A Simple Guide to Lawn Mowing

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Abstract

This paper presents the total time required to mow a two-dimensional rectangular region of grass using a push mower. In deriving the total time, each of the three 'well known' (or intuitive) mowing patterns to cut the entire rectangular grass area is used. Using basic mathematics, analytical and empirical time results for each of the three patterns taken to completely cover this rectangular region are presented, and examples are used to determine which pattern provides an optimal total time to cut a planar rectangular region. This paper provides quantitative information to aid in deciding which mowing pattern to use when cutting one's lawn.

Nomenclature

A Area of the region, $A = L$ x W, m^2

D total distance traversed by the mower, m

l linear distance of each sweep, m

L length of rectangular region, m

i, j indices for sweep number

k number of horizontal sweeps for SS path/route

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n total number of sweeps for a particular path/route

N total number of turns for a particular path/route

S constant or average speed of the lawn mower, m/s

t time, seconds, s

W width of the rectangular region, m

w width of the lawn mower, m

β angular turn within the region (for SS path only), and at the boundaries (all three paths), radians

δ overlap parameter between two successive sweeps, m

σ pertaining to sweep, or i^{th} sweep, σ_i

θ sweep angle ($\tan\theta = \frac{W}{L}$) for the diagonal sweep (DS), radians

Subscript/Superscript

boundary effect of time due to the turns at the boundaries of the region

HS, SS, DS horizontal sweep, spiral sweep, and diagonal sweep, respectively

direction effect of time due to directional/angular turns made within the region

h horizontal sweep from spiral sweep (SS)

k index for length of each DS sweep

l lower ‘triangular’ region

u upper ‘triangular’ region

min minimum

1 Introduction

What is the 'best' way to cut your lawn? By 'best', how can you efficiently cut your lawn in the shortest amount of time? There are a lot of information available, but the recommendations that are available are not supported with quantitative information or that an aid of a computer algorithm is required to implement a pattern to mow one's lawn. This paper provides recommendations based on analyses obtained from basic level mathematics to cut one's lawn effectively.

The time taken to mow a grass region was investigated by middle-school students as part of the questions administered by the Albuquerque Public School (APS)[1] assessment program. Three different patterns/paths to cut a rectangular region were identified, and students were asked to determine which path (or sweep pattern) gave the shortest time to cover the entire two-dimensional grass region. Various problem-solving skill levels were demonstrated by the students in tackling this problem. One student actually measured the total distance with a ruler based on a sketch, while another student derived an arithmetic sequence to determine the total distance required to cover the entire grass area.

In addition to the three different paths analyzed by these APS students, a similar approach is proposed by [ALL21], where a series of horizontal sweeps or 'cuts' by the push mower are used. As the mower approaches the boundary of the region, the mower moves backwards horizontally and continues cutting the grass until the mower reaches the opposite boundary. Then, the mower moves forward and repeats the same process until the entire region is covered. The main reason for this approach is to replace the 180 degree turns at the boundaries (which will incur more time compared to transitioning to a backward sweep) to reduce the overall time. Empirical data (for time) is also provided so that an accurate total time can be obtained. Various examples are used (by [ALL21]) and the examples assume that the grass region is limited to a square region in the calculations. Other authors (for example, [TRI10], [BLA21], and [LAR21]) propose intricate mowing patterns in cutting the lawn efficiently, but suggestions to various mowing patterns lack quantitative information on which mowing patterns are better.

This lawn mowing problem is generally known as the "ice rink problem" or the "milling problem", where both problems fall into a general class of the traveling salesman problems (TSPs). These problems have been investigated by various researchers ([AFM93], [AFM00], and [MCS97]) using advanced graph theoretic algorithms to find an approximate optimal patterns/paths/routes (minimum time taken) to cover any generalized planar region.

In this paper, a straight-forward, analytically and physically-based empirical mod-

[1]Unfortunately, this link does not exist anymore. The authors regret the absence of this reference, as this reference is important. Because of the importance of this reference, the authors will use 'APS' as the reference. The link was: *http://www.rda.aps.edu/mathtaskbank/pdfs/instruct/6−8/i68lenny.pdf*, 2001, *Test Bank*, *Albuquerque Public Schools*.

els will be presented by re-examining and extending the solutions to the lawn mowing problem posed by APS and by [ALL21]. The main reason for choosing the three patterns/paths proposed by APS is due to that fact that analytical (closed-form) solutions can be obtained using basic algebra. It is important to note here that through the derivation process, a good physical understanding of the problem is achieved, and as a result, discoveries called as 'Facts' (or 'nuggets' of information) in this paper are brought to attention and explained without mathematical rigor.

The *total* time taken to mow the entire region will be presented and this will be achieved by using a proposed three-component total time model. A summary of results based on this three-component time model based on the three mowing patterns/ paths/ routes will be presented.

