sciendo International Journal of Mathematics and Computer in Engineering 2(2) (2024) 233–250

International Journal of Mathematics and Computer in Engineering

https://sciendo.com/journal/IJMCE

Original Study

On the complex version of the Cahn–Hilliard–Oono type equation for long interactions phase separation

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Communicated by Juan L.G. Guirao; Received: 18.08.2023; Accepted: 16.11.2023; Online: 10.01.2024

Abstract

This paper focuses on the complex version of the Cahn-Hilliard-Oono equation with Neumann boundary conditions, which is used to capture long-range nonlocal interactions in the phase separation process. The first part of the paper establishes the well-posedness of the corresponding stationary problem associated with the equation. Subsequently, a numerical model is constructed using a finite element discretization in space and a backward Euler scheme in time. We demonstrate the existence of a unique solution to the stationary problem and obtain error estimates for the numerical solution. This, in turn, serves as proof of the convergence of the semi-discrete scheme to the continuous problem. Finally, we establish the convergence of the fully discrete problem to the semi-discrete formulation.

Keywords: Nonlocal Cahn–Hilliard–Oono, well-posedness, steady-state solution, numerical analysis, finite element, phase separation.
AMS 2020 codes: 35Q92; 35J30; 35J60; 65M60; 65M12.

1 Introduction

In this article, we are interested in the following boundary value problem:

$$\frac{\partial \varphi}{\partial t} + \varepsilon \Delta^2 \varphi - \frac{1}{\varepsilon}\Delta f(\varphi) + \alpha \varphi = 0, \qquad \text{in } \Omega \times [0,T], \tag{1}$$

$$\frac{\partial \varphi}{\partial \nu} = \frac{\partial \Delta \varphi}{\partial \nu} = 0, \qquad \text{on } \Gamma, \tag{2}$$

$$\varphi(x,0) = \varphi_0(x), \qquad \text{in } \Omega, \tag{3}$$

in a bounded and regular domain $\Omega \subset \mathbb{R}^n$, $n \leqslant 3$, with boundary Γ and $T > 0$. The initial datum $\varphi_0(x) = \varphi_{0,1}(x) + i\varphi_{0,2}(x)$ satisfies the physical constraint $|\varphi_0| = 1$, where the real part $\varphi_{0,1}(x)$ represents the initial

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sciendo
ISSN 2956-7068 doi:10.2478/ijmce-2024-0018 sciendo

concentration of the metallic components (the concentration of all phases is between 0 and 1), and the imaginary part $\varphi_{0,2}(x) = \sqrt{1-\varphi_{0,1}^2(x)}$. Furthermore, $\varphi = \varphi_1 + i\varphi_2$ is the phase variable.
Equation (1) is the generalization of the original Cahn–Hilliard equation, which plays an essential role in material sciences as it describes the phase separation of binary systems in physics and chemistry. In 1958, Cahn and Hilliard [1–4] presented the equation (1) in the form of free energy, which later led to the development of the Cahn–Hilliard equation as a partial differential equation based on thermodynamic principles [5]. When a binary solution is cooled down sufficiently, the phase separation may occur in two ways: either by nucleation, in which case nuclei of the second phase appear randomly and grow, or the whole solution appears to nucleate at once, and then periodic or semi-periodic structures appear in the so-called spinodal decomposition. The pattern formation resulting from phase separation has been observed in alloys, glasses, and polymer solutions. The Cahn-Hilliard equation has many applications in material science and biology [6–23].
The function $f : \mathbb{C} \to \mathbb{C}$ belongs to $\mathscr{C}^2(\mathbb{C},\mathbb{C})$ and it satisfies the following standard dissipativity assumption:

$$\liminf_{|z|\to\infty} Re(f'(z)) > 0.$$

One typical choice for this function is

$$f(z) = |z|^2 z - z. \tag{4}$$

In this article, we prove the existence of a unique weak solution to the steady-state problem associated with (1)–(2), using the method of fixed-point arguments. Subsequently, we consider a numerical scheme based on a finite element space discretization in space and Backward Euler discretization in time. After obtaining some error estimates for the semi-discrete solution, we demonstrate the convergence of the semi-discrete solution to the continuous one. Finally, we establish the stability of the Backward Euler scheme, which is the key to achieve the convergence of the fully discrete scheme to the continuous problem.
Our primary objective in this article is to propose a straightforward model for a grayscale multi-component phase separation that preserves the advantages of the phase separation achieved with the Cahn–Hilliard model. Specifically, it is computationally efficient and exhibits rapid convergence times. Notably, we can replicate the results of the two-phase separation by computing only two solutions (the real and imaginary parts of the order parameter), regardless of the number of phases in the initial multi-component metal.

Notations

Setting

$$\langle \phi \rangle = \frac{1}{\mathbf{Vol}(\Omega)} \int_\Omega \phi(x)\,dx,$$

we introduce the following spaces:

$$\overline{H^{-1}(\Omega)} = \{\tau \in H^{-1}(\Omega), \langle \tau, 1\rangle_{H^{-1},H^1} = 0\},$$

$$\overline{L(\Omega)} = \{\tau \in L^2(\Omega), \langle \tau \rangle = 0\},$$

and

$$\overline{H^1(\Omega)} = \{\tau \in H^1(\Omega), \langle \tau \rangle = 0\},$$

which are the H^{-1}, L^2 and H^1 spaces with zero spatial average, respectively.

2 Well-posedness of the steady state problem

In this section, we prove the existence of a weak solution for the stationary problem associated with (1)–(2):

$$\varepsilon\Delta^2\varphi - \frac{1}{\varepsilon}\Delta f(\varphi) + \alpha\varphi = 0 \qquad \text{in} \qquad \Omega, \tag{5}$$

$$\frac{\partial\varphi}{\partial\nu} = \frac{\partial\Delta\varphi}{\partial\nu} = 0 \qquad \text{on} \qquad \Gamma. \tag{6}$$

We begin by integrating equation (5) across the domain Ω. Then, taking into account the boundary conditions, we find

$$\langle\alpha\varphi\rangle = 0. \tag{7}$$

We now prove the existence of a solution to the variational problem of (5)-(6) as follows.

We consider the fixed point operator

$$T : L^2(\Omega) \to L^2(\Omega), \qquad \tau \to T(\tau) = \varphi,$$

where τ is chosen from $L^2(\Omega)$, and we consider the following equations:

$$\frac{1}{r}(\varphi-\tau) + \varepsilon\Delta^2\varphi - \frac{1}{\varepsilon}f(\varphi) + \alpha\varphi = 0 \quad \text{in} \quad \Omega, \tag{8}$$

$$\frac{\partial\varphi}{\partial\nu} = \frac{\partial\Delta\varphi}{\partial\nu} = 0 \quad \text{on} \quad \Gamma, \tag{9}$$

where α is a positive constant. Integrating (8) over Ω, we find (7). Therefore, (8) can be rewritten as

$$(\frac{1}{r}+\alpha)\langle\varphi\rangle = \frac{1}{r}\langle\tau\rangle. \tag{10}$$

The variational formulation of (10) reads as follows:

$$\varepsilon((\nabla\varphi,\nabla\rho)) + \frac{1}{\varepsilon}((f(\varphi),\rho)) + \alpha(((-\Delta)^{-\frac{1}{2}}(\varphi - <\varphi>), (-\Delta)^{-\frac{1}{2}}\rho)) = 0,$$

for $\rho \in \overline{H^1(\Omega)}$. In addition, the functional of the variational formulation is given by

$$\mathscr{F}(\varphi,\tau) = \mathscr{J}(\varphi) + \frac{1}{2r}||\varphi - \tau - (\langle\varphi\rangle - \langle\tau\rangle>)||^2_{-1} + \frac{\alpha}{2}||\varphi - \langle\varphi\rangle||^2_{-1}, \tag{11}$$

where $\mathscr{J}(\varphi) = \frac{\varepsilon}{2}\int_\Omega |\nabla\varphi|^2 dx + \frac{1}{\varepsilon}\int_\Omega F(\varphi)dx$ and $||.||_{-1}$ is the norm defined in $\overline{H^{-1}}$.

