Moroccan J. of Pure and Appl. Anal. (MJPAA)
Volume 9(3), 2023, Pages 407–419
ISSN: Online 2351-8227 - Print 2605-6364
DOI: 10.2478/mjpaa-2023-0028

$p(x)$-Kirchhoff bi-nonlocal elliptic problem driven by both $p(x)$-Laplacian and $p(x)$-Biharmonic operators

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Abstract. We investigate the existence of non-trivial weak solutions for the following $p(x)$-Kirchhoff bi-nonlocal elliptic problem driven by both $p(x)$-Laplacian and $p(x)$-Biharmonic operators

$$\begin{cases} M(\sigma)\left(\Delta^2_{p(x)}u - \Delta_{p(x)}u\right) = \lambda\vartheta(x)|u|^{q(x)-2}u\left(\int_\Omega \frac{\vartheta(x)}{q(x)}|u|^{q(x)}dx\right)^r & \text{in } \Omega,\\ u \in W^{2,p(.)}(\Omega)\cap W_0^{1,p(.)}(\Omega), \end{cases}$$

under some suitable conditions on the continuous functions p, q, the non-negative function ϑ and $M(\sigma)$, where

$$\sigma := \int_\Omega \frac{|\Delta u|^{p(x)}}{p(x)} + \frac{|\nabla u|^{p(x)}}{p(x)}\,dx.$$

Our main results is obtained by employing variational techniques and the well-known symmetric mountain pass lemma.

Mathematics Subject Classification (2020). : 35A15, 46E35, 35B38.
Key words and phrases. Variable exponents, Variational methods, $p(x)$-Kirchhoff, Symmetric mountain pass lemma.

Received February 6, 2023 - Accepted: September 29, 2023 .

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1. Introduction

Intense research has been put into the study of variational problems with operators that have variable exponents over the past ten years. These problems, which are frequently referred to as nonhomogeneous eigenvalue problems, have a characteristic that makes it more difficult to use many of the methods employed if the exponent is a positive constant. In many fields, such as image processing [14], heat transfer problems [4], fluid flow problems [19], and structural mechanics problems [18], these nonhomogeneous eigenvalue problems have wide and useful applications.

The main objective of this work is to look into the existence of weak solutions for a $p(x)$-Kirchhoff bi-nonlocal elliptic problem in a bounded domain Ω of $\mathbb{R}^N$ with smooth boundary.

$$(P_\lambda):\begin{cases} M(\sigma)\Big(\Delta^2_{p(x)}u-\Delta_{p(x)}u\Big)=\lambda\vartheta(x)|u|^{q(x)-2}u\Big(\int_\Omega \frac{\vartheta(x)}{q(x)}|u|^{q(x)}dx\Big)^r & \text{in } \Omega,\\ u\in W^{2,p(.)}(\Omega)\cap W_0^{1,p(.)}(\Omega), \end{cases}$$

where p and q are continuous functions on $\overline{\Omega}$, λ and r are a positive reals, $M(\sigma)$ is a continuous function with

$$\sigma:=\int_\Omega \frac{|\Delta u|^{p(x)}}{p(x)}+\frac{|\nabla u|^{p(x)}}{p(x)}\,dx.$$

$\Delta_{p(x)}=div(|\nabla u|^{p(x)-2}\nabla u)$ and $\Delta^2_{p(x)}u=\Delta(|\Delta u|^{p(x)-2}\Delta u)$ are respectively the $p(x)$-Laplacian operator and the $p(x)$-biharmonic, and $\vartheta\in L^{m(x)}$ is a nonnegative function with $m\in C_+(\Omega)$. The terminology "bi-nonlocal" originates from the fact that the equation in (P_λ) contains the following two integral over Ω

$$\int_\Omega \frac{|\Delta u|^{p(x)}}{p(x)}+\frac{|\nabla u|^{p(x)}}{p(x)}\,dx \quad\text{and}\quad \int_\Omega \frac{\vartheta(x)}{q(x)}|u|^{q(x)}dx,$$

with u depicts a process that focuses on the average of itself like the population density, and which no longer exist as pointwise expressions when modeling biological systems. Additionally, they depict several pertinent physical and engineering conditions (such as image processing, describing the theorem of beam vibration, etc) and requires a nontrivial apparatus to solve them. We point out that the research has been active in studying the problems that contain nonlocal terms since the appearance of the work of Kirchhoff [15], in 1883, in which was studied the following hyperbolic problem that expands the traditional D'Alambert's wave equation by taking into account the impact of variations in string length throughout vibrations

$$\rho\frac{\partial^2 u}{\partial t^2}-\left(\frac{\rho_0}{h}+\frac{E}{L}\int_0^L|\frac{\partial u}{\partial x}|^2dx\right)\frac{\partial^2 u}{\partial x^2}=0,$$

such that ρ, ρ_0, h, E, L are constants. During the last decade, there were many works concerning similar problems by involving the variable exponent theory. In 2014, Corrła and ACDR Costa [5] have showed several results concerning the existence of positive solutions of the problem defined by

$$\begin{cases} -M\left(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx \right)\Delta_{p(x)}u = \lambda |u|^{\beta(x)-2}u\left(\int_\Omega \frac{1}{\beta(x)}|u|^{\beta(x)}dx \right)^r \quad \text{in} \quad \Omega, \\ u = 0 \quad \text{on} \quad \partial\Omega. \end{cases} \tag{1.1}$$

