1]_q}{[n+\zeta+1]_q}-\varsigma}$ *is nonincreasing, it follows that*

$$\begin{aligned}h(\varsigma)\geq& h\left(\frac{[\gamma+1]_q}{[n+\gamma+1]_q}\right)=\frac{[n+\gamma+1]_q}{[\gamma+1]_q}\cdot\frac{\frac{[\zeta]_q}{[n+\zeta]_q}-\frac{[\gamma+1]_q}{[n+\gamma+1]_q}}{\frac{[\zeta+1]_q}{[n+\zeta+1]_q}-\frac{[\gamma+1]_q}{[n+\gamma+1]_q}}\\ =&\frac{[n+\gamma+1]_q}{[\gamma+1]_q}\frac{[n+\zeta+1]_q}{[n+\zeta]_q}\frac{[\zeta]_q-[\gamma+1]_q}{[\zeta+1]_q-[\gamma+1]_q}.\end{aligned}$$

Hence, we get

$$\frac{P_{\zeta,n,\gamma}(\varsigma,q)}{P_{\zeta+1,n,\gamma}(\varsigma,q)}\geq\frac{[n+\gamma+1]_q}{[\gamma+1]_q}\frac{[\zeta+1]_q}{[n+\zeta]_q}\frac{[\zeta]_q-[\gamma+1]_q}{[\zeta+1]_q-[\gamma+1]_q}.$$

By taking the hypothesis $[\gamma]_q\leq[\zeta]_q-\sqrt{[\zeta+1]_q+\frac{q^{\gamma}[\gamma+1]_q+[\gamma]_q[\gamma+1]_q}{[n]_q}}$ *which indicates that*

$$[n+\gamma+1]_q[\zeta+1]_q\left([\zeta]_q-[\gamma+1]_q\right)\geq[\gamma+1]_q[n+\zeta]_q\left([\zeta+1]_q-[\gamma+1]_q\right),$$

we obtain

$$\frac{P_{\zeta,n,\gamma}(\varsigma,q)}{P_{\zeta+1,n,\gamma}(\varsigma,q)} \geq 1.$$

(ii) Let $\zeta \in \{0,1,\cdots,\gamma\}$ *and* $[\gamma]_q \leq [\zeta]_q + \sqrt{[\zeta]_q + \frac{[\gamma]_q^2}{[n]_q}}$. *So, we have*

$$\frac{P_{\zeta,n,\gamma}(\varsigma)}{P_{\zeta-1,n,\gamma}(\varsigma)} = \frac{[n+\zeta]_q}{[\zeta]_q} \cdot \varsigma \cdot \frac{\varsigma - \frac{[\zeta]_q}{[n+\zeta]_q}}{\varsigma - \frac{[\zeta-1]_q}{[n+\zeta-1]_q}}.$$

Then, since the function $r(\varsigma) = \frac{[n+\zeta]_q}{[\zeta]_q} \cdot \varsigma \cdot \frac{\varsigma - \frac{[\zeta]_q}{[n+\zeta]_q}}{\varsigma - \frac{[\zeta-1]_q}{[n+\zeta-1]_q}}$ *is nondecreasing on* $\varsigma \in \left[\frac{[\gamma]_q}{[n+\gamma]_q}, \frac{[\gamma+1]_q}{[n+\gamma+1]_q}\right]$, *we get*

$$\begin{aligned} r(\varsigma) \geq & r\left(\frac{[\gamma]_q}{[n+\gamma]_q}\right) = \frac{[n+\zeta-1]_q}{[\zeta]_q} \cdot \frac{[\gamma]_q}{[n+\gamma]_q} \cdot \frac{[n+\zeta]_q[\gamma]_q - [\zeta]_q[n+\gamma]_q}{[\gamma]_q[n+\zeta-1]_q - [\zeta-1]_q[n+\gamma]_q} \\ \geq & \frac{[n+\zeta-1]_q}{[\gamma-\zeta+1]_q} \cdot \frac{[\gamma]_q}{[\zeta]_q} \cdot \frac{[\gamma-\zeta]_q}{[n+\gamma]_q}. \end{aligned}$$

Since the hypothesis $[\gamma]_q \leq [\zeta]_q + \sqrt{[\zeta]_q + \frac{[\gamma]_q^2}{[n]_q}}$, *we easily obtain*

$$\frac{\widehat{P}_{\zeta,n,\gamma}(\varsigma)}{\widehat{P}_{\zeta-1,n,\gamma}(\varsigma)} \geq 1.$$

Therefore, we demonstrate the lemma.