Lemma 1. *Setting* $F(z) = \frac{1}{4}|z|^4 - \frac{1}{2}|z|^2$, *we have*

$$F(z) + F(q) - 2F(\frac{z+q}{2}) > -\frac{1}{4}|z-q|^2,$$

for all $z \in \mathbb{C}^*$ *and* z *is non null.*

Proof. Noting that

$$F(z) + F(q) - 2F(\frac{z+q}{2}) = \frac{1}{4}|z|^4 - \frac{1}{2}|z|^2 + \frac{1}{4}|q|^4 - \frac{1}{2}|q|^2 - \frac{1}{32}|z+q|^4 + \frac{1}{4}|z+q|^2.$$

Furthermore, $|z+q|^4 \leq 4(|z|^2+|q|^2)^2$ and $|z+q|^2 = |z|^2+|q|^2+2Re(\bar{z}q)$,
which yields

$$\begin{aligned} F(z) + F(q) - 2F(\frac{z+q}{2}) &\geq \frac{1}{8}|z|^4 + \frac{1}{8}|q|^4 - \frac{1}{4}|z|^2|q|^2 - \frac{1}{4}(|z|^2+|q|^2-2Re(\bar{z}q)) \\ &\geq \frac{1}{8}(|z|^2-|q|^2)^2 - \frac{1}{4}|z-q|^2 > -\frac{1}{4}|z-q|^2. \end{aligned}$$

Proposition 2. *The equation (8) has a solution in $H^1(\Omega)$. Furthermore, if $r \leq \varepsilon^3$ $(*)$, then this solution is unique.*

Proof. We show that there exists a unique minimizer (say φ^*) of $\mathscr{F}$ provided that (*) holds. First of all, notice that there are two positive constants c_1 and c_2, such that

$$F(\varphi) = \frac{1}{4}|\varphi|^4 - \frac{1}{2}|\varphi|^2 \geq c_1|\varphi|^2 - c_2.$$

Secondly,

$$\begin{aligned}
&\mathscr{F}(\varphi,\tau) \geq \frac{\varepsilon}{2}||\nabla\varphi||^2 + \frac{c_1}{\varepsilon}||\varphi||^2 - \frac{c_2}{\varepsilon} \\
&+\frac{1}{2r}[\frac{1}{2}||\varphi - \langle\varphi\rangle||_{-1}^2 - ||\tau - \langle\tau\rangle||_{-1}^2] + \frac{\alpha}{2}||\varphi - \langle\varphi\rangle||_{-1}^2 \\
&\geq \frac{\varepsilon}{2}||\nabla\varphi||^2 + \frac{c_1}{\varepsilon}||\varphi||^2 + (\frac{1}{4r} + \frac{\alpha}{2})||\varphi - \langle\varphi\rangle||_{-1}^2 + c,
\end{aligned} \tag{12}$$

where c is a constant that depends on Ω, ε and c_2. Consequently, we deduce from (12) that the functional $\mathscr{F}(\varphi,\tau)$ is coercive, and hence $\mathscr{F}$ has a minimizing sequence $\varphi^n \in H^1(\Omega)$. The sequence φ^n is now bounded in $H^1(\Omega)$. Therefore, there exists a subsequence of φ^n that we shall not rename, such that φ^n converges weakly to $\varphi^* \in H^1(\Omega)$. Additionally, φ^n converges strongly to φ^* in $L^2(\Omega)$, due to the fact that $H^1(\Omega)$ is compactly embedded in $L^2(\Omega)$. We now recall that

$$\mathscr{F}(\varphi^n,\tau) = \mathscr{J}(\varphi^n) + \frac{1}{2r}||\varphi^n - \tau - \langle\varphi^n - \tau\rangle||_{-1}^2 + \frac{\alpha}{2}||\varphi^n - \langle\varphi^n\rangle||_{-1}^2, \tag{13}$$

which implies that

$$\begin{aligned}
&||\varphi^n - \tau - \langle\varphi^n - \tau\rangle||_{-1}^2 - ||\varphi^* - \tau - \langle\varphi^* - \tau\rangle||_{-1}^2 \\
&= (((-\Delta^{-1})(\varphi^n - \tau - \langle\varphi^n - \tau\rangle), \varphi^n - \tau - \langle\varphi^n - \tau\rangle)) \\
&-(((-\Delta^{-1})(\varphi^* - \tau - \langle\varphi^* - \tau\rangle), \varphi^* - \tau - \langle\varphi^* - \tau\rangle)) \\
&= (((-\Delta^{-1})(\varphi^n - \varphi^* - \langle\varphi^n - \varphi^*\rangle), \varphi^n - \tau - \langle\varphi^n - \tau\rangle)) \\
&+(((-\Delta^{-1})(\varphi^* - \tau - \langle\varphi^* - \tau\rangle, \varphi^n - \tau - \langle\varphi^n - \tau\rangle)) \\
&\leqslant ||\varphi^n - \varphi^* - \langle\varphi^n - \varphi^*\rangle||.||\varphi^n - \tau - \langle\varphi^n - \tau\rangle|| \\
&+||\varphi^n - \varphi^* - \langle\varphi^n - \varphi^*\rangle||.||\varphi^n - \tau - \langle\varphi^n - \tau\rangle||,
\end{aligned}$$

and

$$\begin{aligned}
&||\varphi^n - \langle\varphi^n\rangle||_{-1}^2 - ||\varphi^* - \langle\varphi^*\rangle||_{-1}^2 \\
&= (((-\Delta^{-1})(\varphi^n - \varphi^* - \langle\varphi^n - \varphi^*\rangle, \varphi^n - \langle\varphi^n\rangle)) \\
&+(((-\Delta^{-1})(\varphi^* - \langle\varphi^*\rangle), \varphi^n - \varphi^* - \langle\varphi^n - \varphi^*\rangle))
\end{aligned}$$

$$\leq ||\varphi^n - \varphi^* - \langle \varphi^n - \varphi^* \rangle||.||\varphi^n - \langle \varphi^n \rangle|| + ||\varphi^n - \varphi^* - \langle \varphi^n - \varphi^* \rangle||.||\varphi^* - \langle \varphi^* \rangle||,$$

and since $\varphi^n \to \varphi^* \in L^2(\Omega)$ strongly, then

$$||\varphi^n - \tau - \langle \varphi^n - \tau \rangle||_{-1}^2 \to ||\varphi^* - \tau - \langle \varphi^* - \tau \rangle||_{-1}^2 \tag{14}$$

strongly, and

$$||\varphi^n - \langle \varphi^n \rangle|| \to ||\varphi^* - \langle \varphi^* \rangle||_{-1}^2 \tag{15}$$

strongly. Furthermore, as F is continuous, we observe that $F(\varphi^n)$ converges to $F(\varphi^*)$. By applying Fatou's Lemma, we can deduce that

$$\mathscr{F}(\varphi^*, \tau) \leq \liminf \mathscr{F}(\varphi^n, \tau). \tag{16}$$

Therefore, $\mathscr{F}$ has a minimizer in $H^1(\Omega)$, i.e., $\exists\, \varphi^* \in H^1(\Omega)$ such that $\varphi^* = \arg\min\, \mathscr{F}(\varphi, \tau)$. Furthermore, with the assistance of the trace function and Neumann boundary conditions, we can easily prove that φ^* satisfies the Neumann boundary condition $\frac{\partial \varphi^*}{\partial v} = 0$, and φ^* serves as a weak solution for the variational problem. Next, we demonstrate the uniqueness of φ^*; for this purpose, we prove that the functional $\mathscr{F}$ is strictly convex. Let v and w belong to $H^1(\Omega)$ such that $\varphi = v - w$. (Note: $\langle \varphi \rangle = \langle v \rangle - \langle w \rangle = 0$). Now, by employing interpolation and Young's inequalities, we obtain

$$\begin{aligned}
&\mathscr{F}(v, \tau) + \mathscr{F}(w, \tau) - 2\mathscr{F}(\frac{v+w}{2}, \tau) \\
&\geq \frac{\varepsilon}{4}||\nabla \varphi||^2 + (\frac{\alpha}{2} + \frac{1}{4r})||\varphi||_{-1}^2 - \frac{1}{4\varepsilon}||\varphi||_{-1}.||\nabla \varphi|| \\
&\geq \frac{\varepsilon}{4}||\nabla \varphi||^2 + (\frac{\alpha}{2} + \frac{1}{4r})||\varphi||_{-1}^2 - \frac{1}{4\varepsilon}(\frac{1}{2\varepsilon^2}||\varphi||_{-1}^2 + \frac{\varepsilon^2}{2}||\nabla \varphi||^2) \\
&\geq (\frac{\alpha}{2} + \frac{1}{4r})||\varphi||_{-1}^2 - \frac{1}{4\varepsilon^3}||\varphi||_{-1}^2 \\
&> 0,
\end{aligned} \tag{17}$$

under the assumption (*). As a result, $\mathscr{F}$ is strictly convex, and the weak solution is unique.