In 2017, Allaoui and Darhouche [1] are focused in the existence of solutions to the problem that contains a $(p_1(x), p_2(x))-$Kirchhoff-type equations with Dirichlet boundary condition, which is presented as follow

$$\begin{cases} -M_1\left(\int_\Omega \frac{1}{p_1(x)}|\nabla u|^{p(x)}dx \right)\Delta_{p_1(x)}u - M_2\left(\int_\Omega \frac{1}{p_2(x)}|\nabla u|^{p(x)}dx \right)\Delta_{p_2(x)}u \; = f(x,u) \quad \text{in} \quad \Omega, \\ u = 0 \quad \text{on} \quad \partial\Omega, \end{cases}$$

and established several conditions on the existence of solutions using variational methods and the theory of the variable exponent Sobolev spaces. Recently, Lee et all [17] studied the following elliptic equation:

$$-M\left(\int_{\mathbb{R}^N} \frac{1}{p(x)}|\nabla u|^{p(x)}dx \right)\Delta_{p(x)}u + \vartheta(x)|u|^{p(x)-2}u \; = \lambda f(x,u) \quad \text{in } \mathbb{R}^N,$$

and used abstract critical point results for an energy functional fulfilling the Cerami condition to calculate the precise positive interval of λ where the problem permits at least two nontrivial solutions. Very recently, Jaafri et all [12] established the existence of a sequence of weak solutions of a similar problem with Navier boundary condition where the expression "$M\left(\int_\Omega \frac{1}{p(x)}|\Delta u|^{p(x)}dx \right)\Delta^2_{p(x)}u$" is used in place of the expression on the left side of (1.1).

Motivated by the works in [5] and [12] we prove the existence of a sequence of weak solutions of (P_λ), and this is according to the conditions from which we proceed. To the best of our knowledge, this work is the first involving both $p(x)$-Laplacian and $p(x)$-Biharmonic operators on one side and nonlocal terms and weight on the other, which could open up new research directions, at both the theoretical and applied levels.

The article is arranged as follows: We first review several fundamental concepts and properties. The existence of non-trivial weak solutions to the problem (P_λ) is the focus of Section 3. Finally, we will compare our results with existing ones.

2. Definitions and fundamental properties

The problem (P_λ) requires the introduction of some fundamental properties of Lebesgue-Sobolev spaces with variable exponent (to learn more, see [16, 11, 10]) and some properties of the operators existing in (P_λ), which permiting our functionals to fulfill the hypotheses, in order to guarantees the mains results.

First let p be a Lipschitz continuous function on $\overline{\Omega}$, verifying

$$1 < p^- := \min_{x\in\overline{\Omega}} p(x) \le p^+ = \max_{x\in\overline{\Omega}} p(x) < \infty.$$

Set
$$C_+(\overline{\Omega}) := \{f : f \in C(\overline{\Omega}), f(x) > 1, \ \text{ for each } x \in \overline{\Omega}\},$$
and define
$$L^{p(x)}(\Omega) = \{v : \Omega \to \mathbb{R} \text{ measurable } : \ \varrho_{p(x)}(|v(x)|^{p(x)}) < \infty\},$$
where $\varrho_{p(.)}$ is the modular functional defined on $L^{p(x)}(\Omega)$ as follows
$$\int_\Omega |v(x)|^{p(x)}\, dx.$$
The norm which is described as follow
$$|v|_{p(.)} = \inf\{\nu > 0 : \ \varrho_{p(x)}\Big(\frac{v}{\nu}\Big) \leq 1\},$$
is endowed in the space $L^{p(x)}(\Omega)$.

Notice that if $p(x)$ is equal to a $p \in \mathbb{R}_+$, then $L^{p(x)}(\Omega)$ is the classical Lebesgue space $L^p(\Omega)$ and the norm $|v|_{p(x)}$ is the standard one $\|v\|_{L^p} = \Big(\int_\Omega |v|^p dx\Big)^{\frac{1}{p}}$ in $L^p(\Omega)$.

Similar to the constant exponent case, we consider for each positive integer k
$$W^{k,p(x)}(\Omega) = \{v \in L^{p(x)}(\Omega) : D^\gamma v \in L^{p(x)}(\Omega), |\gamma| \leq k\},$$
such that $\gamma = (\gamma_1, ..., \gamma_N)$ is a multi-index, $|\gamma| = \sum_{i=1}^N \gamma_i$ and $D^\gamma v = \frac{\partial^{|\gamma|} v}{\partial^{\gamma_1} x_1 ... \partial^{\gamma_N} x_N}$. The space $W^{k,p(x)}(\Omega)$ is equipped with the norm
$$\|v\|_{k,p(x)} = \sum_{|\gamma| \leq k} |D^\gamma v|_{p(x)},$$
is a reflexive and separable Banach space. $W_0^{k,p(x)}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W^{k,p(x)}(\Omega)$.