Lemma 5. *Let's indicate* $t_{n,\zeta}(\varsigma,q) = \begin{bmatrix} n+\zeta \\ \zeta \end{bmatrix}_q \varsigma^{\zeta}$, $q \in (0,1)$, $\gamma \in \{0,1,2,\cdots\}$ *and for all* $\varsigma \in \left[\frac{[\gamma]_q}{[n+\gamma]_q}, \frac{[\gamma+1]_q}{[n+\gamma+1]_q}\right]$ *we get*

$$\bigvee_{\zeta=0}^{n} t_{n,\zeta}(\varsigma,q) = t_{n,\gamma}(\varsigma,q).$$

Proof. Primarily, we demonstrate that $0 \leq \zeta$ and for fixed $n \in \mathbb{N}$, we get

$$0 \leq t_{n,\zeta+1}(\varsigma,q) \leq t_{n,\zeta}(\varsigma,q) \quad \text{if and only if} \quad \varsigma \in \left[0, \frac{[\zeta+1]_q}{[n+\zeta+1]_q}\right].$$

Let's estimate the following inequality

$$0 \leq \begin{bmatrix} n+\zeta+1 \\ \zeta+1 \end{bmatrix}_q \varsigma^{\zeta+1} \leq \begin{bmatrix} n+\zeta \\ \zeta \end{bmatrix}_q \varsigma^{\zeta},$$

after some simplifications by using q-Pascal rules given in (1), the previous inequality can be reduced to

$$0 \leq \varsigma \leq \frac{[\zeta+1]_q}{[n+\zeta+1]_q}.$$

Therefore, if we take $\zeta = 0,1,\cdots,n$ in the inequality above, we get

$$\begin{aligned} t_{n,1}(\varsigma,q) \leq t_{n,0}(\varsigma,q), & \quad \text{if and only if} \quad \varsigma \in \left[0, \frac{1}{[n+1]_q}\right], \\ t_{n,2}(\varsigma,q) \leq t_{n,1}(\varsigma,q), & \quad \text{if and only if} \quad \varsigma \in \left[0, \frac{[2]_q}{[n+2]_q}\right], \\ t_{n,3}(\varsigma,q) \leq t_{n,2}(\varsigma,q), & \quad \text{if and only if} \quad \varsigma \in \left[0, \frac{[3]_q}{[n+3]_q}\right], \end{aligned}$$

and

$$t_{n,\zeta+1}(\varsigma,q)\leq t_{n,\zeta}(\varsigma,q),\quad \text{if and only if}\quad \varsigma\in\left[0,\frac{[\zeta+1]_q}{[n+\zeta+1]_q}\right],$$

and so on. The result of all these inequalities is

$$\begin{aligned}
&\text{if}\quad \varsigma\in\left[0,\frac{1}{[n+1]_q}\right]\text{ then}\quad t_{n,\zeta}(\varsigma,q)\leq t_{n,0}(\varsigma,q),\ \text{for all }\zeta=0,1,\cdots,n;\\
&\text{if}\quad \varsigma\in\left[\frac{1}{[n+1]_q},\frac{[2]_q}{[n+2]_q}\right]\text{ then}\quad t_{n,\zeta}(\varsigma,q)\leq t_{n,1}(\varsigma,q),\ \text{for all }\zeta=0,1,\cdots,n;\\
&\text{if}\quad \varsigma\in\left[\frac{[2]_q}{[n+2]_q},\frac{[3]_q}{[n+3]_q}\right]\text{ then}\quad t_{n,\zeta}(\varsigma,q)\leq t_{n,2}(\varsigma,q),\ \text{for all }\zeta=0,1,\cdots,n;
\end{aligned}$$