Proposition 3. *The operator T has a unique fixed point under the two specified conditions: (*) stated above and (**) defined below,*

$$\frac{1}{2\alpha - \frac{1}{\varepsilon^2}} \leq r \leq \frac{1}{\alpha - \frac{1}{\varepsilon^2} - \frac{a^2\varepsilon}{2}}.$$

Proof. We show that with the help of the two specified conditions, we can restrict the operator T to a compact convex set. By applying Schauder's fixed-point theorem, we can establish the existence of at least one fixed point, denoted as φ^*. Furthermore, we can conclude the uniqueness of φ^* based on the property that the functional $\mathscr{F}$ is strictly convex. To simplify matters, we denote $\varphi = \varphi^*$. Now, let's reframe the problem as follows:

$$\frac{1}{r}((\varphi - \tau, \varphi)) + \varepsilon||\Delta \varphi||^2 + \frac{1}{\varepsilon}((\nabla f(\varphi), \nabla \varphi)) + ((\alpha \varphi, \varphi)) = 0. \tag{18}$$

Take into account (4), we find

$$\frac{1}{r}((\varphi - \tau, \varphi)) + \varepsilon||\Delta \varphi||^2 \leq \frac{1}{\varepsilon}||\nabla \varphi||^2 - \alpha||\varphi||^2. \tag{19}$$

Therefore,

$$(\frac{1}{r}+\alpha)||\varphi||^2+\varepsilon||\Delta\varphi||^2 \le \frac{1}{\varepsilon}||\nabla\varphi||^2+\frac{1}{2r}\int_\Omega |\varphi|^2dx+\frac{1}{2r}\int_\Omega |\tau|^2dx \tag{20}$$
$$\le \frac{1}{\varepsilon}||\nabla\varphi||^2+\frac{1}{2r}||\varphi||^2+k,$$

where k is a constant depending on τ, r, and Ω. It then follows that

$$(\alpha+\frac{1}{2r})||\varphi||^2+\varepsilon||\Delta\varphi||^2 \le \frac{1}{\varepsilon}||\nabla\varphi||^2+k.$$

Furthermore, through the utilization of the interpolation inequality followed by Young's inequality, we obtain

$$||\nabla\varphi||^2 \le ||\varphi||.||\varphi||_{H^2(\Omega)} \le \frac{1}{2\varepsilon^2}||\varphi||^2+\frac{\varepsilon^2}{2}||\Delta\varphi||^2+\frac{\varepsilon^2}{2}\langle\varphi\rangle^2$$
$$\le \frac{1}{2\varepsilon^2}||\varphi||^2+\frac{\varepsilon^2}{2}||\Delta\varphi||^2+\frac{a^2\varepsilon^2}{2}\langle\tau\rangle^2.$$

Consequently,

$$(\alpha+\frac{1}{2r})||\varphi||^2+\varepsilon||\Delta\varphi||^2 \le \frac{1}{2\varepsilon^3}||\varphi||^2+\frac{\varepsilon}{2}||\Delta\varphi||^2+\frac{\varepsilon a^2}{2}\langle\tau\rangle^2+k. \tag{21}$$

Thus,

$$(\alpha+\frac{1}{2r}-\frac{1}{2\varepsilon^3})||\varphi||^2+\frac{\varepsilon}{2}||\Delta\varphi||^2 \le \mathscr{E}||\tau||^2+k, \tag{22}$$

such that $\mathscr{E}=\frac{\varepsilon a^2}{2}$ with $a=\frac{\frac{1}{r}}{\frac{1}{r}+\alpha}$. Under the assumptions (*) and (**), we find

$$||\varphi||^2=||T(\tau)||^2 \le \mathscr{E}'||\tau||^2+\mathscr{K}', \tag{23}$$

where $\mathscr{E}'$ and $\mathscr{K}'<1$. Therefore, φ remains bounded in $L^2(\Omega)$, and T now represents a mapping from the closed ball

$$K=B[0,M]=\varphi,\in,L^2(\Omega);\|\varphi\|_{L^2(\Omega)} \le M$$

to itself, with an appropriate constant $M>0$.
Furthermore, due to the stationary problem, we obtain the following inequality:

$$||\Delta\varphi||^2_{L^2(\Omega)} \le c||\tau||^2_{L^2(\Omega)}+c'. \tag{24}$$

Since $\langle\varphi\rangle$ is null, we conclude that φ is uniformly bounded in $H^2(\Omega)$, and it follows that $B[0,M]$ is compact and convex in $L^2(\Omega)$. It is also clear that T is continuous, which leaves us to show that T is compact. Consider the sequence

$$\tau^n \to \tau \in L^2(\Omega), T(\tau^n)=\varphi^n;$$

φ^n is bounded in $H^1(\Omega)$ for all n. Then, by taking a subsequence (which we do not rename), we have: φ^n weakly converges to $\varphi \in H^1(\Omega)$, and φ^n strongly converges to φ in $L^2(\Omega)$ using the Rellich-Kondrachov compactness theorem. In addition, since f is continuous, $f(\varphi^n)$ converges to $f(\varphi)$ almost everywhere, and $f(\varphi^n)$ is bounded in $L^2(\Omega)$; then, $f(\varphi^n)$ weakly converges to $f(\varphi)$ in $L^2(\Omega)$ due to the weak dominated convergence theorem. Thus, $\varphi=T(\tau)$ is a weak unique solution for (8), thanks to the previous proposition, and T is a continuous operator. Finally, by applying Schauder's Theorem, the operator T has a fixed point in $L^2(\Omega)$, which is the unique solution of the stationary problem.

3 Numerical analysis of the evolution problem

The given problem can be reformulated as follows:

$$\varphi_t = \frac{\partial \varphi}{\partial t} = \Delta w - \alpha\varphi \text{ in } \Omega, \tag{25}$$

$$w = \frac{1}{\varepsilon} f(\varphi) - \varepsilon\Delta\varphi \text{ in } \Omega, \tag{26}$$

$$\frac{\partial \varphi}{\partial \nu} = \frac{\partial \Delta\varphi}{\partial \nu} \text{ on } \Gamma. \tag{27}$$

The variational formulation of (25)-(27) is as follows:

$$((\varphi_t, \phi)) = -((\nabla w, \nabla\phi)) - \alpha((\varphi, \phi)), \tag{28}$$

$$((w, \psi)) = \frac{1}{\varepsilon}((f(\varphi), \psi)) + \varepsilon((\nabla\varphi, \nabla\psi)), \tag{29}$$

for all ϕ, $\psi \in H^1(\Omega)$. In our approach, we employ a quasi-uniform family of decompositions denoted as Ω^h to effectively partition the domain Ω into k-simplices. Within this discretized framework, given a specific triangulation $\Omega^h = \bigcup_{T^h \in \Omega^h} T$, we establish the conventional P^1 conforming finite element space, denoted as V^h. This space, characterized by functions m^h belonging to $C^0(\overline{\Omega})$ with the property that $m^h|T$ is affine for all $T \in \Omega^h$, plays a critical role in our numerical analysis. Notably, we observe that V^h is a subset of the more general function space $H^1(\Omega)$. To facilitate our computations, we introduce the function I^h_φ, which represents a unique element within V^h and precisely replicates the values of the function φ at the nodes of the triangulation. It is important to note that our methodology aligns with the following well-established standard approximation result, affirming the reliability of our numerical approach

$$\|\varphi - I^h_\varphi\|_{L^2(\Omega)} + h\|\varphi - I^h_\varphi\|_{H^1(\Omega)} \leq Ch^2\|\varphi\|_{H^2(\Omega)} \text{ for all } \varphi \in H^2(\Omega). \tag{30}$$

Here, $C > 0$ is a constant that solely depends on Ω^h. Additionally, the inverse estimate below still remains valid (refer to [24]).

$$||m^h||_{C^0(\bar{\Omega})} \leq Ch^{\frac{-n}{2}}||m^h||_{L^2(\Omega)} \quad \text{for all } m^h \in V^h. \tag{31}$$

Setting

$$\overline{V^h} = V^h \cap \overline{L(\Omega)}.$$

The discrete version of (28)-(29) can be written as follows: Find $(\varphi^h, w^h) : [0,T] \longrightarrow V^h \times V^h$ such that they satisfy the following conditions:

$$((\varphi^h_t, \phi)) = -((\nabla w^h, \nabla\phi)) - \alpha((\varphi^h, \phi), \tag{32}$$

$$((w^h, \psi)) = \frac{1}{\varepsilon}((f(\varphi^h), \psi)) - \varepsilon((\nabla\varphi^h, \nabla\psi)), \tag{33}$$

for all $\phi, \psi \in V^h$.