In what follows, we set $X := W^{2,p(.)}(\Omega) \cap W_0^{1,p(.)}(\Omega)$ and see that weak solutions of problem (P_λ) are taken in X, with the following norm
$$\|v\|_{p(.)} = |\Delta v|_{p(.)} + |\nabla v|_{p(.)}.$$
According to [20], let us notice that the norms $\|v\|_{p(.)}$ and $|\Delta v|_{p(.)}$ are equivalent. They are also equivalent to the norm defined by
$$\|v\| = inf\Big\{\kappa > 0 : \int_\Omega \Big(\Big|\frac{\Delta v(x)}{\kappa}\Big|^{p(x)} + \Big|\frac{\nabla v(x)}{\kappa}\Big|^{p(x)}\Big) dx \leq 1\Big\}.$$

Proposition 1. *Assume that for each $v \in X$*
$$\Lambda_p(v) = \int_\Omega \big(|\Delta v(x)|^{p(x)} + |\nabla v(x)|^{p(x)}\big) dx.$$
Then

- $\|v\| < 1 (= 1, > 1) \Leftrightarrow \Lambda_p(v) < 1 (= 1, > 1)$,
- $\|v\| < 1 \Rightarrow \|v\|_{p(x)}^{p^+} \leq \Lambda_p(v) \leq \|v\|_{p(x)}^{p^-}$,
- $\|v\| > 1 \Rightarrow \|v\|_{p(x)}^{p^-} \leq \Lambda_p(v) \leq \|v\|_{p(x)}^{p^+}$,

- $\|v_n\| \to 0 \Leftrightarrow \Lambda_p(v_n) \to 0$,
- $\|v_n\| \to \infty \Leftrightarrow \Lambda_p(v_n) \to \infty$.

The above statements can be proven in the same way as in [8, Theorem 1.3].

Proposition 2. [6] *Let p and q be two measurable functions verifying $p \in L^\infty(\Omega)$ and for all $x \in \Omega$ one has $1 < p(x)q(x) \le \infty$. Then we have for each $v \in L^{q(x)}(\Omega), v \neq 0$*

$$|v|_{p(x)} \le 1 \Rightarrow |v|^{p^+}_{p(x)q(x)} \le \left||v|^{p(x)}\right|_{q(x)} \le |v|^{p^-}_{p(x)q(x)},$$
$$|v|_{p(x)} \ge 1 \Rightarrow |v|^{p^-}_{p(x)q(x)} \le \left||v|^{p(x)}\right|_{q(x)} \le |v|^{p^+}_{p(x)q(x)}.$$

Lemma 3. [7] $\Delta^2_{p(.)} : W_0^{2,p(.)}(\Omega) \to W_0^{-2,p'(.)}(\Omega)$ *is belong to the class (S_+); that is, if $v_n \rightharpoonup v$ in $W_0^{2,p(.)}(\Omega)$ and $\limsup_{n\to\infty}\langle \Delta^2_{p(.)} v_n, v_n - v\rangle \le 0$, therefore $v_n \to v$ in $W_0^{2,p(.)}(\Omega)$.*

Lemma 4. [9] $-\Delta_{p(.)} : W_0^{1,p(.)}(\Omega) \to W_0^{-1,p'(.)}(\Omega)$ *is belong to the class (S_+).*

Proposition 5. *(Hölder inequality) Assume that p' is the conjugate function of p. Then for each $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$ one has*

$$\int_\Omega |uv|dx \le \left(\frac{1}{p^-} + \frac{1}{p'^-}\right) |u|_{p(x)}|v|_{p'(x)} \le 2|u|_{p(x)}|v|_{p'(x)}.$$

Furthermore, if p_1, p_2, $p_3 \in C_+(\Omega)$ such that $\frac{1}{p_1(x)} + \frac{1}{p_2(x)} + \frac{1}{p_3(x)} = 1$, then according to [9, Proposition 2.5] *we have for all $u \in L^{p_1(x)}(\Omega)$, $v \in L^{p_2(x)}$et $w \in L^{p_3(x)}$*

$$\int_\Omega |uvw|dx \le \left(\frac{1}{p_1^-} + \frac{1}{p_2^-} + \frac{1}{p_3^-}\right) |u|_{p_1(x)}|v|_{p_2(x)}|w|_{p_3(x)}.$$

Theorem 2.1. [3] Assume that $p, q \in C_+(\Omega)$.