and

$$\text{if}\quad \varsigma\in\left[\frac{[\gamma]_q}{[n+\gamma]_q},\frac{[\gamma+1]_q}{[n+\gamma+1]_q}\right]\text{ then}\quad t_{n,\zeta}(\varsigma,q)\leq t_{n,\gamma}(\varsigma,q),\ \text{for all }\zeta=0,1,\cdots,n,$$

which completes the proof.

4 Approximation degree of $Z_n^{(M)}(f)(x;q)$

The Shisha-Mond theorem, which is applicable to nonlinear max-product kind operators and is presented in [13, 17], is used in this section to provide an error estimate for the operators $Z_n^{(M)}(f)(\varsigma;q)$ which is defined in (2), with regard to the modulus of continuity.

Theorem 6. *Let's $q\in(0,1)$ and the function f is a bounded and continuous on $[0,1]$. Then, we have*

$$\left|Z_n^{(M)}(f)(\varsigma;q)-f(\varsigma)\right|\leq 18\omega_1\left(f;\frac{(1-\varsigma)\sqrt{\varsigma}}{\sqrt{[n]_q}}\right),$$

where $n\geq 4$, $\varsigma\in[0,1]$ and $\omega_1(f;\delta)=\sup\{|f(\varsigma)-f(\zeta)|;\varsigma,\zeta\in[0,1],|\varsigma-\zeta|\leq\delta\}$.

Proof. Since the max-product Meyer-König and Zeller operators based on q-integers supply the conditions in Corollary 2 and we get the following

$$\left|Z_n^{(M)}(f)(\varsigma;q)-f(\varsigma)\right|\leq\left(1+\frac{1}{\delta_n}Z_n^{(M)}(\eta_\varsigma,\varsigma;q)\right)\omega_1(f;\delta_n),\tag{6}$$

where $\eta_\varsigma(t)=|t-\varsigma|$. Estimation of the following term is enough for the proof of lemma:

$$Z_n^{(M)}(\eta_\varsigma,\varsigma;q)=\frac{\bigvee_{\zeta=0}^{\infty}t_{n,\zeta}(\varsigma,q)\left|\frac{[\zeta]_q}{[n+\zeta]_q}-\varsigma\right|}{\bigvee_{\zeta=0}^{\infty}t_{n,\gamma}(\varsigma,q)}.$$

Let's assume that $\varsigma\in\left[\frac{[\gamma]_q}{[n+\gamma]_q},\frac{\zeta_n[\gamma+1]_q}{[n+\gamma+1]_q}\right]$, where $\gamma\in\{0,1,\cdots\}$ is fixed and arbitrary. From Lemma 5, we get

$$Z_n^{(M)}(\eta_\varsigma,\varsigma;q)=\bigvee_{\zeta=0}^{\infty}P_{\zeta,n,\gamma}(\varsigma,q).$$

Firstly, for $\gamma=0$ we obtain $Z_n^{(M)}(\eta_\varsigma,\varsigma;q)\leq\frac{[2](1-\varsigma)\sqrt{\varsigma}}{\sqrt{[n]_q}}$ for all $\varsigma\in\left[0,\frac{1}{[n+1]_q}\right]$, so we can claim that $\gamma=\{1,2,\cdots\}$.

