Arithmetic billiards

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Abstract

Arithmetic billiards show a nice interplay between arithmetics and geometry. The billiard table is a rectangle with integer side lengths. A pointwise ball moves with constant speed along segments making a 45° angle with the sides and bounces on these. In the classical setting, the ball is shooted from a corner and lands in a corner. We allow the ball to start at any point with integer distances from the sides: either the ball lands in a corner or the trajectory is periodic. The length of the path and of certain segments in the path are precisely (up to the factor $\sqrt{2}$ or $2\sqrt{2}$) the least common multiple and the greatest common divisor of the side lengths.

1 Introduction

Arithmetic billiards, also known under the name *Paper Pool*, show a nice interplay between arithmetics and geometry. They are a mathematical model for a billiard with which one can visualize the greatest common divisor and the least common multiple of two natural numbers (more general models for billiards exist in the branch of mathematics called dynamical systems).

The billiard table is a rectangle with integer side lengths. The ball is a point that bounces on the billiard sides and moves with constant speed on segments that make a 45° angle with the sides. We are interested in the geometric properties of the path that the ball traces on the table, regardless of the actual movements of the ball.

In the classical setting, the ball is shooted from a corner of the billiard table, and the ball necessarily lands in a different corner. We call the resulting path *corner path*: these have been studied by various authors including Martin Gardner [Gar84, Ste99, Tan12, Per18]. If a and b are the side lengths of the billiard table, then a corner path is the

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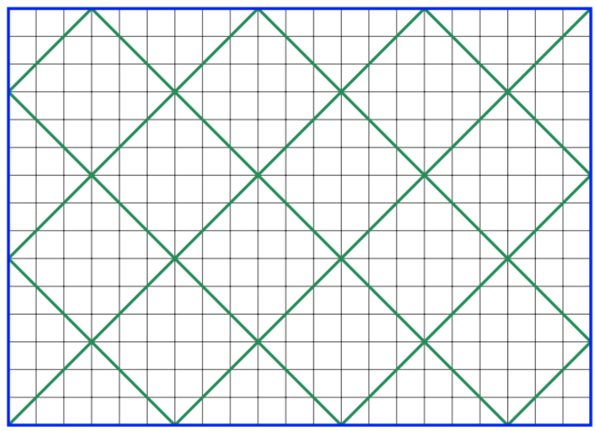

Figure 1: Example of corner path for the 21×15 billiard table.

intersection of the billiard table with a grid of squares of side length $\gcd(a,b)\sqrt{2}$ (the grid is diagonally oriented, and the starting corner is a vertex for one of the squares), and the length of a corner path is $\operatorname{lcm}(a,b)\sqrt{2}$. In the next section we collect the results about corner paths.

We also investigate the analogous paths that start at any point of the billiard table with integer distances from the sides. If the starting point does not belong to a corner path, then the ball does not land in a corner but it periodically bounces on the billiard sides: we call such paths *closed paths*. In the last two sections we prove various results about closed paths (to the best of our knowledge these are original, with the exception of the formula for the length of the path, see [Wik]).

With a closed path, we can again visualize the greatest common divisor and the least common multiple of the side lengths of the billiard table. Indeed, closed paths have the following properties:

- The length of the path is $\operatorname{lcm}(a,b)2\sqrt{2}$.
- The path is the intersection of the billiard table with two grids of diagonally oriented squares of side length $\gcd(a,b)\sqrt{2}$ which only differ by a shift parallel to one of the billiard sides.
- The path is symmetric (point-symmetric w.r.t. the center of the billiard table or symmetric with respect to the perpendicular bisector of two billiard sides).
- The path is periodic, with period $2\gcd(a,b)$ in both directions parallel to the billiard sides. Usually this is the minimal period (only in a very special case the minimal period is $\gcd(a,b)$).