If $q(x) < p^*(x)$ where

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-2p(x)}, & p(x) < \frac{N}{2}, \\ +\infty, & p(x) \ge \frac{N}{2}, \end{cases}$$

then there exists a compact and continuous embedding $X \hookrightarrow L^{q(x)}(\Omega)$.

In what follows, it is assumed that $\Phi \in C^1(X, \mathbb{R})$ with X is a Banach space.

Definition 2.1. Let $u_0 \in X$ and $k = \Phi(u_0)$. If $\Phi'(u_0) = 0$, then k and u_0 are called respectively a critical value of Φ and a critical point of Φ.

Definition 2.2. The Palais-Smale condition (PSC) is said to be satisfied by a functional Φ at the level k ($k \in \mathbb{R}$) if for all sequence $\{u_n\}_{n\in\mathbb{N}}$ of X verifying $\Phi(u_n) \to k$ in $\mathbb{R}$ and $\Phi'(u_n) \to 0$ in X^* (the dual space of X) it is possible to extract a sub-sequence strongly converging to a critical point of Φ in X.

Lemma 2.1. *If Φ satisfies the (PSC) at the k level ($k \in \mathbb{R}$) and is bounded from below, then Φ reaches its minimum k.*

Theorem 2.2. [2](Mountain pass Theorem) Consider a real infinite dimensional Banach space X. If $\Phi \in C^1(X, \mathbb{R})$ checks the following conditions

(1) Φ' is Lipschitz continuous on bounded subsets of X,
(2) $\Phi(u)$ verifies the (PSC),
(3) $\Phi(0) = 0$,
(4) there are positive constants R and C that satisfy $\Phi(u) \geq C$ if $\|u\| = R$, and there exists $w \in X$ with $\|w\| > R$ such that $\Phi(w) \leq 0$.

Then

$$c = \inf_{\tau \in \Gamma} \sup_{t \in [0,1]} \Phi\big(\tau(t)\big),$$

is a critical value of Φ, where

$$\Gamma = \{\tau \in C([0,1], X) : \tau(0) = 0, \tau(1) = w\}.$$

Theorem 2.3. [13](Symmetrical mountain pass lemma) Let Γ_n be the family of closed symmetric subsets H of X with $0 \notin H$ and $\gamma(H) \geq n$ where $\gamma(H)$ is the genus of H, i.e.,

$$\gamma(H) = inf\{n \in \mathbb{N} : \exists g : H \to \mathbb{R}^n \setminus \{0\} \text{ such that } g \text{ is an odd continuous mapping }\}.$$

If Φ checks the following conditions

(1) $\Phi(u)$ is even,
(2) $\Phi(u)$ is bounded from below,
(3) $\Phi(0) = 0$,
(4) $\Phi(u)$ verifies the (PSC),
(5) $\forall n \in \mathbb{N}$, $\exists H_n \in \Gamma_n : \sup_{u \in H_n} \Phi(u) < 0$.

Then, each $c_n := \inf_{H \in \Gamma_n} \sup_{u \in H} \Phi(u)$ is a critical value of Φ. Furthermore, Φ admits a sequence of non-trivial critical points $\{u_n\}$ verifying

$$\Phi'(u_n) = 0, \quad \Phi(u_n) \leq 0 \quad \textit{and} \quad \lim_n u_n = 0.$$

3. Main results

In this section, using mountain pass theorem and under some conditions we show in the first result that the problem (P_λ) has a non-trivial weak solution, and using symmetrical mountain pass lemma and other conditions we prove in second result that there is existence of a sequence of non-trivial weak solutions of (P_λ), and this for each strictly positive λ and subject to the following conditions

$$\begin{array}{ll} (A_1) & 1 < q(x) < p(x) < \frac{N}{2} < m(x) \text{ with } q(x) < p^*(x) \textit{ for each } x \in \overline{\Omega}. \\ (A_2) & \vartheta \in L^{m(x)}(\Omega) \text{ such that there is a measurable set } \Omega_0 \subset \Omega \text{ verifying} \\ & \vartheta(x) > 0, \textit{ for each } x \in \overline{\Omega}_0. \\ (A_3) & \text{There exist } \theta_1 \geq \theta_0 > 0 \text{ such that } \theta_0 \leq M(t) \leq \theta_1 \textit{ for each } t \in \mathbb{R}^+. \end{array} \tag{3.1}$$

Let $m'(x)$ be the conjugate exponent of $m(x)$ and consider $\eta(x) := \frac{m(x)q(x)}{m(x)-q(x)}$. By (A_1) one get for all $x \in \overline{\Omega}$, $\eta(x) < p^*(x)$ and $m'(x)q(x) < p^*(x)$. Hence, according to Theorem 2.1, the embeddings $X \hookrightarrow L^{m'(x)q(x)}(\Omega)$ and $X \hookrightarrow L^{\eta(x)}(\Omega)$ are compact and continuous.