Indeed, for each $P_{\zeta,n,\gamma}(\varsigma,q)$ we determine an upper estimate, for $\gamma=0$, $\varsigma\in\left[0,\frac{1}{[n+1]_q}\right]$ and $\zeta\in\{0,1,\cdots,n\}$. Besides, Lemma 4(i) which indicates that for $\zeta\geq 2$ one gets $P_{\zeta,n,0}(\varsigma,q)\geq P_{\zeta+1,n,0}(\varsigma,q)$ which means that $Z_n^{(M)}\left(\eta_\varsigma,\varsigma;q\right)=max_{\zeta\in\{0,1,2\}}\left\{P_{\zeta,n,0}(\varsigma,q)\right\},\varsigma\in\left[0,\frac{1}{[n+1]_q}\right]$. For $\zeta=0$, we have

$$\begin{aligned}P_{\zeta,n,0}(\varsigma,q)=&\varsigma=\sqrt{\varsigma}\cdot\sqrt{\varsigma}\leq\sqrt{\varsigma}\cdot\frac{1}{\sqrt{[n+1]_q}}\leq\sqrt{\varsigma}\cdot\frac{1}{\sqrt{[n]_q}}\\ \leq&(1-\varsigma)\sqrt{\varsigma}\cdot\frac{1}{\sqrt{[n]_q}}\frac{1}{1-\varsigma}\leq(1-\varsigma)\sqrt{\varsigma}\cdot\frac{1}{\sqrt{[n]_q}}\frac{[n+1]_q}{[n]_q}\\ \leq&\frac{[2](1-\varsigma)\sqrt{\varsigma}}{\sqrt{[n]_q}}.\end{aligned}$$

For $\zeta=1$, we have

$$P_{1,n,0}(\varsigma,q)=\begin{bmatrix}n+1\\1\end{bmatrix}_q\varsigma\left|\frac{1}{[n+1]_q}-\varsigma\right|\leq\varsigma\leq\frac{[2](1-\varsigma)\sqrt{\varsigma}}{\sqrt{[n]_q}}.$$

For $\zeta=2$, we obtain

$$\begin{aligned}P_{2,n,0}(\varsigma,q)=&\begin{bmatrix}n+2\\2\end{bmatrix}_q\varsigma^2\left|\frac{2}{[n+2]_q}-\varsigma\right|\leq\frac{[n+1]_q[n+2]_q}{2}\varsigma^2\frac{2}{[n+2]_q}\\ \leq&[n+1]_q\cdot\varsigma\cdot\frac{1}{[n+1]_q}\leq\varsigma\leq\frac{[2](1-\varsigma)\sqrt{\varsigma}}{\sqrt{[n]_q}}.\end{aligned}$$

Now, let's take $\gamma=1,2,\cdots$ is fixed, $\varsigma\in\left[\frac{[\gamma]_q}{[n+\gamma]_q},\frac{[\gamma+1]_q}{[n+\gamma+1]_q}\right]$ and $\zeta=0,1,\cdots$, then we get an upper estimate for each $P_{\zeta,n,\gamma}(\varsigma,q)$. Under these circumstances, the proof will be separated into the following cases:

Case 1) $\zeta\geq\gamma+1$

Subcase a) From the hypothesis $[\gamma]_q\geq[\zeta]_q-\sqrt{[\zeta+1]_q+\frac{[\gamma+1]_q^2}{[n]_q}}$ which refers that $[\zeta]_q\leq[\gamma]_q+\sqrt{[\zeta+1]_q+\frac{[\gamma+1]_q^2}{[n]_q}}$, we get

$$\begin{aligned}P_{\zeta,n,\gamma}(\varsigma;q)=&p_{\zeta,n,\gamma}(\varsigma;q)\left(\frac{[\zeta]_q}{[n+\zeta]_q}-\varsigma\right)\leq\frac{[\zeta]_q}{[n+\zeta]_q}-\varsigma\\ \leq&\frac{[\zeta]_q}{[n+\zeta]_q}-\frac{[\gamma]_q}{[n+\gamma]_q}\leq\frac{[\gamma]_q+\sqrt{[\zeta+1]_q+\frac{[\gamma+1]_q^2}{[n]_q}}}{[n+\gamma]_q+\sqrt{[\zeta+1]_q+\frac{[\gamma+1]_q^2}{[n]_q}}}-\frac{[\gamma]_q}{[n+\gamma]_q}\\ \leq&\frac{[n]_q\sqrt{[\zeta+1]_q+\frac{[\gamma+1]_q^2}{[n]_q}}}{\left([n+\gamma]_q+\sqrt{[\zeta+1]_q+\frac{[\gamma+1]_q^2}{[n]_q}}\right)[n+\gamma]_q}.\end{aligned}$$