2 Description of the Problem

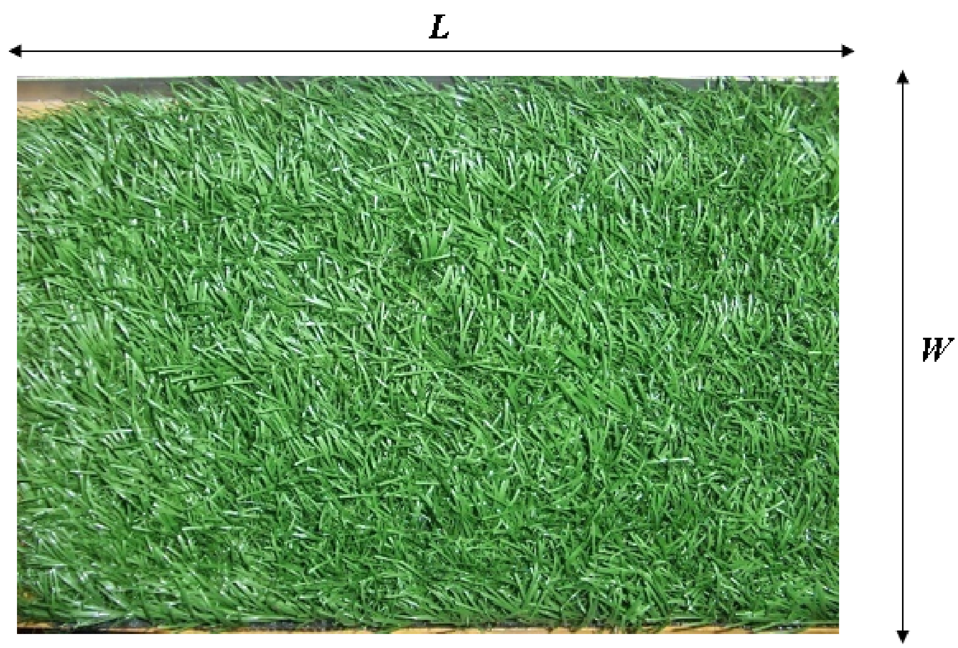

Figure 1: Two-Dimensional Rectangular Region of Grass.

A two-dimensional rectangular region is shown in Figure 1. The region is simply-connected (that is, the region does not have any obstacles, such as patches of dirt, garden, shrubs, trees, bushes, pool, etc. all within the region), with length L and width W (with $L \geqslant W$) which contains entirely of grass of uniform height. The blade diameter/length of the push mower is w, so if the mower travels a linear distance l, the area covered is assumed to be lw.

In the analyses presented in this paper, three different patterns/paths/routes have been identified to cover the entire area (which was also proposed by APS), and they are shown on Figures 2 through 4. The mower will only use one of the three paths in cutting the entire grass area:

1) a series of horizontal sweeps (HS[2], which is similar to the sequence of alternating forward and backward sweeps or 'cuts' analyzed by APS; or

2) a series of diagonal sweeps (DS) parallel to the slope W/L, which is the slope of an imaginary diagonal line constructed from the lower left corner of the region to the top right corner; or

3) a series of alternating horizontal and vertical sweeps (or also called the 'spiral' sweep, SS) in the clock-wise direction.

The following assumptions have been assumed in the analyses:

1. the mechanical push mower works flawlessly throughout the entire mowing process, and a riding push mower is not used;

2. the mower possesses discipline by mowing the region with *constant speed* S by using a prescribed route/path (and not deviating from it) until the entire region has been cut/covered (that is, the mower does not take a break); and

3. during the mowing process, as the mower approaches the edges (of the boundaries), the mower will stop (instantaneously), change direction, and then proceed in taking further sweeps. That is, the mower does *not* move forward while turning, but *turns only when stopped instantaneously.*

The total time taken to cover the entire area for each of the three routes will be demonstrated later using various examples.

For each pattern/path/route chosen, a series of cuts called *sweeps* are used until the entire region has been covered. A *sweep* is defined, in this paper, as the *linear* [3] distance traversed by the mower, and that current sweep is terminated when the direction of the current sweep has changed. For a given pattern/route/path, each i^{th} linear sweep denoted by σ_i is shown in Figures 2 to 4.

[2]Although, a series of vertical sweeps can be used, a series of horizontal sweeps will be used as preference throughout the paper.

[3]Only linear sweeps will be considered in this paper as curved sweeps will not be considered.

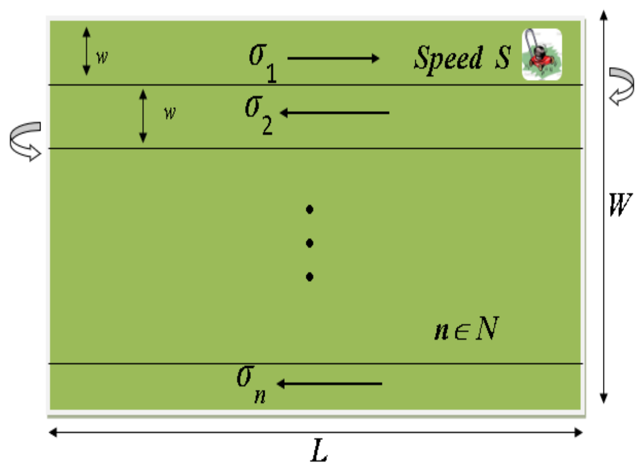

Figure 2: A Series of Alternating Horizontal Sweeps (HS).