Definition 3.1. A fixed point $u \in X$ is called a weak solution of (P_λ) if for each $v \in X$ one has

$$M(\sigma)\bigg(\int_\Omega (|\Delta u|^{p(x)-2}\Delta u\Delta v + |\nabla u|^{p(x)-2}\nabla u\nabla v)\,dx\bigg) = \lambda\int_\Omega |u|^{q(x)-2}uv\,dx\bigg(\int_\Omega \frac{\vartheta(x)}{q(x)}|u|^{q(x)}\,dx\bigg)^r.$$

In the case where $u \neq 0$, it is said that λ is the eigenvalue of (P_λ).

The energy functional $\Phi_\lambda : X \to \mathbb{R}$ associated to (P_λ) is set as follows

$$\Phi_\lambda(u) = \widetilde{M}\bigg(\int_\Omega \frac{|\Delta u|^{p(x)}}{p(x)} + \frac{|\nabla u|^{p(x)}}{p(x)}\,dx\bigg) - \frac{\lambda}{r+1}\bigg(\int_\Omega \frac{\vartheta(x)}{q(x)}|u|^{q(x)}dx\bigg)^{r+1},$$

such that $\widetilde{M}(\sigma) = \int_0^\sigma M(s)\,ds$.

It is easy to show that $\Phi_\lambda \in C^1(X, \mathbb{R})$ and that for each $u, v \in X$ one has

$$\begin{aligned}\Phi'_\lambda(u)(v) =& M(\sigma)\bigg(\int_\Omega (|\Delta u|^{p(x)-2}\Delta u\Delta v + |\nabla u|^{p(x)-2}\nabla u\nabla v)\,dx\bigg)\\ &- \lambda\bigg(\int_\Omega \frac{\vartheta(x)}{q(x)}|u|^{q(x)}dx\bigg)^r\int_\Omega \vartheta(x)|u|^{q(x)-2}uvdx.\end{aligned}$$

Consequently, the weak solution of problem (P_λ), coincides with the critical point of Φ_λ.

We consider the functionals $\varphi, \chi : X \to X^*$ defined as follows

$$\varphi(u) = \widetilde{M}(u)F(x)(u),$$

and

$$\chi(u) = B(u)E(u),$$

where for each $u, v \in X$

$$\langle F(u), v\rangle = \int_\Omega \big(|\Delta u|^{p(x)-2}\Delta u\Delta v + |\nabla u|^{p(x)-2}\nabla u\nabla v\big)\,dx,$$

$$B(u) = \bigg(\int_\Omega \frac{\vartheta(x)}{q(x)}|u|^{q(x)}dx\bigg)^r \quad \textit{and} \quad \langle E(u), v\rangle = \int_\Omega \vartheta(x)|u|^{q(x)-2}uvdx.$$

We are able to write Φ_λ as follows:

$$\Phi_\lambda(u) = \varphi(u) - \lambda\chi(u).$$

Now, we can introduce our first main result.

Theorem 3.1. Assume that

(i) conditions (A_1), (A_2) and (A_3) stated in (3.1) holds,
(ii) $p^+ < q^-(r+1)$,
(iii) $\frac{\theta_1 p^+}{\theta_0} < \frac{(q^-)^{(r+1)}(r+1)}{(q^+)^r}$.

Then for each λ strictly positive, the problem (P_λ) admits a non-trivial weak solution.

Proof. The proof consists of three steps:

Step 1: There are two strictly positive reals R and C satisfying

$$\Phi_\lambda(u) \geq C \text{ for each } u \text{ in } X \text{ with } \|u\| = R. \tag{3.2}$$

Recalling that

$$\Phi_\lambda(u) \geq \frac{\theta_0}{p^+}\int_\Omega (|\Delta u|^{p(x)} + |\nabla u|^{p(x)})\, dx - \frac{\lambda}{r+1}\left[\int_\Omega \frac{\vartheta(x)}{q(x)}|u|^{q(x)}dx\right]^{r+1}.$$

First, through the continuity of the embedding $X \hookrightarrow L^{m'(x)q(x)}(\Omega)$ we get the existence of $C_1 > 0$ such that for all $u \in X$ one has

$$|u|_{m'(x)q(x)} \leq C_1\|u\|. \tag{3.3}$$

Let us fix $R \in (0,1)$ with $R < \frac{1}{C_1}$. Hence by (3.3) one has for any $u \in X$, $|u|_{m'(x)q(x)} < 1$ with $\|u\| = R$. Then, one get

$$\int_\Omega \vartheta(x)|u|^{q(x)}dx \leq |\vartheta|_{m(x)}\left||u|^{q(x)}\right|_{m'(x)} \leq |\vartheta|_{m(x)}|u|^{q^-}_{m'(x)q(x)}, \tag{3.4}$$

for any $u \in X$ with $\|u\| = R$.

Using (3.3) and (3.4) together, we deduce that

$$\int_\Omega \vartheta(x)|u|^{q(x)}dx \leq C_1^{q^-}|\vartheta|_{m(x)}\|u\|^{q^-},$$

for any $u \in X$ such that $\|u\| = R$.