One can easily see that $[\zeta+1]_q\leq 2[\gamma+1]_q$, then $\sqrt{[\zeta+1]_q+\frac{[\gamma+1]_q^2}{[n]_q}}\leq\sqrt{2[\gamma+1]_q+\frac{[\gamma+1]_q^2}{[n]_q}}$,

$$P_{\zeta,n,\gamma}(\varsigma;q)\leq\frac{[n]_q\sqrt{[\gamma+1]_q\left(2+\frac{[\gamma+1]_q}{[n]_q}\right)}}{\left([n+\gamma]_q+\sqrt{[\gamma+1]_q\left(2+\frac{[\gamma+1]_q}{[n]_q}\right)}\right)[n+\gamma]_q}.$$

From $x \in \left[\frac{[\gamma]_q}{[n+\gamma]_q}, \frac{[\gamma+1]_q}{[n+\gamma+1]_q}\right]$, we obtain $\frac{[n]_q+\varsigma.q^{n+\gamma}-q^{\gamma}}{1-\varsigma} \leq [n+\gamma]_q$ and $[\gamma+1]_q \leq 2[\gamma]_q \leq \frac{2[n]_q\varsigma}{1-\varsigma}$. Because the function $\frac{[n]_q[z]_q}{([n+\gamma]_q+[z]_q)[n+\gamma]_q}$ is decreasing according to γ and increasing according to z, we get

$$P_{\zeta,n,\gamma}(x;q) \leq \frac{2[n]_q\sqrt{\frac{[n]_q\varsigma}{1-\varsigma}\left(1+\frac{[n]_q\varsigma}{(1-\varsigma)[n]_q}\right)}}{\left(\frac{[n]_q+\varsigma.q^{n+\gamma}-q^{\gamma}}{1-\varsigma}+2\sqrt{\frac{[n]_q\varsigma}{1-\varsigma}\left(1+\frac{[n]_q\varsigma}{(1-\varsigma)[n]_q}\right)}\right)\frac{[n]_q+\varsigma.q^{n+\gamma}-q^{\gamma}}{1-\varsigma}}.$$

By using simple calculation and taking $n \geq 4$, we obtain

$$P_{\zeta,n,\gamma}(\varsigma;q) \leq \frac{2[n]_q(1-\varsigma)\sqrt{[n]_q\varsigma}}{[n-1]_q^2} \leq 4\frac{(1-\varsigma)\sqrt{\varsigma}}{\sqrt{[n]_q}}.$$

Subcase b) Let $[\gamma]_q \leq [\zeta]_q - \sqrt{[\zeta+1]_q + \frac{[\gamma+1]_q^2}{[n]_q}}$. Since the function $g(\zeta) = [\zeta]_q - \sqrt{[\zeta+1]_q + \frac{[\gamma+1]_q^2}{[n]_q}}$ is non-decreasing according to $\zeta \geq 0$, it results that there exists $\overline{\zeta} \in \{1,2,\cdots\}$, of maximum value such that $[\overline{\zeta}]_q - \sqrt{[\overline{\zeta}+1]_q + \frac{[\gamma+1]_q^2}{[n]_q}} < [\gamma]_q$. Then for $\zeta_1 = \overline{\zeta}+1$ we get $[\zeta_1]_q - \sqrt{[\zeta_1+1]_q + \frac{[\gamma+1]_q^2}{[n]_q}} \geq [\gamma]_q$. Additionally, by choosing $\overline{\zeta}$ means that $\zeta_1 \leq 2[\gamma+1]_q$. Hence, as in the prior case we obtain

$$P_{\overline{\zeta},n,\gamma}(\varsigma;q) = p_{\overline{\zeta},n,\gamma}(\varsigma;q)\left(\frac{[\overline{\zeta}+1]_q}{[n+\overline{\zeta}+1]_q} - \varsigma\right) \leq 4\frac{(1-\varsigma)\sqrt{\varsigma}}{\sqrt{[n]_q}}.$$