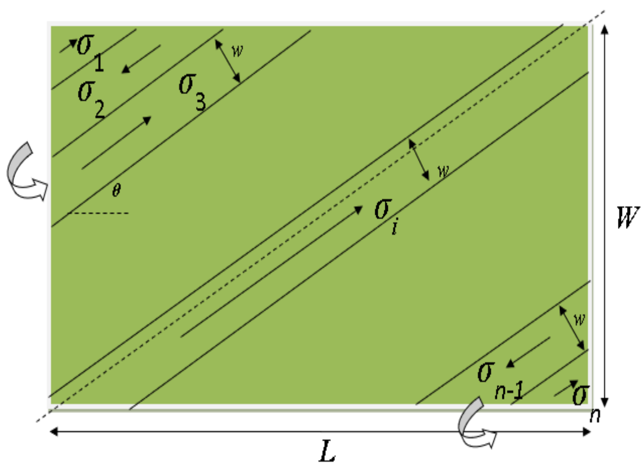

Figure 3: A Series of Diagonal Sweeps (DS) Parallel to the Slope W/L.

3 Scope of the Analyses

For each of the three routes/paths, time required, denoted by t_{min}, will be presented first. This time is the minimum time required to *simply* cover the entire area, and does not take into account additional time incurred due to directional changes made by the mower during the mowing process. In reality, changing directions will require the mower to slow down, similar to when a car slows down when it turns. The solutions to the lawn mowing problem administered by APS, and by [TRI10], [BLA21], and [LAR21] mainly account for this t_{min}, and do not consider additional time incurred due to directional

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changes. To obtain t_{min} , the total distance D traveled by the mower for each of the chosen patterns/paths/routes is first obtained, and then t_{min} is obtained using the relationship $t_{min} = D/S$, where S is the average constant speed of the push mower throughout the mowing process.

Also, to model a 'physically' realistic mowing situation, an overlap parameter δ is introduced. This overlap parameter is the portion of the sweep area where the push mower covers the *same* area between the two successive sweeps. For example, if an area lw is covered on the first sweep (assuming that the mower travelled a linear distance of l), the mower will not be able to precisely align one edge of the push mower during the subsequent sweep with the same edge from the previous sweep. In general, the area covered after the i^{th} sweep, that is, the $(i+1)^{th}$ sweep, will be not be lw, but $l(w-\delta)$, where $0 < \delta < w$. It is assumed here that the mower will be able to maintain a constant overlap δ between any two consecutive sweeps and also during the entire mowing process.

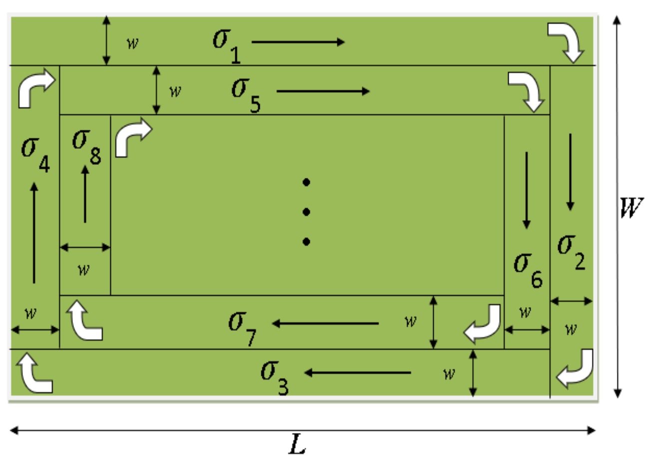

Figure 4: A Series of Horizontal and Vertical Sweeps in the Clock-wise Direction (SS).

3.1 Case 1: Horizontal Sweeps (HS)

Although arbitrary, the mowing process will start at the top left corner of the region (see Figure 2). From here, the mower will move horizontally from 'left to right' flush against the top boundary until the mower reaches the boundary at the opposite side. After stopping, the mower turns 180 degrees and travels from "right to left" (in the opposite direction) until the mower reaches the boundary at the opposite side. This process is continued until the entire region has been covered.

3.1.1 No Overlap Between Consecutive Sweeps ($\delta = 0$)

For each i^{th} sweep, σ_i , the linear distance traveled l_i is given by $l_i = L$, and for this σ_i , the area covered is wL (except for the last sweep). The total distance D_{HS} traveled by the mower is

$$D_{HS} = \sum_{i=1}^{n_{HS}} l_i = n_{HS}L, \tag{1}$$

where $n_{HS} = \lceil \frac{W}{w} \rceil$ is the number of horizontal sweeps required to cover the entire area, and $\lceil \cdot \rceil$ is the ceiling function. The number of turns N_{HS} the mower makes to cover the entire area is simply given by $N_{HS} = n_{HS} - 1$, and this will be used later when additional time is incurred when the mower changes directions.