It follows that,

$$\begin{aligned}\Phi_\lambda(u) &\geq \frac{\theta_0}{p^+}R^{p^+} - \frac{\lambda}{(r+1)(q^-)^{r+}}C_1^{q^-(r+1)}|\vartheta|^{r+1}_{s(x)}R^{q^-(r+1)}\\ &\geq R^{p^+}\left(\frac{\theta_0}{p^+} - \frac{\lambda}{(r+1)(q^-)^{r+}}C_1^{q^-(r+1)}|\vartheta|^{r+1}_{m(x)}R^{q^-(r+1)-p^+}\right).\end{aligned}$$

From (ii), we deduce that, in a neighborhood of 0, the function

$$R \mapsto \frac{\theta_0}{p^+} - \frac{\lambda}{(r+1)(q^-)^{r+}}C_1^{q^-(r+1)}|\vartheta|^{r+1}_{s(x)}R^{q^-(r+1)-p^+}$$

is strictly positive. This implies the existence of the positive numbers R, C satisfying (3.2).

Step 2: There is $w \in X$ with $\|w\| > R$ and $\Phi_\lambda(w) \leq 0$.

Let $\psi \in C_0^\infty$ with $\psi \neq 0$. For $t > 1$, one has

$$\Phi_\lambda(t\psi) \leq \frac{\theta_1 t^{p^+}}{p^-}\int_\Omega (|\Delta \psi|^{p(x)} + |\nabla \psi|^{p(x)})\, dx - \frac{\lambda}{r+1}\frac{t^{q^-(r+1)}}{(q^+)^{r+1}}\left(\int_\Omega \vartheta(x)|\psi|^{q(x)}dx\right)^{r+1}.$$

From (ii) one get $\Phi_\lambda(t\psi) \to -\infty$ as $t \to +\infty$. Consequently, there exists $w = t\psi$ which verifies that $\|w\| > R$ and $\Phi_\lambda(w) \leq 0$.

Step 3: The function Φ_λ satisfies the (PSC).

Consider a sequence $(u_n) \subset X$ verifying $\Phi_\lambda(u_n) \to d$ and $\Phi'_\lambda(u_n) \to 0$ in X'. Assume that (u_n) is not bounded and $\|u_n\| > 1$, $\forall n \in \mathbb{N}$.
Hence, by choosing $\frac{\theta_1 p^+}{\theta_0} < \tau < \frac{(q^-)^{r+1}(r+1)}{(q^+)^r}$, we obtain

$$\begin{aligned} d + 1 + \|u_n\| &\geq \Phi_\lambda(u_n) - \frac{1}{\tau}\Phi'_\lambda(u_n)u_n \\ &\geq \left(\frac{\theta_0}{p^+} - \frac{\theta_1}{\tau}\right)\Big(\int_\Omega (|\Delta u_n|^{p(x)} + |\nabla u_n|^{p(x)})\,dx\Big), \\ &+ \lambda\left(\frac{1}{\tau(q^+)^r} - \frac{1}{(q^-)^{r+1}(r+1)}\right)\Big(\int_\Omega \vartheta(x)|u_n|^{q(x)}dx\Big)^{r+1}, \\ &\geq \left(\frac{\theta_0}{p^+} - \frac{\theta_1}{\tau}\right)\big(\Lambda_p(u_n)\big)^{p-}, \\ &\geq \left(\frac{\theta_0}{p^+} - \frac{\theta_1}{\tau}\right)\big(\|u_n\|\big)^{p-}, \end{aligned}$$

which contradicts that $p^- > 1$. Thus, the sequence (u_n) is bounded in X, therefore as X is reflexive, it follows that there is a subsequence still denoted (u_n), that satisfies $u_n \rightharpoonup u$ in X. Since

$$\Phi'_\lambda(u_n) \to 0,$$

then

$$\Phi'_\lambda(u_n)(u_n - u) \to 0,$$

and so

$$\begin{aligned} \widetilde{M}(u_n)\Big(\int_\Omega |(\Delta u_n|^{p(x)-2}\Delta u_n \Delta(u_n - u) + |\nabla u_n|^{p(x)-2}\nabla u_n \nabla(u_n - u))\,dx\Big) \\ -\lambda B(u_n)\int_\Omega |u_n|^{q(x)-2}u_n(u_n - u)dx \to 0. \end{aligned}$$

From Proposition 5, one get

$$\Big|\int_\Omega |u_n|^{q(x)-2}u_n(u_n - u)\,dx\Big| \leq \int_\Omega |u_n|^{q(x)-1}|(u_n - u)\,dx| \leq C|u_n|^{q(x)-1}_{\frac{q(x)}{q(x)-1}}|(u_n - u)|_{q(x)}.$$