Since $[\zeta_1]_q > [\zeta_1]_q - \sqrt{[\zeta_1+1]_q + \frac{[\gamma+1]_q^2}{[n]_q}} \geq [\gamma]_q$, it follows $[\zeta_1]_q \geq [\gamma+1]_q$ and Lemma 4 (i), it means that

$$P_{\overline{\zeta}+1,n,\gamma}(\varsigma;q) \geq P_{\overline{\zeta}+2,n,\gamma}(\varsigma;q) \geq \cdots.$$

Therefore, we get $P_{\zeta,n,\gamma}(\varsigma;q) \leq 4\frac{(1-\varsigma)\sqrt{\varsigma}}{\sqrt{[n]_q}}$ for any $\zeta \in \left\{\overline{\zeta}+1, \overline{\zeta}+2, \cdots\right\}$.

Case 2) Let's suppose $\zeta \in \{0,1,\cdots,\gamma\}$.

Subcase a) Firstly, we suppose that $[\gamma]_q \leq [\zeta]_q + \sqrt{[\zeta]_q + \frac{[\gamma]_q^2}{[n]_q}}$ and this condition implies that $[\zeta]_q \geq [\gamma]_q - \sqrt{[\zeta]_q + \frac{[\gamma]_q^2}{[n]_q}}$. Hence, we get

$$\begin{aligned} P_{\zeta,n,\gamma}(\varsigma;q) =& p_{\zeta,n,\gamma}(\varsigma;q)\left(\varsigma - \frac{[\zeta]_q}{[n+\zeta]_q}\right) \leq \frac{[\gamma+1]_q}{[n+\gamma+1]_q} - \frac{[\zeta]_q}{[n+\zeta]_q} \\ \leq& \frac{[\gamma+1]_q}{[n+\gamma+1]_q} - \frac{[\gamma]_q - \sqrt{[\zeta]_q + \frac{[\gamma]_q^2}{[n]_q}}}{[n]_q + [\gamma]_q - \sqrt{[\zeta]_q + \frac{[\gamma]_q^2}{[n]_q}}} \\ =& \frac{[n]_q\left(\sqrt{[\zeta]_q + \frac{[\gamma]_q^2}{[n]_q}} + q^{\gamma}\right)}{[n+\gamma+1]_q\left([n]_q + [\gamma]_q - \sqrt{[\zeta]_q + \frac{[\gamma]_q^2}{[n]_q}}\right)}. \end{aligned}$$

By taking $\zeta \leq \gamma$, we get

$$P_{\zeta,n,\gamma}(\varsigma;q) = \frac{[n]_q\left(\sqrt{[\gamma]_q + \frac{[\gamma]_q^2}{[n]_q}} + q^\gamma\right)}{[n+\gamma+1]_q\left([n]_q + [\gamma]_q - \sqrt{[\gamma]_q + \frac{[\gamma]_q^2}{[n]_q}}\right)}$$
$$\leq \frac{[n]_q\left(\sqrt{\frac{[n]_q\varsigma}{1-\varsigma}\frac{1}{1-\varsigma}} + q^\gamma\right)}{[n+\gamma+1]_q\left([n]_q + [\gamma]_q - \sqrt{\frac{[n]_q\varsigma}{1-\varsigma}\frac{1}{1-\varsigma}}\right)}.$$

From $\frac{[n]_q+\varsigma.q^{n+\gamma}-q^\gamma}{1-\varsigma} \leq [n+\gamma]_q$,

$$P_{\zeta,n,\gamma}(\varsigma;q) \leq \frac{(1-\varsigma)[n]_q\left(\sqrt{[n]_q\varsigma} + q^\gamma(1-\varsigma)\right)}{([n]_q + (q^n-1)q^\gamma)\left([n]_q + \varsigma q^{n+\gamma} - q^\gamma - \sqrt{[n]_q\varsigma}\right)}$$
$$\leq \frac{(1-\varsigma)\left(\sqrt{[n]_q\varsigma} + q^\gamma(1-\varsigma)\right)}{[n]_q + \varsigma q^{n+\gamma} - q^\gamma - \sqrt{[n]_q\varsigma}}.$$