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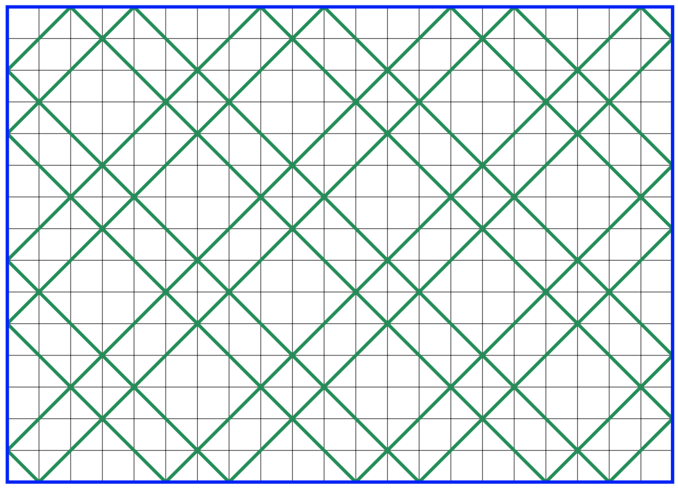

Figure 2: Example of closed path for the 21×15 billiard table.

Up to a symmetry of the billiard table, there is exactly one corner path in a given billiard table. However there can be many non symmetric closed paths (and there are no closed paths if a and b are coprime).

- The number of closed paths up to symmetry is the integer part of $\gcd(a, b)/2$.
- For any integer r in the range from 1 to $\gcd(a, b) - 1$ consider the point P_r on a fixed billiard side at distance r from one fixed corner. Then any closed path contains precisely one of the points P_r.

There are several quantities that are the same for all closed paths inside a given billiard table. We have already mentioned the length of the path and the side length of the squares in the two grids. But there is also — as we will see — the number of *boundary points*, i.e. the points of the path which are on the billiard sides. And the number of *self-intersection points*, i.e. the points where the path crosses itself. Moreover, the path partitions the billiard table into rectangles and triangles: also the number of rectangles and the number of triangles do not depend on the closed path.

Key to our investigation is the following: we can write a formula for the coordinates of the boundary points. Notice that we prove or sketch the proof for the results concerning closed paths, however one may opt to use the pictures as guidance because in most cases they are sufficiently generic (and usually only a case distinction concerning the parity of $a/\gcd(a, b)$ and $b/\gcd(a, b)$ is needed).

The exploration of arithmetic billiards is a source of activities for pupils [GF13, Tho08, Zuc07, Lap04, Pap]. Indeed, the pupils are asked to find out by themselves some of the known results. It is also possible to go one step further and investigate open questions: as a research direction we suggest considering other shapes for the billiard table (for example, an L-shaped figure with an axial symmetry, or a square with a square hole in the middle).

2 Preliminary remarks, Corner paths

Setting

We fix two positive integers a and b and call g their greatest common divisor. We take as a *billiard table* a rectangle with side lengths a and b, and we choose coordinates by placing the origin in a billiard corner and letting the opposite corner be the point (a, b).

We consider the trajectory of one point (the *ball*) inside the billiard table such that the path consists of segments that make a 45° angle with the sides. The speed does not matter, so we may suppose that it is constant. The ball bounces on the billiard sides (making either a left or a right 90° turn) and stops only if it lands in a corner. A *step* in the path results when the ball moves from a point with integer coordinates to the next one (each coordinate changes by 1).

There are *corner paths*, where the ball is shooted from a billiard corner and necessarily lands in a different corner. If we start at a point with integer coordinates that is not on a corner path, then we get a *closed path*, which corresponds to a periodic trajectory.

We call *boundary points* the points of the path which are on the rectangle sides. Most paths have *self-intersection points*, where two path segments cross perpendicularly.

Corner paths

We now collect the known results about corner paths, see [Per18]. Some observation may be new, but it is an easy exercise given the known results. The reader can also get inspiration from the analogous statements for closed paths in the oncoming sections (everything is simpler for corner paths).

A corner path starts in any billiard corner, and we can predict what the ending corner will be: if a/g and b/g are odd, then the starting corner and the ending corner are opposite; if a/g is even and b/g is odd, then the starting and the ending corner are adjacent to one a-side; if a/g is odd and b/g is even, then the starting and the ending corner are adjacent to one b-side.

Neglecting their orientation, there are two corner paths. Moreover, there is a symmetry of the billiard table mapping one path to the other, namely the symmetry mapping the starting and ending corner of one path to those of the other. The length of the path is $\mathrm{lcm}(a, b)\sqrt{2}$ (because there are $\mathrm{lcm}(a, b)$ steps) and the path crosses $\mathrm{lcm}(a, b)$ unit squares.

The path is symmetric: if the starting and the ending corner are opposite, then the path is point symmetric w.r.t. the center of the rectangle, else it is symmetric with respect to the perpendicular bisector of the side connecting the starting and the ending corner.