3.1.2 Constant Overlap Between Consecutive Sweeps ($0 < \delta < w$)

Using this *HS* pattern, the total distance $D_{HS,}$ traveled by the mower is given by

$$D_{HS,\delta} = n_{HS,\delta}L, \tag{2}$$

where $n_{HS,\delta} = 1 + \left\lceil \frac{W-w}{w-\delta} \right\rceil$ is the number of sweeps (by including sweep overlap δ to cover the entire regions. It has been assumed here that in the first sweep, the area covered is wL, and the area covered during subsequent sweeps is $(w - \delta)L$, except possibly for the final sweep. The total number of turns is given by $N_{HS\ ,\delta} = n_{HS,\delta} - 1$.

3.2 Case 2: Spiral Sweeps (SS)

Starting once again at the top-left corner of the region, the first sweep is a horizontal sweep to the right until the mower reaches the boundary at the opposite side (see Figure 4). Then the mower travels vertically downward until the front of the push mower reaches the lower boundary. A horizontal sweep to the left and a vertical sweep 'up' are used until the mower reaches a distance w away from the upper boundary. These four successive sweeps form one complete cycle (or spiral) of a set of similar cycle of sweeps (spirals) to cover the entire region.

The distance traveled on the 1^{st} (horizontal) sweep is $l_1 = L$, the 2^{nd} sweep (1^{st} vertical sweep) is $l_2 = W - w$, the 3^{rd} sweep (2^{nd} horizontal sweep) is $l_3 = L - w$, 4^{th} sweep (2^{nd} vertical sweep) is $l_4 = W - 2w$, and so on until the entire area is covered by the sequence of these horizontal and vertical sweeps. When the sweep number is odd, horizontal sweeps have been used, and when the sweep number is even, vertical sweeps

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have been used. Thus, the i^{th} linear distance l_i traveled for sweep σ_i, it is given by:

$$l_i = \begin{cases} L - \frac{1}{2}(i-1)w, & i = 1, 3, 5, \ldots, i \in N_n \\ W - \frac{1}{2}iw, & i = 2, 4, 6, \ldots, i \in N_n \end{cases} \tag{3}$$

We see here that for each of the horizontal and vertical sweeps, arithmetic sequences are established.

Here are some 'observations/discoveries' made based on the analyses presented above.

Fact #1: When starting from the top-left corner of the region, it will be assumed that the first sweep is the horizontal sweep. Under this situation, the final/last sweep will always be a horizontal sweep. This is so because there will be less 'number of horizontal layers/slices' compared to 'the number of vertical layers/slices', since $W \leqslant L$. Namely, if $W = n_h w$, where $n_h = \lceil \frac{W}{w} \rceil$, and n_h is odd, the final sweep is a horizontal sweep from "right to left"; otherwise, the final horizontal sweep is sweep from "left to right".

The total distance traversed to completely cover the region (with the use of equations (3)) is then given by

$$D_{SS} = \sum_{i=1}^{n_{SS}} l_i = n_h \left[L + W - w(n_h - 1)\right] - W \tag{4}$$

where $n_h = \lceil \frac{W}{w} \rceil$. The number of vertical sweeps $n_v = n_h - 1$. Therefore, the total number of sweeps $n_{S\ S}$ is given by $n_{SS} = n_h + n_v = 2n_h - 1$.

Fact #2a: When $W \pmod{w} \equiv 0$, $D_{SS} = D_{HS}$ (see equations (1) and (4)). In both patterns of *HS* and *DS*, all sweeps are contained inside the region, so an exact area of *WL* is covered. The total distance traveled is simply $WL/w = A/w$ (for both patterns).

Fact #2b: When $W \pmod{w} \neq 0$, the total distance covered using *SS* is less than the total distance traveled using *HS*; that is, $D_{SS} < D_{HS}$. Since $\lceil \frac{W}{w} \rceil = \frac{W}{w} + \varepsilon$, where $0 \leq \varepsilon < 1$, then for $W (\text{mod } w) \neq 0$, $\varepsilon > 0$ and D_{SS} , equation (4), can be simplified to $D_{SS} = n_h L - w(n_h - 1)\varepsilon < D_{HS}$ since $n_h = n_{HS}$. The simple work provided here addresses the question posed by [ALL21] that indeed $D_{SS} \leqslant D_{HS}$.

Fact #3: From Fact #1, the total number of sweeps n_{SS} is *always odd*. There are four (4) sweeps required to complete one cycle (that is, a series of horizontal – vertical – horizontal – vertical sweeps). Since the final sweep is a horizontal sweep, for a given cycle, either the first horizontal sweep of the cycle or the third horizontal sweep of the same cycle will complete the final cycle. That is, the total number of horizontal sweeps is given by n_h , and the total number of vertical sweeps, n_v is *one* less or $n_h - 1$. Therefore, the total number of sweeps is given by $n_{SS} = n_h + n_v$ or $n_{SS} = 2n_h - 1$.