Besides, from $\gamma \geq \frac{1}{n+1}$ and $n \geq 4$ we get $1-\varsigma+\sqrt{[n]_q\varsigma} \leq \sqrt{[n]_q}$ and $\sqrt{[n]_q\varsigma}+1 \leq \frac{5}{2}\sqrt{[n]_q\varsigma}$. Therefore,

$$P_{\zeta,n,\gamma}(\varsigma;q) \leq \frac{(1-\varsigma)\left(\sqrt{[n]_q\varsigma} + q^\gamma\right)}{[n]_q - \sqrt{[n]_q}} \leq \frac{5}{2}\frac{(1-\varsigma)\sqrt{[n]_q}\sqrt{\varsigma}}{[n]_q - \sqrt{[n]_q}}$$
$$\leq \frac{5(1-\varsigma)\sqrt{\varsigma}}{\sqrt{[n]_q}}.$$

Subcase b) Let $[\gamma]_q \geq [\zeta]_q + \sqrt{[\zeta]_q + \frac{[\gamma]_q^2}{[n]_q}}$ and $\widetilde{\zeta} \in \{1,2,\cdots,\gamma\}$ be the minimum value such that $[\widetilde{\zeta}]_q + \sqrt{[\widetilde{\zeta}]_q + \frac{[\gamma]_q^2}{[n]_q}} > [\gamma]_q$. So, $\zeta_2 = \widetilde{\zeta} - 1$ verifies $[\zeta_2]_q + \sqrt{[\zeta_2]_q + \frac{[\gamma]_q^2}{[n]_q}} \leq [\gamma]_q$ and similar to sub case (a), we get

$$P_{\widetilde{\zeta}-1,n,\gamma}(\varsigma;q) = p_{\widetilde{\zeta}-1,n,\gamma}(\varsigma;q)\left(\varsigma - \frac{[\widetilde{\zeta}-1]_q}{[n+\widetilde{\zeta}-1]_q}\right) \leq \frac{[\gamma+1]_q}{[n+\gamma+1]_q} - \frac{[\widetilde{\zeta}-1]_q}{[n+\widetilde{\zeta}-1]_q}$$
$$\leq \frac{[\gamma+1]_q}{[n+\gamma+1]_q} - \frac{[\gamma]_q - \sqrt{[\widetilde{\zeta}]_q + \frac{[\gamma]_q^2}{[n]_q}} - 1}{[n]_q + [\gamma]_q - \sqrt{[\widetilde{\zeta}]_q + \frac{[\gamma]_q^2}{[n]_q}} - 1}$$
$$\leq \frac{[n]_q\left(\sqrt{[\widetilde{\zeta}]_q + \frac{[\gamma]_q^2}{[n]_q}} + 1 + q^\gamma\right)}{[n+\gamma+1]_q\left([n]_q + [\gamma]_q - \sqrt{[\widetilde{\zeta}]_q + \frac{[\gamma]_q^2}{[n]_q}} - 1\right)}$$
$$\leq \frac{[n]_q\left(\sqrt{[\gamma]_q + \frac{[\gamma]_q^2}{[n]_q}} + 1 + q^\gamma\right)}{[n+\gamma+1]_q\left([n]_q + [\gamma]_q - \sqrt{[\gamma]_q + \frac{[\gamma]_q^2}{[n]_q}} - 1\right)}$$

$$\leq \frac{[n]_q\left(\sqrt{\frac{[n]_q\varsigma}{1-\varsigma}\frac{1}{1-\varsigma}+\frac{[\gamma]_q^2}{[n]_q}}+1+q^\gamma\right)}{[n+\gamma+1]_q\left([n]_q+[\gamma]_q-\sqrt{\frac{[n]_q\varsigma}{1-\varsigma}\frac{1}{1-\varsigma}+\frac{[\gamma]_q^2}{[n]_q}}-1\right)}$$

$$\leq \frac{(1-\varsigma)(\sqrt{[n]_q\varsigma}+1+q^\gamma)}{[n]_q-1-q^{n+\gamma}+(1+q^{n+\gamma})\varsigma-\sqrt{[n]_q x}}.$$