3.1 Error estimates

Setting

$$\varphi^h(t)-\varphi(t)=\theta^\varphi+\beta^\varphi,\ \text{with}\ \theta^\varphi=\varphi^h-{\varphi_e}^h\ \text{and}\ \beta^\varphi={\varphi_e}^h-\varphi, \tag{34}$$

$$w^h(t)-w(t)=\theta^w+\beta^w,\ \text{with}\ \theta^w=w^h-{w_e}^h\ \text{and}\ \beta^w={w_e}^h-w, \tag{35}$$

for all $t\in[0,T]$, where ${w_e}^h={w_e}^h(t)$ represents the elliptic projection of $w=w(t)$, and ${\varphi_e}^h={\varphi_e}^h(t)$ is the elliptic projection of $\varphi=\varphi(t)$. These projections satisfy the following conditions:

$$((\nabla {w_e}^h,\nabla\psi))=((\nabla w,\nabla\psi))\qquad \text{for all}\ \psi\in\overline{H^1(\Omega)}, \tag{36}$$

$$(({w_e}^h,1))=((w,1)), \tag{37}$$

$$((\nabla {\varphi_e}^h,\nabla\psi))=((\nabla \varphi,\nabla\psi))\qquad \text{for all}\ \psi\in\overline{H^1(\Omega)}, \tag{38}$$

$$((\varphi_e^h,1))=((\varphi,1)). \tag{39}$$

Using the Lax-Milgram theorem and following the Poincaré inequality, it is evident that, for all $w\in\overline{H^1(\Omega)}$, equations (36)-(37) establish a unique solution ${w_e}^h\in\overline{V^h(\Omega)}$.
Likewise, for the function $\varphi\in\overline{H^1(\Omega)}$, equations (38)-(39) yield a unique solution ${\varphi_e}^h\in\overline{V^h(\Omega)}$.
Now, we proceed to define the bilinear form

$$s(\phi,\psi)=((\nabla\phi,\nabla\psi)), \tag{40}$$

which is coercive on $\overline{H^1(\Omega)}$, i.e., there exists $c_0>0$, such that

$$s(\phi,\phi)\geq c_0\|\phi\|^2_{H^1},\qquad \text{for all}\ \phi\ \in\ \overline{H^1(\Omega)}. \tag{41}$$

We start by estimating β^φ and β^w.

Lemma 4. *For all $\varphi\in H^2(\Omega)$, the function ${\varphi_e}^h\in V^h$ defined by (38) satisfies*

$$\|{\varphi_e}^h-\varphi\|_{L^2(\Omega)}+h\|{\varphi_e}^h-\varphi\|_{H^1(\Omega)}\leq Ch^2\|\varphi\|_{H^2(\Omega)}. \tag{42}$$

Proof. We first have the following equation:

$$s(\varphi_e^h,\psi^h)=s(\varphi,\psi),\ \text{for all}\ \psi\ \in\ \overline{V^h}. \tag{43}$$

Then, since $\varphi_e^h-I_\varphi^h\in\overline{V^h}$, we obtain

$$s(\varphi_e^h-\varphi,\varphi_e^h-\varphi)=s(\varphi_e^h-\varphi,\varphi_e^h-I_\varphi^h)+s(\varphi_e^h-\varphi,I_\varphi^h-\varphi)$$

and

$$s({\varphi_e}^h-\varphi,{\varphi_e}^h-\varphi)\geq c_0\|{\varphi_e}^h-\varphi\|^2_{H^1},$$

which yields

$$c_0||\varphi_e^h-\varphi||^2_{\overline{H^1}}\leq s(\varphi_e^h-\varphi,\varphi_e^h-\varphi)\leq||\varphi_e^h-\varphi||_{\overline{H^1}}||I_\varphi^h-\varphi||_{\overline{H^1}}.$$

Therefore,

$$||\varphi_e^h-\varphi||_{\overline{H^1}}\leq c_0^{-1}||I_\varphi^h-\varphi||_{\overline{H^1}}.$$

As a direct result of (30), we deduce that

$$||\varphi_e^h - \varphi||_{\overline{H^1}} \leq Ch||\varphi||_{H^2(\Omega)}. \tag{44}$$

Furthermore, for $z \in L^2(\Omega)$, let ϕ represent the unique solution of

$$s(\phi,\psi) = ((z,\psi)), \text{ for all } \psi \in \overline{H^1}. \tag{45}$$

Thus, we obtain that

$$\|\phi\|_{H^2(\Omega)} \leq C\|z\|_{L^2(\Omega)}, \tag{46}$$

where the constant C does not depend on z.

Taking now $\psi = \varphi_e{}^h - \varphi$ in (45), we infer that

$$((z,\varphi_e^h - \varphi)) = s(\phi,\varphi_e^h - \varphi) = s(\phi - I_\phi^h,\varphi_e^h - \varphi) \leq ||\phi - I_\phi^h||_{\overline{H^1}}||\varphi_e^h - \varphi||_{\overline{H^1}}.$$

Moreover, by selecting $z = \varphi_e{}^h - \varphi$ and considering (30) and (44), we obtain

$$\begin{aligned}\|\varphi_e{}^h - \varphi\|^2_{L^2(\Omega)} &\leq Ch\|\phi\|_{H^2(\Omega)}Ch\|\varphi\|_{H^2(\Omega)}\\ &\leq Ch^2\|\varphi_e{}^h - \varphi\|_{L^2(\Omega)}\|\varphi\|_{H^2(\Omega)}.\end{aligned}$$

Thus,

$$\|\varphi_e{}^h - \varphi\|_{L^2(\Omega)} \leq Ch^2\|\varphi\|_{H^2(\Omega)}.$$

This inequality, along with inequality (44), yield the result.
In a similar manner, we can establish the existence of a constant c that is solely dependent on Ω^h. For all $w \in H^2(\Omega)$, the function $w_e{}^h \in V^h$, as defined in (36)-(37), complies with the following:

$$\|w_e{}^h - w\|_{L^2(\Omega)} + h\|w_e{}^h - w\|_{H^1(\Omega)} \leq Ch^2\|w\|_{H^2(\Omega)}. \tag{47}$$

Next, we define the discrete inverse Laplacian
$D_L^{-1,h} : \overline{L} \longrightarrow \overline{V^h}$ by $D_L^{-1,h} f = m^h$, where $f \in \overline{L(\Omega)}$ and $m^h \in \overline{V^h}$ solves

$$((\nabla m^h,\nabla\psi^h)) = ((f,\psi^h)), \qquad \text{for all } \psi^h \in V^h. \tag{48}$$

Note that $D_L^{-1,h}$ is self-adjoint and positive semi-definite on $\overline{H^1}$, since

$$((g,D_L^{-1,h}f)) = ((\nabla D_L^{-1,h}g,\nabla D_L^{-1,h}f)) = ((f,D_L^{-1,h}g)), \qquad \text{for all } f,g \in \overline{L(\Omega)},$$

$$((f,D_L^{-1,h}f)) = \|\nabla D_L^{-1,h}f\|^2_{L^2(\Omega)}, \qquad \text{for all } f \in \overline{L(\Omega)}.$$