Please note: For this *SS* route, overlap has not been considered due to multiple ways in which overlap can be introduced.

3.3 Case 3: Diagonal Sweeps (DS)

A series of "diagonal" sweeps are used to cover the entire area. Again, the first sweep starts at the upper left corner of the region, and then a series of diagonal sweeps parallel and in alternating direction relative to the imaginary diagonal line (with slope $\frac{W}{L}$) is used until the entire area is covered (see Figure (3)).

3.3.1 No Overlap Between Consecutive Sweeps ($\delta = 0$)

In analyzing the total distance covered, the region is divided into two regions: an "upper" and "lower" triangular regions which are separated by the aforementioned imaginary diagonal line (constructed from the bottom left corner to the top right corner). For each sweep within the upper triangular *portion of the region*, the i^{th} linear distance l_i covered is given by

$$l_i = \sqrt{\left(\frac{iw}{cos\theta}\right)^2 + \left(\frac{iw}{\sin\theta}\right)^2} = \frac{iw}{\sin\theta\ cos\theta} = iw\left(\frac{W}{L} + \frac{L}{W}\right), \quad i = 1, 2, \ldots, n_u \quad \text{(5a)}$$

where $tan\theta = \frac{W}{L}$, $n_u = \left\lfloor \frac{W cos\theta}{w} \right\rfloor$ is the number of sweeps in the upper triangular region *before* sweeping/traveling along the main diagonal, and $\lfloor\cdot\rfloor$ is the floor function.

The distance covered on the j^{th} sweep *within the lower triangular region* is similar to equation (5a), and is given by

$$l_j = 2\sqrt{W^2 + L^2} - w\left(n_u + j\right)\left(\frac{W}{L} + \frac{L}{W}\right),\ j = 1, 2, \ldots, n_l \quad \text{(5b)}$$

where $n_u = \left\lfloor \frac{W\cos\theta}{w} \right\rfloor$ is the number of sweeps in the upper triangular region *before* sweeping/traveling along the diagonal, n_l is the number of sweeps needed to cover the lower triangular portion of the region, and $\lfloor\cdot\rfloor$ is the floor function.

On the $(n_u + 1)^{th}$ sweep, that is for $i = n_u + 1$, the mower will travel a distance of $\sqrt{L^2 + W^2}$.

The distance traveled by the mower is summarized below:

$$l_k = \begin{cases} iw\left(\frac{W}{L} + \frac{L}{W}\right), & k = i = 1, 2, \ldots, n_u \\ \sqrt{L^2 + W^2}, & k = i = n_u + 1 \\ 2\sqrt{W^2 + L^2} - w\left(n_u + j\right)\left(\frac{W}{L} + \frac{L}{W}\right), & k = j = 1, 2, \ldots, n_l \end{cases} \quad \text{(6a)}$$

Based on equations (5a) and (5b), the derivation of the total distance traveled D_{DS} can be obtained by considering: the total distance traveled in the upper triangular region, D_{DS}^{u}, the distance traveled along the main diagonal D_{diag}, and the total distance traveled in the lower triangular region D_{DS}^{l}. Thus,

$$\begin{aligned} D_{DS} &= D_{DS}^{l} + D_{diag} + D_{DS}^{l} \\ &= (2n_l + 1)\sqrt{W^2 + L^2} + \frac{w}{2}\left(\frac{L}{W} + \frac{W}{L}\right)\left(n_u^2 + n_u - 2n_u n_l - n_l^2 - n_l\right) \end{aligned} \tag{6b}$$

where n_l is the number of sweeps needed to cover the lower triangular portion of the region, given by

$$n_l = \left\lceil \frac{2W\cos\theta}{w} \right\rceil - (n_u + 1) \tag{6c}$$

and, once again, $n_u = \left\lfloor \frac{W\cos\theta}{w} \right\rfloor$.

It is important to note here that the total number of sweeps is given by $n_{DS} = \left\lceil \frac{2W cos\theta}{w} \right\rceil$.

Fact #4a: A way to understand the total number of sweeps n_{DS} for this diagonal sweep can be explained as follows: each sweep covers a *vertical* distance of $w/\cos\theta$, and the total *vertical* distance of $2\,W$ (vertical distance W is covered in the upper-triangular region, and another vertical distance W in the lower-triangular region) is required to mow the entire region. So, the total number of sweeps required to cover the entire area is given by $n_{DS} = \left\lceil \frac{2W}{w/\cos\theta} \right\rceil = \left\lceil \frac{2W\cos\theta}{w} \right\rceil$.

Fact #4b: The total number of sweeps n_{DS} generally is made up of the number of sweeps in the upper triangular region, n_u , one sweep along the main diagonal, and the number of sweeps in the lower triangular region n_l . So, $n_{DS} = n_u + 1 + n_l$.