From $\gamma \geq \frac{1}{n+1}$ and $n \geq 4$, we immediately get $\sqrt{[n]_q x}+1+q^\gamma \leq (1+\sqrt{5})\sqrt{[n]_q\varsigma}$ and it follows that

$$P_{\tilde{\zeta}-1,n,\gamma}(\varsigma;q) \leq \frac{(1+\sqrt{5})\sqrt{[n]_q\varsigma}(1-\varsigma)}{[n]_q-1-q^{n+\gamma}+(1+q^{n+\gamma})\varsigma-\sqrt{[n]_q\varsigma}}.$$

Let $h(\varsigma)=[n]_q-1-q^{n+\gamma}+(1+q^{n+\gamma})\varsigma-\sqrt{[n]_q\varsigma}$, $\varsigma \geq 0$. It is easy to show that h has a global minimum in $\varsigma_0=\frac{n}{4(1+q^{n+\gamma})^2}$. It means that $h(\varsigma) \geq h(\frac{n}{4(1+q^{n+\gamma})^2})=\frac{(3+4q^{n+\gamma})[n]_q-4(1+q^{n+\gamma})^2}{4(1+q^{n+\gamma})^2}$. Therefore, we obtain

$$P_{\tilde{\zeta}-1,n,\gamma}(\varsigma;q) \leq \frac{4(1+q^{n+\gamma})(1+\sqrt{5})}{3}\frac{(1-\varsigma)\sqrt{\varsigma}}{\sqrt{n}}.$$

Lemma 4 (ii) gives us to $P_{\tilde{\zeta}-1,n,\gamma}(\varsigma;q) \geq P_{\tilde{\zeta}-2,n,\gamma}(\varsigma;q) \geq \cdots \geq P_{0,n,\gamma}(\varsigma;q)$. Hence, we have

$$P_{\zeta,n,\gamma}(x;q) \leq \frac{4(1+q^{n+\gamma})(1+\sqrt{5})}{3}\frac{(1-\varsigma)\sqrt{\varsigma}}{\sqrt{n}},$$

for any $\zeta \leq \gamma$. Collecting all the estimates obtained above, we have $Z_n^{(M)}\left(\eta_\varsigma,\varsigma;q\right) \leq \frac{4(1+q^{n+\gamma})(1+\sqrt{5})}{3}\frac{(1-\varsigma)\sqrt{\varsigma}}{\sqrt{n}}$ for all $\varsigma \in [0,1]$ and choosing $\delta_n=\frac{4(1+q^{n+\gamma})(1+\sqrt{5})}{3}\frac{(1-\varsigma)\sqrt{\varsigma}}{\sqrt{n}}$ in the inequality given in (6), we get the proof of the theorem.

5 Conclusion

In this paper, nonlinear max-product type q-Meyer-König and Zeller operators have been introduced. Additionally, the modulus of continuity has been used to investigate the degree of approximation and the rate of convergence of the operators. As a result, the max-product type q-Meyer-König and Zeller operators approximated better than the classical linear q-Meyer-König and Zeller operators. In future studies, the shape preservation properties of these operators may be studied, and comparable research may be incorporated into more practical operator frameworks.

6 Declarations

6.1 Competing interests:

The authors declare that there is no conflict of interest regarding the publication of this paper.

6.2 Author's contributions:

E.A.-Conceptualization, Methodology, Formal Analysis, Writing-Review and Editing. Ö.Ö.G.-Formal Analysis, Validation, Writing-Original Draft. S.K.S.-Formal Analysis, Validation, Resources. All authors read and approved the final submitted version of this manuscript.

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All data that support the findings of this study are included within the article.

6.6 Using of AI tools:

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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