There are a/g boundary points (including the corners) on the two a-sides, and b/g on the two b-sides. Moreover, the boundary points are evenly distributed along the rectangle perimeter: the distance along the perimeter (i.e. possibly going around the corner) between two such neighbouring points equals $2g$.

The corner path starting at the origin is the intersection of the billiard table with the grid of squares whose corners are the points (xg, yg), where x, y are integers with the same parity (the squares are oriented at 45° w.r.t. the billiard sides). The grid partitions the billiard tables into squares, triangles at the boundary which are half of the squares, and two triangles at the corners which are a quarter of the squares.

Since each of the large triangles occupies $2g$ of the billiard perimeter, we can see that there are $(a+b)/g - 2$ such triangles. By noticing that the billiard area ab consists of the area of the triangles and the area of the squares, we easily deduce that there are $(a-g)(b-g)/2g^2$ squares.

Unless a is a multiple or a divisor of b there are self-intersection points, and more precisely there are $(a-g)(b-g)/2g^2$ of them (to derive this formula consider that every g steps there is a boundary point or a self-intersection point, and we find each self-intersection point twice). Moreover, there are self-intersection points on the first segment of the path, and the ball arrives at such a point after g steps: the least distance between a corner and a self-intersection point is $\gcd(a, b)\sqrt{2}$. Unless $a = b$, the integer g is the least distance between a corner and a boundary point which is not a corner (if $a = b$, then the path is just a diagonal of the billiard table).

If we would let the ball bounce at the corners, then a corner path would correspond to a periodic trajectory: the ball would go twice through the path (forwards and backwards) in every period.

3 Boundary points for closed paths

We now turn our attention to closed paths. These by definition do not contain corners, and the ball never stops. The trajectory is periodic because the ball can only move on the finitely many segments touching the billiard sides at points with integer coordinates. We are going to study the path, concentrating on one period.

Notice that if $g = 1$, then all points in the billiard table with integer coordinates lie on the corner paths and there is no closed path, so *we will suppose that* $g > 1$.

Length of the path, number of boundary points

There are boundary points on each billiard side so we may suppose that the starting point is on the bottom a-side and the starting direction is rightwards, i.e. both coordinates are increasing.

• *The length of a closed path is* $\mathrm{lcm}(a,b)2\sqrt{2}$. Indeed, we are back to the bottom a-side after any number of steps which is a multiple of $2b$. Moreover, since we start and end the path by going rightwards, then we can be back to the starting point only after a number of steps which is a multiple of $2a$. So the total number of steps in a period is $2\,\mathrm{lcm}(a,b)$.

• *There are a/g boundary points on each a-side and b/g boundary points on each b-side.* Indeed, we touch one same a-side every $2b$ steps, and $2\,\mathrm{lcm}(a,b)/2b = a/g$. For the b-sides, we reason analogously.

The boundary points

In what follows we determine the set of boundary points. The formulas for the coordinates of these points will depend on a, b, and the smallest positive integer r such that the point $(r,0)$ belongs to the path.

Let r be an integer in the range from 1 to $g-1$. In particular, the path containing the point $(r,0)$ is not a corner path. The boundary points of this path are as in the following tables, where we specify the x-coordinate for the a-sides and the y-coordinate for the b-sides. Keep in mind that a/g and b/g cannot be both even and that, up to exchanging the role of a and b, we may suppose that a/g is odd.

bottom a-side	$r, 2g-r, \ldots, n2g+r, (n+1)2g-r, \ldots, \frac{a-g}{2g}2g+r$
right b-side	$g-r, g+r, \ldots, n2g+g-r, n2g+g+r, \ldots, \frac{b-g}{2g}2g+g-r$
upper a-side	$g-r, g+r, \ldots, n2g+g-r, n2g+g+r, \ldots, \frac{a-g}{2g}2g+g-r$
left b-side	$r, 2g-r, \ldots, n2g+r, (n+1)2g-r, \ldots, \frac{b-g}{2g}2g+r$

Figure 3: Boundary points if a/g and b/g are odd.

bottom a-side	$r, 2g-r, \ldots, n2g+r, (n+1)2g-r, \ldots, \frac{a-g}{2g}2g+r$
right b-side	$g-r, g+r, \ldots, (\frac{b}{2g}-1)2g+g-r, (\frac{b}{2g}-1)2g+g+r$
upper a-side	$r, 2g-r, \ldots, n2g+r, (n+1)2g-r, \ldots, \frac{a-g}{2g}2g+r$
left b-side	$r, 2g-r, \ldots, (\frac{b}{2g}-1)2g+r, (\frac{b}{2g}-1)2g+2g-r$

Figure 4: Boundary points if a/g is odd and b/g is even.