3.3.2 Constant Overlap Between Consecutive Sweeps ($0 < \delta < w$)

Similar to the case when $\delta = 0$, the distance covered by each sweep in the upper triangular region, along the diagonal, and the lower triangular region but now allowing overlap ($0 < \delta < w$), is given by

$$l_{k,\delta} = \begin{cases} [w + (i-1)(w-\delta)\left(\frac{W}{L} + \frac{L}{W}\right), & k = i = 1, 2, \dots,\ n_{u,\delta} \\ \sqrt{L^2 + W^2}, & k = i = n_{u,\delta} + 1 \\ 2\sqrt{W^2 + L^2} - [(w-\delta)(n_{u,\delta} + j) + \delta]\left(\frac{W}{L} + \frac{L}{W}\right), & k = j = 1, 2, \dots, n_{l,\delta} \end{cases} \tag{7}$$

where $n_{u,\delta} = 1 + \left\lfloor \frac{W\cos\theta - w}{w-\delta} \right\rfloor$ and $n_{l,\delta} = \left\lceil \frac{2W\cos\theta - w}{w-\delta} \right\rceil - n_{u,\delta}$ are, respectively, the number of sweeps in the upper and lower triangular portion of the region. It has been assumed here that on the *first* sweep, the distance covered is $l_1 = w\left(\frac{W}{L} + \frac{L}{W}\right)$. The total number of sweeps $n_{DS,\delta}$ is given by $n_{DS,\delta} = 1 + \left\lceil \frac{2W\cos\theta\ - w}{w-\delta} \right\rceil = n_{u,\delta} + 1 + n_{l,\delta}$.

The total distance covered but now allowing overlap, $0 < \delta < w$, is given by

$$D_{DS,\delta} = (2n_{l,\delta} + 1)\sqrt{W^2 + L^2} + \left(\frac{L}{W} + \frac{W}{L}\right)\left[\begin{array}{c} \delta\left(n_{u,\delta} - n_{l,\delta}\right) + \\ \frac{(w-\delta)}{2}\left(n_{u,\delta}^2 + n_{u,\delta} - 2n_{u,\delta}n_{l,\delta} - n_{l,\delta}^2 - n_{l,\delta}\right) \end{array}\right] \tag{8}$$

Table 1 below summarizes the analytical results: the number of sweeps required and the total distance traveled for each of the three patterns.

Route/Path	**Total Number of Sweeps, n**	**Total Distance, D (m)**
HS	$\left\lceil \frac{W}{w} \right\rceil$	$\left\lceil \frac{W}{w} \right\rceil L$
DS	$\left\lceil \frac{2W\cos\theta}{w} \right\rceil$ $cos\theta = L/\sqrt{L^2 + W^2}$	Eq.(6b)
SS	$2\left\lceil \frac{W}{w} \right\rceil - 1$	$\left\lceil \frac{W}{w} \right\rceil \left[L + W - w\left(\left\lceil \frac{W}{w} \right\rceil - 1\right)\right] - W$

Table 1: Summary of Analytical Results: Total Number of Sweeps and the Total Distance Traveled D for Each Path/Route (when $\delta = 0$).

3.4 Time Model for Directional Changes within the Region and at the Boundary

Additional time incurred due to directional changes, $t_{\beta i}$, from one sweep to the next is proposed. A proportional model is used, and it is defined as the *fraction* of the time taken to rotate β_i with respect to the time taken to rotate 2π radians. That is, for each angular turn β_i between two successive sweeps, $t_{\beta i} = \frac{\beta_i}{2\pi} t_{2\pi}$, where $t_{2\pi}$is the time taken, in seconds, to rotate the push mower 2π radians at any position of the region. This proportional model will be used for each of the path chosen. Thus, the total time incurred due to directional changes is the sum of the time incurred for each directional/angular turns β_i . That is,

$$t_{direction},\ t_{boundary} = \sum_{i=1}^{N} t_{\beta i} = t_{2\pi} \sum_{i=1}^{N} \frac{\beta_i}{2\pi} = \frac{t_{2\pi}}{2\pi} \sum_{i=1}^{N} \beta_i \tag{9}$$

where N is the total number of turns for a path chosen. It is noted here that $t_{direction}$ and $t_{boundary}$ are the total time incurred due to directional changes within the region, and at the boundaries, respectively, and the same time model equation (9) for both $t_{direction}$

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and $t_{boundary}$ are used. Please note: $t_{direction}$ only appears for the *SS* option, and for both the *HS* and *DS* options, $t_{direction} = 0$.

The above model given by equation (9) will be used to determine the total time required for each of the three paths.

3.5 Total Time Model for this Mowing Problem

For this lawn mower problem, the total time taken, t_{total} , to mow the entire area is the sum of three time components:

1. t_{min}, the time required to simply cover the grass area via one of three proposed sweeping recommendations. $t_{min} = D/S$, where D is the total distance traversed using either of the three routes chosen, and S is the average speed of person pushing the mower;

2. $t_{direction}$, the additional time incurred due to directional changes made by the push mower within the region (which is dependent on the chosen route); and

3. $t_{boundary}$, another additional time incurred due to directional changes of the push mower at the boundaries/edges of the region (which in this case, generally independent of the chosen route of the push mower).