Hence X is compactly embedded in $L^{q(x)}$ since for each $x \in \Omega : q(x) < p^*(x)$. Therefore (u_n) converges strongly to u in $L^{q(x)}$. Consequently

$$\int_\Omega |u_n|^{q(x)-2}u_n(u_n - u)dx \to 0.$$

Because (u_n) is bounded, there is $c_1, c_2 > 0$ such that

$$c_1 \leq B(u_n) \leq c_2.$$

Hence $G(u_n)(u_n - u) \to 0$. We can suppose that, there is $c_3, c_4 > 0$ satisfying

$$c_3 \leq \widetilde{M}(u_n) \leq c_4.$$

In the other hand, one has

$$L_{p(x)}(u_n)(u_n-u)=\int_\Omega |\Delta u_n|^{p(x)-2}\Delta u_n \Delta(u_n-u)dx \to 0,$$

and

$$K_{p(x)}(u_n)(u_n-u)=\int_\Omega |\nabla u_n|^{p(x)-2}\nabla u_n \nabla(u_n-u)dx \to 0.$$

Since $L_{p(x)}$ and $K_{p(x)}$ is belong to the class (S_+), one obtains $u_n \to u$ in X.

Now, by **Step 1** and **Step 2**, we obtain

$$\max(\Phi_\lambda(0),\Phi_\lambda(w))=\Phi_\lambda(0)<\inf_{\|u\|=R}\Phi_\lambda(u)=\beta.$$

Thus, from **Step 3** and the mountain pass theorem, we reach the conclusion that there exist a non-trivial weak solution u of (P_λ) related to the critical value of Φ_λ provided by

$$c:=\inf_{\tau\in\Gamma}\sup_{t\in[0.1]}\Phi_\lambda(\tau(t))\geq\beta,$$

where

$$\Gamma=\{\tau\in C([0,1],X):\tau(0)=0,\tau(1)=w\}.$$

In the following, our second main result where we show that it exists of a sequence of non-trivial weak solutions of (P_λ).

Theorem 3.2. Suppose that conditions (A_1), (A_2) and (A_3) hold true. If $p^- < q^+(r+1)$, then for each λ strictly positive there is a sequence (u_n) of non-trivial weak solutions of (P_λ) which converges strongly to 0 in X.

Proof. **Step 1:** Φ_λ is even, $\Phi_\lambda(0)=0$, bounded from below and verifies the (PSC).

It is easy to show the first two assertions concerning the functional Φ_λ; for the third we get from Proposition 5

$$\int_\Omega \vartheta(x)|u|^{q(x)}dx \leq |\vartheta|_{m(x)}\left||u|^{q(x)}\right|_{m'(x)} \leq |\vartheta|_{m(x)}|u|^{q^j}_{m'(x)q(x)}, \quad \forall u\in X, \tag{3.5}$$

such that

$$\begin{cases} j=+ \quad, |u|_{m'(x)q(x)}>1, \\ j=- \quad, |u|_{m'(x)q(x)}<1. \end{cases}$$

Taking into account that X is continuously embedded in $L^{m'(x)q(x)}(\Omega)$, it yields the existence of $C_2>0$ satisfying

$$|u|_{m'(x)q(x)}\leq C_2|u|, \quad \forall u\in X. \tag{3.6}$$

Combining (3.5) and (3.6), we obtain

$$\int_\Omega \vartheta(x)|u|^{q(x)}dx \leq C_2^{q^j}|\omega|_{m(x)}\|u\|^{q^j}. \tag{3.7}$$

Therefore, from (3.7), it follows that

$$\begin{aligned}\Phi_\lambda(u) =& \widetilde{M}\left(\int_\Omega \frac{|\Delta u|^{p(x)}}{p(x)} + \frac{|\nabla u|^{p(x)}}{p(x)}\,dx\right) - \frac{\lambda}{r+1}\left[\int_\Omega \frac{\vartheta(x)}{q(x)}|u|^{q(x)}dx\right]^{r+1},\\ &\geq \frac{\theta_0}{p^+}\Lambda_p(u) - \frac{\lambda C_2^{q^j}}{(r+1)(q^-)^{r+1}}|\vartheta|_{m(x)}\|u\|^{q^j},\\ &\geq \frac{\theta_0}{p^+}\xi(\|u\|) - \frac{\lambda C_2^{q^j}}{(r+1)(q^-)^{r+1}}|\vartheta|_{m(x)}\|u\|^{q^j},\end{aligned}$$

where $\xi : [0,+\infty[\to \mathbb{R}$ is given as

$$\xi(t) = \begin{cases} t^{p^+}, & \text{if } t \leq 1,\\ t^{p^-}, & \text{if } t > 1.\end{cases}$$

On one hand by $q^j < p^-$, we obtain that Φ_λ is bounded from below. On the other hand since $\Phi_\lambda(u) \to \infty$ when $\|u\| \to \infty$, then Φ_λ is coercive.