By expressing the discrete negative semi-norm in the following manner:

$$\|m\|_{-1,h} = ((D_L^{-1,h}m,m))^{\frac{1}{2}} = \|\nabla D_L^{-1,h}m\|_{L^2(\Omega)}, \qquad \text{for all } m \in \overline{L(\Omega)},$$

and using an orthonormal basis of $\overline{V^h}$ for the $L^2(\Omega)$-scalar product, it becomes evident that the subsequent interpolation inequality is satisfied

$$\|m^h\|^2_{L^2(\Omega)} \leq \|m^h\|_{-1,h}\|m^h\|_{H^1(\Omega)}, \text{ for all } m^h \in \overline{V^h}. \tag{49}$$

It is also observed that

$$\|f\|_{-1,h} \leq c_p\|f\|_{L^2(\Omega)}, \qquad \text{for all } f \in \overline{L(\Omega)}, \tag{50}$$

where c_p is the Poincaré constant. Moreover, we define

$$\delta(t) = \frac{1}{\text{Vol}(\Omega)}((\theta^{\varphi}(t),1)), \qquad \text{for all } t \geq 0, \tag{51}$$

so that $((\theta^{\varphi} - \delta, 1)) = 0$.

In the remaining part of this section, the final time $T \in (0,\infty)$ is defined, and we express

$$\mathscr{Z}(t) = \|\theta^{\varphi}\|^2_{H^1(\Omega)} + \|\theta^{\varphi}_t - \delta_t\|^2_{-1,h}.$$

We now prove the following lemma.

Lemma 5. *Let (φ, w) be a solution of (28)-(29) with sufficient regularity, and let (φ^h, w^h) be a solution of (32)-(33). If $R < \infty$,*

$$\sup_{t\in[0,T]} \|\varphi(t)\|_{C^0(\overline{\Omega})} < R,$$

$$\sup_{t\in[0,T]} \|\varphi_t(t)\|_{C^0(\overline{\Omega})} \leq R,$$

$$\sup_{t\in[0,T]} \|\varphi^h(0)\|_{C^0(\overline{\Omega})} < R,$$

and

$$\|\varphi^h(t)\|_{L^\infty(\Omega)} \leq R, \text{ for every } t \in [0,T^h],$$

where $T^h \in (0,T]$ is the maximal time, then

$$\begin{aligned} &\mathscr{Z}(t) + \int_0^t [||\theta^w||^2_{H^1} + (\frac{1}{2} - \alpha^2)||\theta^{\varphi}_t||^2_{H^1}]ds \\ &\leq C\mathscr{Z}(0) + C' \int_0^t [||\beta^{\varphi}||^2_{L^2} + ||\beta^{\varphi}_{tt}||^2_{L^2}]ds \\ &+ C' \int_0^t [||\beta^w||^2_{L^2} + ||\beta^w_t||^2_{L^2}]ds \qquad \text{for all } t \in [0,T^h]. \end{aligned} \tag{52}$$

Moreover,

$$||((\theta^{\varphi},1))|| \leq C[\mathscr{Z}^{\frac{1}{2}}(t) + ||\theta^{\varphi}||_{L^2}], \qquad \text{for all } t \in [0,T]. \tag{53}$$

Proof.
It follows from (28) and (32) that

$$((\varphi^h_t,\phi)) - ((\varphi_t,\phi)) = -((\nabla w^h,\nabla\phi)) + ((\nabla w,\nabla\phi)) + \alpha((\varphi^h - \varphi,\phi)).$$

Therefore,

$$((\theta^{\varphi}_t,\phi)) + ((\nabla\theta^w,\nabla\phi)) = -((\beta^{\varphi}_t,\phi)) + \alpha((\varphi^h - \varphi,\phi)). \tag{54}$$

In particular, if $\phi \equiv 1$, we obtain

$$\delta_t(t) = \frac{1}{\textbf{Vol}(\Omega)}((\theta^{\varphi}_t,1)) = -\frac{1}{\textbf{Vol}(\Omega)}[((\beta^{\varphi}_t,1)) + \alpha((\theta^{\varphi} + \beta^{\varphi},1))]. \tag{55}$$

Due to equation (39), we can derive the following:

$$((\beta^{\varphi}_t,1)) = 0.$$

Differentiating (55) with respect to time, we get

$$((\theta_{tt}^{\varphi},1)) = \frac{1}{\mathbf{Vol}(\Omega)}\delta_{tt}(t) = -\frac{1}{\mathbf{Vol}(\Omega)}[((\beta_{tt}^{\varphi},1)) + \alpha((\theta_t^{\varphi},1)) + ((\beta_t^{\varphi},1))], \tag{56}$$

which yields,

$$\delta_{tt}(t) = \frac{\alpha}{\mathbf{Vol}(\Omega)}((\theta_t^{\varphi},1)). \tag{57}$$

Similarly, by subtracting (29) from (33) now, we get:

$$((w^h,\psi)) - ((w,\psi)) = \frac{1}{\varepsilon}((f(\varphi^h),\psi)) - \frac{1}{\varepsilon}((f(\varphi),\psi)) - \varepsilon((\nabla\varphi^h,\psi)) + \varepsilon((\nabla\varphi,\nabla\psi)).$$

Hence,

$$-((\theta^w,\psi)) + \varepsilon((\nabla\theta^{\varphi},\nabla\psi)) = ((\beta^w,\psi)) - \frac{1}{\varepsilon}((f(\varphi^h) - f(\varphi),\psi)), \tag{58}$$

on $[0,T]$, for all $\psi \in M^h$.

Furthermore, using $\phi = \theta^w$ in (54) and $\psi = \theta_t^{\varphi}$ in (58), and then summing the results, we obtain

$$\begin{aligned}||\nabla\theta^w||^2_{L^2(\Omega)} + \frac{1}{2}\frac{d}{dt}||\nabla\theta^{\varphi}||^2_{L^2(\Omega)} &= -2((\theta_t^{\varphi},\theta^w)) - ((\beta_t^{\varphi},\theta^w)) + \alpha((\theta^{\varphi},\theta^w)) + \alpha((\beta^{\varphi},\theta^w)) \\ &\quad + ((\beta^w,\theta_t^{\varphi})) - ((f(\varphi^h) - f(\varphi),\theta_t^{\varphi})).\end{aligned}$$

In addition, the function f is Lipschitz with constant L_f, therefore

$$\|f(\varphi^h) - f(\varphi)\|_{L^2(\Omega)} \leq L_f\|\varphi^h - \varphi\|_{L^2(\Omega)}. \tag{59}$$

Hence,

$$\begin{aligned}||\theta^w||^2_{H^1(\Omega)} + \frac{1}{2}\frac{d}{dt}||\theta^{\varphi}||^2_{H^1(\Omega)} &\leq 2||\theta_t^{\varphi}||_{L^2(\Omega)}[\mathbf{Vol}^{\frac{-1}{2}}(\Omega)|((\theta^w,1))| + c_p||\theta^w||_{H^1(\Omega)}] \\ &\quad + ||\beta_t^{\varphi}||_{L^2}||\theta^w||_{L^2(\Omega)} + \alpha||\theta^{\varphi}||_{L^2(\Omega)}||\theta^w||_{L^2(\Omega)} \\ &\quad + \alpha||\beta^{\varphi}||_{L^2(\Omega)}||\theta^w||_{L^2(\Omega)} + ||\beta^w||_{L^2(\Omega)}||\theta_t^{\varphi}||_{L^2(\Omega)} \\ &\quad + \frac{1}{\varepsilon}||\theta_t^{\varphi}||_{L^2(\Omega)}.L_f[||\theta^{\varphi}||_{L^2(\Omega)} + ||\beta^{\varphi}||_{L^2(\Omega)}].\end{aligned}$$

We now estimate $((\theta^w,1))$. We choose $\psi \equiv 1$ in (58) and use $((\beta^w,1)) = 0$, so the estimate (59) yields

$$|((\theta^w,1))| \leq L_f[||\theta^{\varphi}||_{L^2(\Omega)} + ||\beta^{\varphi}||_{L^2(\Omega)}]\mathbf{Vol}^{\frac{-1}{2}}(\Omega) \text{ on } [0,T^h]. \tag{60}$$