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The boundary points are thus the points on the billiard sides whose side coordinate leaves remainder r or $2g-r$ (respectively, $g-r$ or $g+r$) after division by $2g$, and r is minimal such that the point $(r,0)$ belongs to the given path. By varying r from 1 to $g-1$ we obtain all points of the sides whose coordinate is not a multiple of g (those other points lie on the corner paths). Also notice that the boundary points are evenly distributed along the rectangle perimeter (i.e. possibly going around the corner) because the distance between any two of them is alternatively $2r$ and $2g-2r$.

How can we prove that we have written down the correct set of boundary points? Since in the tables we have the correct amount of boundary points, it suffices to show that the "next" boundary point is again in the set. For example, suppose that a/g and b/g are odd and consider the boundary point $(p,0)$, where p is any integer from 1 to $a-1$ whose remainder after division by $2g$ is r or $2g-r$: the next boundary point can only be one of $(a,a-p)$, $(p\pm b,b)$, $(0,p)$ so it belongs to the given set.

Symmetry

From the distribution of the boundary points, we may deduce that a closed path is symmetric. Indeed, if a/g and b/g are odd, then the path is point-symmetric w.r.t. the center of the billiard table while if w.l.o.g. a/g is odd and b/g is even, then the path is symmetric with respect to the perpendicular bisector of the b-sides. In any case, the closed path through $(r,0)$ and the closed path through $(g-r,0)$ are symmetric because the formulas for the boundary points imply that the path through $(r,0)$ also goes through the point $(a-(g-r),0)$ or the point $(0,b-(g-r))$.

4 Shape of a closed path

Consider the closed path containing the point $(r,0)$, where r is an integer from 1 to $g-1$, and recall from the previous section that we know the set of boundary points.

The grid structure

The closed path is formed by segments with slope 1 or -1 connecting two boundary points. By the formulas in Figures 3 and 4, the distances between parallel path segments are alternatively $r\sqrt{2}$ and $(g-r)\sqrt{2}$. Then the path segments form a grid which partitions the billiard table into squares having side lengths $r\sqrt{2}$ and $(g-r)\sqrt{2}$, rectangles with side lengths $r\sqrt{2}$ and $(g-r)\sqrt{2}$ (which are squares if and only if $r=g/2$), triangles around the border which are half of one of the squares, and triangles at the corners which are a quarter of one of the squares. We call *corner triangles* the triangles containing the corners and *side triangles* the further triangles along the boundary.

Notice that the path is also the intersection with the billiard table of two parallel grids of diagonally oriented squares of side length $g\sqrt{2}$, one grid being the horizontal or vertical shift of the other by $2r$ (or, equivalently, by $2g - 2r$). There is some freedom in choosing the two grids, and according to this choice we will have a horizontal or vertical shift, see Figure 2 (and Figure 5). A self-intersection point (respectively, a boundary point) of the path is either the vertex of a square in one of the two grids or a point of intersection between the two grids. In fact, if we fix one of these points, then we can choose the two grids such that the point is a vertex. Finally notice that each grid of diagonally oriented squares is invariant by the horizontal and vertical shift by $2g$, so in fact the pattern that we see inside the billiard table is periodic with period $2g$ (this is clearly the minimal period unless $r = g/2$, the period being g in this case).

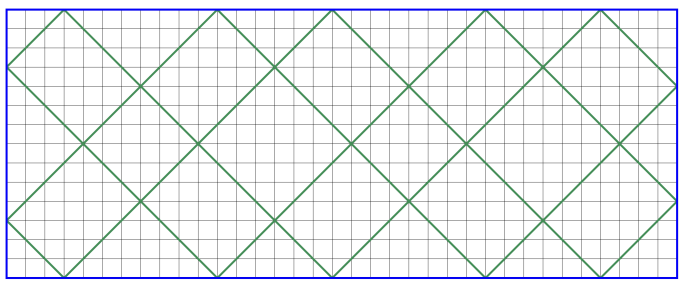

Figure 5: Example of closed path for the 35×14 billiard table.