This total time model is given by:

$$t_{total} = t_{min} + t_{direction} + t_{boundary} \tag{10}$$

The first term in equation (10), t_{min} can be minimized based on the chosen route taken by the mower, but cannot be eliminated since a finite amount of time is required to simply cover the entire area. In fact, this term was the only term quantified by APS, and by [AFM93] and [AFM00].

As it will be shown in the next section, t_{min} depends on the path taken, the dimensions of the region, width of the lawn mower, speed of the lawn mower, and the height of the grass (which was not considered in the analysis).

The second term, $t_{direction}$, is the time incurred due to the change of the mower's direction within the domain (only for *SS* route) to cover the area. The consideration of this term, as noted earlier, has been omitted by APS and others ([TRI10], [AFM93], [AFM00], and [MCS97]). This effect is noticeable for the *SS* path only, where the mower was required to change directions by 90^o (or $\pi/2$ clockwise) after each sweep. For the *HS* and *DS* paths, $t_{direction} = 0$, as previously mentioned.

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4 Discussion of Results

Example 1. *Let $L = W$ (square region) with $\sqrt{L^2 + W^2} = 3.5w$, a very small grass region. Using the analytical results for D_{HS}, D_{SS}, and D_{DS}, the table below summarizes the results for no overlap (that is, for $\delta = 0$). Also, the number of turns, denoted by N, is given by $N = n - 1$, where n is the number of sweeps required to cover the entire region for a particular path/pattern (HS, DS, or SS).*

Path/ Route	Total Number of Sweeps, n	Total Number of Turns, N	Total Distance, D	t_{min}	$t_{direction}$	$t_{boundary}$	t_{total}
HS	**3**	**2**	**7.42w**	**7.42w/S**	**0**	**3**	**7.42w/S + 3**
SS	**5**	**4**	**6.37w**	**6.37w/S**	**0.75**	**2.25**	**6.37w/S + 3**
DS	**4**	**3**	**9.5w**	**9.5w/S**	**0**	**4.5**	**9.5w/S + 4.5**

Table 2a: Summary of Results for Example 1 for No Overlap ($\delta = 0$).

Example 2. *Let $L = 3W$ (rectangular region) with $\sqrt{L^2 + W^2} = 100w$, a very large grass region. Using the analytical results for D_{HS}, D_{SS}, and D_{DS}, the table below summarizes the results for no overlap ($\delta = 0$).*

Path/ Route	Total Number of Sweeps, n	Total Number of Turns, N	Total Distance, D	t_{min}	$t_{direction}$	$t_{boundary}$	t_{total}
HS	**32**	**31**	**3035.8w**	**3035.8w/S**	**0**	**46.5**	**3035.8w/S + 46.5**
SS	**63**	**62**	**3024.1w**	**3024.1w/S**	**44.25**	**2.25**	**3024.1w/S + 46.5**
DS	**60**	**59**	**3100w**	**3100w/S**	**0**	**88.5**	**3100w/S + 88.5**

Table 2b: Summary of Results for Example 2 for $\delta = 0$.

Based on the results listed in Table and above for Examples 1 and 2, respectively, the *DS* route incurs the most time when cutting the lawn. The main reason is due to the largest distance traveled by the mower compared to the *HS* and *SS* routes. Area covered during each sweep from the *DS* pattern does not conform to the overall geometry of the region, and thus 'excess' area from each sweep is covered. Whereas, for the *HS* and *SS* patterns, areas covered during each sweep are consistent with the geometry of the entire region. The use of the *SS* route provides the optimal option out of the three routes.

With increased average speed S, t_{min} will decrease as the mower covers the same area faster, but time incurred due to directional changes will remain fixed under this

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situation. As a result, the percentage of added time due to directional changes relative to the total time to mow the grass will increase as S increases since t_{min} decreases (thus, the total time t_{tot}). Also, if the mower can turn faster (that is, as $t_{2\pi}$ decreases), the percentage of $t_{boundary}$ and $t_{direction}$ will decrease based on the overall/total time since the same coverage of the region can be done faster. It is important to note here that $t_{2\pi}$ is independent of t_{min}.

For larger regions, despite L/W remains unchanged, the percentage of added time due to the directional changes at the boundaries and in the interior region relative to the total time will decrease as most of the time will be spent simply cutting the area. That is, under this situation, although the number of turns increases, the effect of the time required to sweep across the *large* area will out-weigh the time incurred due to directional changes within the domain and at the boundaries.

Example 3. *The geometry of the problem and parameters used in the present study are as follows: L = 40 m, W = 25 m, S = 0.60 m/s, w = 0.55 m, and* $t_{2\pi}$*= 3.0 s (based on the author's property, length of the lawn mower blade, and 'measured' data, respectively). Overlap is not considered (that is, for* $\delta = 0$*) at first, but overlap will be considered for these parameters in Example 4.*

Path/Route	**Total Number of Sweeps, n**	**Total Distance, D (m)**
HS	46	1840.0
SS	91	1826.5
DS	78	1865.1

Table 3: Summary of Results for t_{min} without Sweep Overlap ($\delta = 0$).