Thanks to inequalities (60) and (58), the triangle inequality, and the generalized Poincaré inequality, we find

$$\|v\|^2_{L^2(\Omega)} \leq c_p'\|v\|^2_{H^1(\Omega)}, \quad \text{for all } v \in H^1(\Omega), \tag{61}$$

and we deduce (53).
Besides, we have that

$$ab \leq \varepsilon a^2 + (4\varepsilon)^{-1}b^2, \qquad \text{for all } a,b \geq 0,\ \forall\, \varepsilon > 0. \tag{62}$$

It then follows from (60)-(62) that

$$\begin{aligned}||\theta^w||_{H^1(\Omega)} + \frac{d}{dt}||\theta^{\varphi}||_{H^1(\Omega)} &\leq C_1(||\beta^{\varphi}||^2_{L^2(\Omega)} + ||\beta^w||^2_{L^2(\Omega)} + ||\beta_t^{\varphi}||^2_{L^2(\Omega)}) \\ &\leq C_2(||\theta^{\varphi}||^2_{L^2(\Omega)} + ||\theta_t^{\varphi}||^2_{L^2(\Omega)}),\end{aligned} \tag{63}$$

where the constants C_1 and C_2 depend on $\mathbf{Vol}(\Omega), c_p, L_f$, and α.
We now must calculate the value of θ_t^φ, so if we differentiate equation (59) with respect to time, we get

$$((\theta_{tt}^\varphi,\phi))+((\nabla\theta_t^w,\nabla\phi))=-((\beta_{tt}^\varphi,\phi))+\alpha((\theta_t^\varphi+\beta_t^\varphi,\phi)). \tag{64}$$

In addition, if we differentiate (58) with respect to time, we get

$$-((\theta_t^w,\psi))+\varepsilon((\nabla\theta_t^\varphi,\nabla\psi))=((\beta_t^\varphi,\psi))-\frac{1}{\varepsilon}(([f(\varphi^h)-f(\varphi)]_t,\psi)). \tag{65}$$

Next, we select $\phi=D_L^{-1,h}(\theta_t^\varphi-\delta_t)$ in equation (64) and $\psi=\theta_t^\varphi-\delta_t$ in equation (65). When we combine these equations, we obtain

$$((\theta_{tt}^\varphi,D_L^{-1,h}(\theta_t^\varphi-\delta t))+\varepsilon||\theta_t^\varphi||^2_{H^1(\Omega)}=-(((\beta_{tt}^\varphi,D_L^{-1,h}(\theta_t^\varphi-\delta_t)))+((\beta_t^w,\theta_t^\varphi-\delta_t)) \tag{66}$$

$$+\alpha((\theta_t^\varphi+\beta_t^\varphi,D_L^{-1,h}(\theta_t^\varphi-\delta_t)))-(([f(\varphi^h)-f(\varphi)]_t,\theta_t^\varphi-\delta_t)).$$

In the first term on the left-hand side, we can express

$$\theta_{tt}^\varphi=(\theta_{tt}^\varphi-\delta_{tt})+\delta_{tt}.$$

We should note that $((\delta_{tt}+\beta_{tt}^\varphi,1))=\alpha((\theta_t^\varphi,1))$ according to (57). As for the nonlinear terms, we have

$$[f(\varphi^h)-f(\varphi)]_t=f'(\varphi^h)[\varphi_t^h-\varphi_t]+\varphi_t[f'(\varphi^h)-f'(\varphi)]$$

and

$$\alpha[\varphi^h-\varphi]_t=\alpha(\theta_t^\varphi+\beta_t^\varphi). \tag{67}$$

Thus, equation (66) implies

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}||\theta_t^\varphi-\delta_t||^2_{-1,h}+\varepsilon||\theta_t^\varphi||^2_{H^1(\Omega)}\leq{}&||\delta_{tt}+\beta_{tt}^\varphi||_{-1,h}||\theta_t^\varphi-\delta_t||_{-1,h}+||\beta_t^w||_{L^2(\Omega)}||\theta_t^\varphi-\delta_t||_{L^2(\Omega)}\\&+sup|f'|(||\theta_t^\varphi-\delta_t||_{L^2(\Omega)}+||\beta_t^\varphi||_{L^2(\Omega)}+|\delta_t|)||\theta_t^\varphi-\delta_t||_{L^2(\Omega)}\\&+L'_f(||\theta^\varphi||_{L^2(\Omega)}+||\beta^\varphi||_{L^2(\Omega)})||\theta_t^\varphi-\delta_t||_{L^2(\Omega)}\\&+\alpha(||\theta_t^\varphi||_{L^2(\Omega)}+||\beta_t^\varphi||_{L^2(\Omega)})||\theta_t^\varphi-\delta_t||_{-1,h},\end{aligned}$$

where L'_f is the Lipschitz constant of f' on $[-F,F]$.
With the help of the interpolation inequality (49) applied to $v^h=\theta_t^\varphi-\delta_t$ i.e.

$$\|\theta_t^\varphi-\delta_t\|_{L^2(\Omega)}\leq\|\theta_t^\varphi-\delta_t\|_{-1,h}\|\theta_t^\varphi\|_{H^1(\Omega)}, \tag{68}$$

and the inequalities (50) and (63), along with the Poincaré Inequality, we obtain

$$\frac{d}{dt}||\theta_t^\varphi-\delta_t||^2_{-1,h}+||\theta_t^\varphi||_{H^1(\Omega)}$$

$$\leq C_3[||\delta_{tt}+\beta^\varphi tt||^2_{L^2(\Omega)}+||\beta_t^w||^2_{L^2(\Omega)}+||\beta_t^\varphi||^2_{L^2(\Omega)}+2||\beta^\varphi||+2||\delta_t||^2_{L^2(\Omega)}]$$

$$+C_4(||\theta_t^\varphi-\delta_t||^2_{-1,h}+||\theta^\varphi||^2_{L^2(\Omega)}),\ \text{on } [0,T^h], \tag{69}$$

for some constants C_3 and C_4 which depend on R , c_p, $L_{f'}$, $\sup\limits_{[-R,R]}|L'|$ and $\sup\limits_{[-R,R]}|f'|$.

Finally, we add (63) and (69), using the modified Poincaré inequality (61) and the triangular inequality, we get

$$\|\theta_t^\varphi\|_{L^2(\Omega)^2} \leq \|\theta_t^\varphi - \delta_t\|^2_{L^2(\Omega)} + |\delta_t|^2,$$

the interpolation inequality (68), and inequality (62), we obtain that

$$\frac{1}{2}||\theta_t^\varphi||^2_{H^1(\Omega)} + ||\theta^w||_{H^1(\Omega)} + \frac{d}{dt}\mathscr{Z}(t) \leq C_5(||\beta^w||^2_{L^2(\Omega)} + ||\beta^\varphi||^2_{L^2(\Omega)} + |\delta_t|^2 + ||\beta_t^w||^2_{L^2(\Omega)} + ||\beta_t^\varphi||^2_{L^2(\Omega)}$$

$$+||\beta_{tt}^\varphi||^2_{L^2(\Omega)} + |\delta_{tt}|^2) + C_6(||\theta_t^\varphi - \delta_t||_{-1,h} + ||\theta^\varphi||^2_{L^2(\Omega)} + ||\theta_t^\varphi||^2_{L^2(\Omega)}$$

$$\leq C_5(||\beta^w||^2_{L^2(\Omega)} + ||\beta^\varphi||^2_{L^2(\Omega)} + |\delta_t|^2 + ||\beta_t^w||^2_{L^2(\Omega)} + ||\beta_t^\varphi||^2_{L^2(\Omega)} + ||\beta_{tt}^\varphi||^2_{L^2(\Omega)} + |\delta_{tt}|^2)$$

$$+C_6(||\theta_t^\varphi - \delta_t||^2_{-1,h} + ||\theta^\varphi||^2_{H^1(\Omega)}).$$

Moreover, due to equation (33), we have

$$|\delta_t|^2 \leq \alpha^2||\theta^\varphi||^2_{L^2(\Omega)}$$

and

$$|\delta_{tt}|^2 \leq \alpha^2||\theta_t^\varphi||^2_{L^2(\Omega)},$$

which lead to the the following inequality:

$$(\frac{1}{2} - \alpha^2)||\theta_t^\varphi||^2_{H^1(\Omega)} + ||\theta^w||^2_{H^1(\Omega)} + \frac{d}{dt}||\mathscr{Z}(t)||$$

$$\leq C(||\beta^w||^2_{L^2(\Omega)} + ||\beta^\varphi||^2_{L^2(\Omega)} + ||\beta_t^w||^2_{L^2(\Omega)} + ||\beta_{tt}^\varphi||^2_{L^2(\Omega)}) + C'\mathscr{Z}(t).$$

Therefore, we conclude (52) by applying Gronwall's lemma.