It remains in this step to demonstrate that Φ_λ satisfies the (PSC). For this we consider a sequence $(u_n) \subset X$ verifying $\Phi_\lambda(u_n) \to d$ with $\Phi'_\lambda(u_n) \to 0$ in X^*. According to the coercivity of Φ_λ, we deduce that (u_n) is bounded in X, and so we can finish this proof by employing arguments that are identical to those that were used in Step 3 of the previous proof.

Step 2: For any $n \in \mathbb{N}^*$, there is $H_n \in \Gamma_n$ satisfying

$$\sup_{u\in H_n} \Phi_\lambda(u) < 0.$$

To show this we consider $a_1, a_2, \ldots, a_n \in C_0^\infty(\Omega)$ which verify for all $i, j \in \{1, 2, \ldots, n\}$, $i \neq j$, the condition $\operatorname{supp}(a_i) \cap \operatorname{supp}(a_j) = \emptyset$ and the Lebesgue measure of $\operatorname{supp}(a_i)$ and $\operatorname{supp}(a_j)$ is strictly positive.

Let us note by A_n the set $\operatorname{span}\{a_1, a_2, \ldots, a_n\}$. It is easy to show that $\dim A_n = n$ and for each $\mu \in A_n \setminus \{0\}$ we obtain

$$\int_\Omega \vartheta(x)|\mu(x)|^{q(x)}dx > 0.$$

Consider $S = \{\mu \in X : \|\mu\| = 1\}$ and for all $0 < t \leq 1$ $H_n(t) = t(S \cap A_n)$.

Clearly, we show that for any $t \in]0,1]$ we have $\gamma(H_n(t)) = n$ and

$$\begin{aligned}\sup_{u\in H_n(t)} \Phi_\lambda(u) \leq& \sup_{\mu\in S\cap A_n} \Phi_\lambda(t\mu)\\ &= \sup_{\mu\in S\cap A_n}\left\{\widetilde{M}\left(\int_\Omega t^{p(x)}\left(\frac{|\Delta\mu|^{p(x)}}{p(x)} + \frac{|\nabla\mu|^{p(x)}}{p(x)}\right)dx\right) - \frac{\lambda}{r+1}\left(\int_\Omega \frac{\vartheta(x)}{q(x)}|t\mu|^{q(x)}dx\right)^{r+1}\right\}\\ &\leq \sup_{\mu\in S\cap A_n}\left\{\frac{\theta_1 t^{p^-}}{p^-}\Lambda_p(x) - \frac{\lambda t^{q^+(r+1)}}{(r+1)(q^+)^{r+1}}\left(\int_\Omega \vartheta(x)|\mu|^{q(x)}dx\right)^{r+1}\right\}\end{aligned}$$

$$\leq \sup_{\mu \in S \cap A_n} \left\{ t^{p^-} \left[\frac{\theta_1}{p^-} - \frac{\lambda}{(r+1)(q^+)^{r+1} t^{p^- - q^+(r+1)}} \left(\int_\Omega \vartheta(x) |\mu|^{q(x)}\, dx \right)^{r+1} \right] \right\}.$$

Since $\delta := \min_{\mu \in S \cap A_n} \int_\Omega \vartheta(x) |\mu(x)|^{q(x)}\, dx > 0$, we can find a rather small value of t which verifies

$$\frac{\theta_1}{p^-} - \frac{\lambda}{(r+1)(q^+)^{r+1} t^{p^- - q^+(r+1)}} \delta < 0.$$

Consequently

$$\sup_{u \in H_n(t_n)} \Phi_\lambda(u) < 0.$$

Step 3: We conclude that $\Phi_\lambda(u)$ admits a sequence of non-trivial weak solutions $(u_n)_n$ because all assumptions of the symmetric mountain pass lemma have been verified, and for all n, one has

$$u_n \neq 0, \quad \Phi'_\lambda(u_n) = 0, \quad \Phi_\lambda(u_n) \leq 0, \quad \lim_n u_n = 0.$$

4. Final comments

We need to make it clear that if $\vartheta(x) = 1$, then (P_λ) reduces to the following bi-nonlocal elliptic problem

$$\begin{cases} M(\sigma) \left(\Delta^2_{p(x)} u - \Delta_{p(x)} u \right) = \lambda |u|^{q(x)-2} u \left(\int_\Omega \frac{1}{q(x)} |u|^{q(x)} dx \right)^r & \text{in } \Omega, \\ u \in W^{2,p(.)}(\Omega) \cap W_0^{1,p(.)}(\Omega), \end{cases} \tag{4.1}$$

then our theorems still valid in this case. If we replace in the equation's left side of (4.1) the positive real σ by the expression $'' - M\left(\int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} dx \right) \Delta_{p(x)} u''$ we get the same equation in problem presented in [5], and if we replace it with the expression $''M\left(\int_\Omega \frac{1}{p(x)} |\Delta u|^{p(x)} dx \right) \Delta^2_{p(x)} u''$ we get the same that one studied in [12]. Hence, our results improve the corresponding results obtained in [5, 12]. The findings presented in this study can serve as a theoretical foundation for further exploration of similar problems.

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