A summary of results is presented in Tables 3 through 6 based on the aforementioned parameters assumed. For the three paths, total distance traveled as shown in Table 3 above is approximately 1.8 km (or just over one mile) when no overlap is considered (δ = 0). The total time incurred is approximately 52 minutes, and the best pattern to use, once again, is the SS pattern as shown in Table 4 below. The total distance incurred using the HS and SS paths are relatively close (to within 40 m) compared to the total distance using the DS path.

Example 4. *The same parameters as in Example 3 are used but with overlap* $\delta = (1/4)w$. *Tables 5 and 6 above show that the total distance traveled and the times incurred by the mower when this overlap is considered. The total distance traveled for this defined problem is approximately 2.4 km. This increase in total distance traveled is a result of less area covered per sweep (or an increase in the number of sweeps) to cover the same*

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Path/Route	t_{min} (s)	$t_{direction}$ (s)	$t_{boundary}$ (s)	t_{total} (s)
HS	3066.7	0	67.5	3134.2
SS	3044.2	65.25	2.25	3111.7
DS	3108.5	0	115.5	3224.0

Table 4: Summary of Results for the Three-Component Total Time Model t_{total} without Sweep Overlap ($\delta 0$).

Path/Route	**Total Number of Sweeps, n**	**Total Distance, D (m)**
HS	61	2440.0
DS	103	2471.3

Table 5: Summary of Results for Example 3 for t_{min} with Sweep Overlap of $\delta = (1/4)w$.

Path/Route	t_{min} (s)	$t_{direction}$ (s)	$t_{boundary}$ (s)	t_{total} (s)
HS	4066.7	0	90.0	4156.7
DS	4118.8	0	153.0	4271.8

Table 6: Summary of Results for the Three-Component Total Time Model t_{total} with Sweep Overlap, $\delta = (1/4)w$.

area for a given path. In both cases for $\delta = 0$ and $\delta = (1/4)w$, DS incurred the largest distance traveled (and thus time required) to cover the entire region. This is due to that fact that "excess" coverage (in area) has incurred for each sweep at the boundaries, that is, some area **outside** *of the boundaries is covered for each sweep. For each sweep, one can see that the geometry of the area swept is not consistent with the geometry of the rectangular region.*

As shown in Tables 4 and 6, for this example, the additional time incurred due to the directional changes at the boundary and within the domain account for approximately 2% – 4% of the overall total time taken to cover the entire region for the three mowing patterns. For larger regions, the percentage of added time due to the directional changes at the boundaries and in the interior region relative to the total time will decrease as most of the time is spent simply cutting the interior region.

Fact #5: The total time incurred due to directional changes for the *HS* pattern, $(t_{direction} + t_{boundary})_{HS}$, is equal to the total time incurred due to directional changes for the *SS* pattern, $(t_{direction} + t_{boundary})_{SS}$. The same time results are shown in the 3^{rd} and 4^{th} columns in Table 4. For the *HS* pattern, 180 degree turns are made by the mower at the boundaries. For the *SS* pattern, the turns are only 90 degrees throughout. While transitioning from one cycle to the next cycle for the *SS* pattern, five sweeps are needed (*horizonal – vertical – horizontal – vertical – horizontal*). So this is equivalent to four 90 degree turns which is equivalent to the *two* turns of 180 degrees for the *HS* pattern[4].

5 Concluding Remarks

The mathematics of the mowing problem was presented in detail. The contents of this paper can be used a guide in choosing which mowing patterns/sweeps should be used in cutting one's own lawn. A 2D rectangular region was used in the analyses by investigating the total time taken to mow the entire 2D region using three different prescribed patterns. By using basic high school algebra, the time taken to simply cover the region is obtained. A general three-component total time model was introduced, and this model was used to compare the total time taken to mow a 2D region using the prescribed three paths/routes. Two of the three components in the total time model use empirical directional time parameters to illustrate how time is accrued due to the fact that the mower has to change directions within the region and at (near) the boundaries during the mowing process. Also, a physically realistic situation by considering overlap between successive sweeps has also been presented. As is the case for considering overlap, the total time increased for each of the paths chosen compared to when no overlap was introduced. Through examples, it was demonstrated that there were *moderate* differences in terms of the total time taken based on the three different routes/paths. The minimum total time taken to mow the entire 2D region is based on the spiral sweep (*SS*).

Because of the moderate differences incurred between the three routes, the authors suggest to use of any one of the three patterns in mowing one's lawn, and to alternate the routes to break the monotony of the chore. The use of a series of diagonal sweeps (*DS*) will enhance the curb appeal of the lawn despite incurring the largest (although not significant) of the total time taken based on the other two patterns.

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