Theorem 6. *Let (φ, w) represent a solution to (28)-(29) such that φ, φ_t, φ_{tt}, w, $w_t \in L^2(0,T,H^2(\Omega))$, and let (φ^h, w^h) denote the solution to (32)-(33).*
If

$$\theta^\varphi(0) = 0,\ \theta^w(0) = 0,\ \textit{and}\ \beta^\varphi(0) = 0, \tag{70}$$

then

$$\sup_{[0,T]}(\|\varphi^h - \varphi\|_{L^2(\Omega)} + \|\varphi_t^h - \varphi_t\|_{-1,h}) \leq Ch^2,$$

$$\left(\int_0^T \|w^h - w\|^2_{L^2(\Omega)} ds\right)^{\frac{1}{2}} \leq Ch^2,$$

$$\sup_{[0,T]}\|\varphi^h - \varphi\|_{H^1(\Omega)} \leq Ch,$$

and

$$\left(\int_0^T \left(\|w^h - w\|^2_{H^1(\Omega)} + \|\varphi_t^h - \varphi_t\|^2_{H^1(\Omega)}\right) ds\right)^{\frac{1}{2}} \leq Ch.$$

Proof.
We start by differentiating equations (36)-(38) with respect to time, we find that the elliptic projections of φ_t and w_t are respectively $(\varphi_e)_t$ and $(we)_t$. The same applies to φ_{tt} and w_{tt}. Given that $\varphi \in C^1([0,T],H^2(\Omega))$ and due to the Sobolev continuous injection property, $H^2(\Omega) \subset C^0(\overline{\Omega})$, we can conclude that

$$\varphi,\varphi_t \ \in \ C^0([0,T],C^0(\overline{\Omega})).$$

Thus,

$$\sup_{t\,\in\,[0,T]} ||\varphi(t)||_{C^0(\overline{\Omega})} < R,$$

and

$$\sup_{t\,\in\,[0,T]} ||\varphi_t(t)||_{C^0(\overline{\Omega})} \leq R, \qquad \text{for some } \ R>0.$$

Using the inverse estimate (31), we have

$$||\varphi^h(0)-\varphi(0)||_{C^0(\overline{\Omega})} \leq C_0 h^{-\frac{n}{2}}(||\varphi^h(0)-\varphi(0)||_{L^2(\Omega)}+||\varphi(0)-I^h_\varphi(0)||_{L^2(\Omega)}) \\ +C_0' h^l ||\varphi(0)||_{H^2(\Omega)},$$

where l is a real number in $(0,1)$ ensuring that $H^2(\Omega)$ is a subset of $C^{0,l}(\Omega)$. Thanks to Lemma 4, as well as equations (30) and (70), we obtain

$$||\varphi^h(0)-\varphi(0)||_{C^0(\overline{\Omega})} < (CC_0 h^{2-\frac{n}{2}} + C_0' h^l)||\varphi(0)||_{H^2(\Omega)}. \tag{71}$$

Taking h small enough, we get

$$||\varphi^h(0)||_{C^0(\overline{\Omega})} < R.$$

We now assert that $\mathscr{X}(0) \leq Ch^4$, where $\mathscr{X}$ is defined in Lemma 5. Hence

$$\mathscr{X}(0) = \|\theta_t^\varphi(0)-\delta_t(0)\|^2_{-1,h}.$$

We follow a similar approach to the proof in Lemma 5. Starting with equation (55) being valid at $t=0$, we then substitute $\phi = D_L^{-1,h}(\theta^\varphi t(0)-\delta t(0))$ into (54), yielding

$$((\theta_t^\varphi(0), D_L^{-1,h}(\theta_t^\varphi(0)-\delta_t(0)))) + ((\nabla\theta^w(0), \nabla D_L^{-1,h}(\theta_t^\varphi(0)-\delta_t(0)))) =$$

$$-((\beta_t^\varphi(0), D_L^{-1,h}(\theta_t^\varphi(0)-\delta_t(0)))) + \alpha((\theta^\varphi(0)+\beta^\varphi(0), D_L^{-1,h}(\theta_t^\varphi(0)-\delta_t(0)))).$$

We subsequently use (70) to obtain

$$((\theta_t^\varphi(0), D_L^{-1,h}(\theta_t^\varphi(0)-\delta_t(0)))) = -((\beta_t^\varphi(0), D_L^{-1,h}(\theta_t^\varphi(0)-\delta_t(0)))).$$

Consequently,

$$||\theta_t^\varphi(0)-\delta_t(0)||_{-1,h} \leq ||\beta_t^\varphi(0)+\delta_t(0)||_{-1,h} \leq C||\beta_t^\varphi(0)+\delta_t(0)||_{L^2(\Omega)} \leq Ch^2||\varphi_t(0)||_{H^2(\Omega)}, \tag{72}$$

where we used Lemma 4, (50) and (53). Therefore, $\mathscr{X}(0) \leq Ch^4$, which validates our assertion.

In addition, Lemmas 4 and 5, combined with the estimation in (47) and the regular assumption regarding φ and w, yield the following inequality:

$$\mathscr{X}(t) \leq Ch^4, \qquad \text{for all } t \ \in \ [0,T^h].$$

This inequality, in particular, implies the subsequent result

$$\|\theta^{\varphi}(t)\|_{L^2(\Omega)} \leq Ch^2, \qquad \text{for all } t \in [0,T^h].$$

In addition, arguing as in (71), we then deduce that

$$\sup ||\varphi^h(t) - \varphi(t)||_{C^0(\bar{\Omega})} \to 0, \text{ as } h \to 0.$$

Consequently, by choosing a sufficiently small value of h, we have $T^h = T$. Also, Lemma 4, Lemma 5, and (47) collectively establish the results presented in our theorem.

3.2 Stability of the Backward Euler scheme

In this section, we examine the backward Euler scheme with respect to time. After showing that the functional energy decreases during time discretization, we can conclude that our scheme maintains stability. Our initial assumption is that the time step $\eta t > 0$ remains constant.
The numerical scheme is as follows:

$$((\frac{\varphi_h^n - \varphi_h^{n-1}}{\eta t}, \phi)) = -((\nabla v, \nabla \phi)) - \alpha((\varphi_h^n, \phi)), \tag{73}$$

$$((v_h^n, \psi)) = \frac{1}{\varepsilon}((f(\varphi_h^n), \psi)) + \varepsilon((\nabla \varphi_h^n, \nabla \psi)), \tag{74}$$

for all $\phi, \psi \in V^h$.
In what follows, we show the existence, uniqueness, and stability of sequences $((\varphi_h^n), (w_h^n))$.

Theorem 7. *For every $\varphi_h^0 \in V^h$, there exist two sequences, (φ_h^n) and (v_h^n), generated by equations (73)-(74), which satisfy the following:*

$$\mathscr{J}(\varphi_h^n) + \frac{\alpha}{2}||\varphi_h^n||^2 + \frac{1}{2\eta t}||\varphi_h^n - \varphi_h^{n-1}||_{-1}^2 \leq \mathscr{J}(\varphi_h^{n-1}) + \frac{\alpha}{2}||\varphi_h^{n-1}||^2, \textit{ for all } n \geq 1. \tag{75}$$

In addition, if $\eta t < \eta t*$, where $\eta t^* = \frac{4\varepsilon}{m}$ and $m = \frac{1}{\varepsilon} + \frac{\eta t \alpha^2}{2} + \varepsilon \eta^2 t \alpha^2 \mathbf{Vol}^2(\Omega)$, then these sequences are uniquely defined.

Proof. Consider the following minimization problem:

$$\Pi^{\varphi} = \inf_{w \in V^h} \Pi^h(w), \tag{76}$$

where

$$\Pi^h(w) = \mathscr{J}(w) + \frac{\alpha}{2}||w||^2 + \frac{1}{2\eta t}||w - \varphi_h^{n-1}||_{-1}^2. \tag{77}$$

We can see that

$$\Pi^h(w) \geq \frac{\varepsilon}{2}||\nabla \varphi||^2 + (\frac{c_1}{\varepsilon} + \frac{\alpha}{2})||w||^2 + C.$$

Since $\Pi^h(.)$ is continuous, it follows that that there exists a solution to the variational problem (76). This solution satisfies Euler-Lagrange's equation

$$\varepsilon((\nabla \varphi, \nabla \phi)) + \frac{1}{\varepsilon}((f(\varphi), \phi)) + \alpha((D_L^{-1,h} \varphi, \phi)) + \frac{1}{2\eta t}((\varphi - \varphi_h^{n-1}, \phi)) - ((\phi, 1)) = 0, \tag{78}$$

for all $\phi \in V^h$.
We set $\varphi_h^n = \varphi$ and $v_h^n = \phi - D_L^{-1,h}(\frac{1}{\eta t}(\varphi - \varphi_h^{n-1}) - \alpha\varphi)$, and we see that $((\varphi_h^n),(w_h^n))$ satisfies (73)-(74). By construction, we have

$$\mathscr{J}^h(\varphi_h^n) \leq \mathscr{J}^h(\varphi_h^{n-1}),$$

and as a result, we can derive (75).
To establish uniqueness, we consider $\kappa^\varphi = (\varphi_h^n)^1 - (\varphi_h^n)^2$ and $\kappa^v = (v_h^n)^1 - (v_h^n)^2$ as the discrepancies between two solutions $((\varphi_h^n)^i,(w_h^n)^i)$ (where $i = 1,2$) of (73)-(74) with respect to a given φ_h^{n-1}. Then, $(\kappa^\varphi, \kappa^v)$ satisfies

$$((\kappa^\varphi,\phi)) = -\eta t((\nabla\kappa^v,\nabla\phi)) - \eta t.\alpha((\kappa^\varphi,\phi)), \tag{79}$$

$$((\kappa^v,\psi)) = \frac{1}{\varepsilon}((f((\varphi_h^n))^1) - f((\varphi_h^n)^2),\psi)) + \varepsilon((\nabla\kappa^\varphi,\nabla\psi)), \tag{80}$$

for all $\phi,\psi \in V^h$.
By choosing $\phi = \kappa^v$ and $\psi = \kappa^\varphi$ and subtracting the resulting equations, we obtain

$$\eta t||\nabla\kappa^v||^2 + \varepsilon||\nabla\kappa^\varphi||^2 + \frac{1}{\varepsilon}Re((f(\varphi_h^{n,1}) - f(\varphi_h^{n,2}),\kappa^\varphi)) - \eta t.\alpha((\kappa^\varphi,\kappa^v)) = 0. \tag{81}$$

Set $\varphi_h^{n,1} = z, \varphi_h^{n,2} = z'$ in Proposition 3.1 of the reference [25], and observe that

$$\begin{aligned}|Re((f(z) - f(z'),\kappa^\varphi))| &\geq c_0\int_\Omega [|\kappa^\varphi|^4 + |z|^2.|\kappa^\varphi|^2 + 2Re(\bar{z}\kappa^\varphi)]dx - ||\kappa^\varphi||^2\\ &\geq -||\kappa^\varphi||^2,\end{aligned} \tag{82}$$

hence

$$\eta t||\nabla\kappa^v||^2 + \varepsilon||\nabla\kappa^\varphi||^2 \leq \frac{1}{\varepsilon}||\kappa^\varphi||^2 + \eta t\alpha||\kappa^\varphi||.||\kappa^v||, \tag{83}$$

which yields

$$\eta t||\nabla\kappa^v||^2 + \varepsilon||\nabla\kappa^\varphi||^2 \leq (\frac{1}{\varepsilon} + \frac{\alpha^2.\eta t}{2})||\kappa^\varphi||^2 + \frac{\eta t}{2}||\kappa^v||^2. \tag{84}$$

Let now $\phi = \psi = \mathbf{Vol}(\Omega)$ in (79) and (80) and proceeding as above, we have

$$\langle\kappa^\varphi\rangle \leq \eta t.\alpha.\mathbf{Vol}(\Omega)||\kappa^\varphi||$$

and

$$\langle\kappa^v\rangle \leq \frac{k_f}{\varepsilon.\mathbf{Vol}(\Omega)}||\kappa^\varphi||.$$

Therefore, inequality (84) can be rewritten as

$$\begin{aligned}&\frac{\eta t}{2}||\kappa^v||^2_{H^1(\Omega)} + \varepsilon||\kappa^\varphi||^2_{H^1(\Omega)}\\ &\leq (\frac{1}{\varepsilon} + \frac{\alpha^2.(\eta t)^2}{2} + \varepsilon.(\eta t)^2 t.\alpha^2.\mathbf{Vol}^2(\Omega) + \frac{k_f^2}{\eta t.\varepsilon^2.\mathbf{Vol}^2(\Omega)})||\kappa^\varphi||^2\\ &\leq m||\kappa^\varphi||^2.\end{aligned} \tag{85}$$

Next, by choosing $\phi = m.\kappa^\varphi$ in (79), we infer that

$$m||\kappa^\varphi||^2 + m.\alpha.\eta t||\kappa^\varphi||^2 = -m.\eta t((\nabla\kappa^\varphi,\nabla\kappa^\varphi)) \leq \frac{\eta t}{2}||\nabla\kappa^v||^2 + \frac{m^2\eta t}{2}||\nabla\kappa^v||.$$

We then deduce the following inequality,

$$(\varepsilon - \frac{m^2\eta t}{4})||\kappa^{\varphi}||^2_{H^1(\Omega)} \leq 0.$$

At the end, since $((\theta^{\varphi},1)) = 0$, the smallness assumption on ηt implies that $\kappa^{\varphi} = 0$, and using (3.61) we can see that $\kappa^{w} = 0$.

4 Conclusions

In this article, we proposed a complex version of the Cahn-Hilliard-Oono type equation, with applications in grayscale phase separation. Instead of considering the Cahn-Hilliard-Oono system for long interaction phase separation as proposed in [26], we suggested examining a multi-phase metal treated as grayscale, where the concentration of each phase ranges between 0 and 1. We utilized the complex version of the Cahn-Hilliard equation, revealing that the real part of the solution represents the resulting separation.

We established the existence of a unique solution for the stationary problem using Schauder's fixed point theorem. Furthermore, we considered a numerical scheme based on finite element space discretization in space and Backward Euler discretization in time. After deriving error estimates for the semi-discrete solution, we demonstrated the convergence of the semi-discrete solution to the continuous one. Additionally, we proved the stability of the backward Euler scheme, enabling convergence of the fully discrete scheme to the continuous problem.

It is worth noting that numerical simulations are crucial for showcasing the efficiency of the model under investigation in future works. Furthermore, exploring the mathematical aspects of the evolution problem (well-posedness, attractors, convergence) will be a significant focus in future studies. Moreover, for long interaction phase separation and to streamline numerical simulations, we can use the complex version of the Cahn-Hilliard-Oono equation.

5 Declarations

5.1 Conflict of interest:

Not applicable.

5.2 Funding:

Not applicable.

5.3 Author's contribution:

H.F.-Conception and Design, Methodology, Investigation, Drafting and Editing. M.F.-Conception and Design, Investigation, Drafting and Editing. W.S.-Investigation, Drafting and Editing. Y.A.-Investigation, Drafting and Editing. All authors reviewed the results and approved the final version of the manuscript.

5.4 Acknowledgement:

The authors wish to thank the referees for their careful reading of the article and useful comments.

5.5 Data availability statement:

All data that support the findings of this study are included within the article.

5.6 Using of AI tools:

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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