Self-intersection points

If $a = b$, then there are no self-intersection points because the path is the rectangle with corners $(r, 0)$, $(a, a - r)$, $(a - r, a)$, $(0, r)$. On the other hand if $a \neq b$, then there are self-intersection points. Indeed, supposing w.l.o.g. that $a > b$, the path contains the segment from $(r, 0)$ to $(r + b, b)$: this segment cuts the billiard table into two parts and there are self-intersection points on it. Below we will prove that the number of self-intersection points is

$$2ab/g^2 - (a + b)/g$$

The triangles

Using the formulas in Figures 3 and 4 for the boundary points we can study the triangles.

In the special case where $r = g/2$, all four corner triangles have legs r. Moreover, all side triangles have hypothenuse g: there are $a/g - 1$ side triangles along each a-side and $b/g - 1$ side triangles along each b-side.

Now suppose that $r \neq g/2$. Two corner triangles have legs r, the other two have legs $g - r$, and we have: if a/g and b/g are odd, then the corner triangles at two opposite

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corners are congruent; if w.l.o.g. a/g is odd and b/g is even, then the corner triangles adjacent to one same b-side are congruent.

Moreover, the side triangles have hypothenuse $2r$ and $2g-2r$. There are $a/g-1$ side triangles along each a-side, and their size alternates. If a/g is odd, then there are $(a-g)/2g$ side triangles of each type on each a-side; if a/g is even, then there are $a/g-1$ side triangles of each type on the two a-sides (more precisely, there are $a/2g-1$ large side triangles on the side whose triangle corners are small, and $a/2g$ on the other). The analogous formulas hold for the b-sides.

The squares in the special case $r = g/2$

If $r = g/2$, then the intersection of the two grids of diagonally oriented squares gives just one grid with smaller squares. So the path is the intersection of the billiard table with the grid of squares of side length $g/\sqrt{2}$ having a vertex at the point $(r, 0)$. In this case the set of self-intersection points together with the boundary points is the union of two sets that are easy to describe:

$$\begin{array}{lll} (r+ng, mg) & \text{with} \quad 0 \leqslant n \leqslant a/g-1 & 0 \leqslant m \leqslant b/g \\ (ng, r+mg) & \text{with} \quad 0 \leqslant n \leqslant a/g & 0 \leqslant m \leqslant b/g-1 \end{array}$$

These are $2ab/g^2 + (a+b)/g$ points, and hence (recalling that there are $2(a+b)/g$ boundary points) the number of self-intersection points is $2ab/g^2 - (a+b)/g$.

We now count the squares in the partition of the billiard table given by the path: there are clearly b/g horizontal stripes of squares with a/g squares, and $b/g-1$ horizontal stripes of squares with $a/g-1$ squares, so the total number of squares is

$$\frac{2ab}{g^2} - \frac{a+b}{g} + 1$$

The rectangles in the generic case

We suppose that $r \neq g/2$ because this other case has been treated above. We can choose the two grids of diagonally oriented squares to have vertices at the points $(r, 0)$ and $(2g-r, 0)$ respectively. Then we can write down the self-intersection points together with the boundary points: they are the points in the billiard table whose coordinates are of the form $(n2g, m2g)$ for some integers n, m plus any of the following eight points:

$$(\pm r, 0) \quad (g \pm r, g) \quad (0, \pm r) \quad (g, g \pm r)$$

By writing the sets explicitly (this is a straightforward exercise) we may then count the the self-intersection points together with the boundary points as done in the special case $r = g/2$, and we obtain again that there are $2ab/g^2 - (a+b)/g$ self-intersection points.

We now count the non-square rectangles in the partition of the billiard table given by the path. To do this we exploit the periodicity of the pattern in the arithmetic billiard. The centers of the non-square rectangles are the points in the billiard table whose coordinates are of the form (ng, mg) for some integers n, m plus the point $(g/2, g/2)$. Consequently, there are

$$ab/g^2$$

non-square rectangles.

In the partition of the billiard table given by the path there are squares with side lengths $r\sqrt{2}$ and $(g-r)\sqrt{2}$. There is the same amount of squares of the two types because, replacing r with $g-r$, we obtain a second path which only differs from the first by a symmetry of the billiard table. Reasoning as above, the number of squares of side length $r\sqrt{2}$ is

$$(a-g)(b-g)/2g^2$$

Namely, the centers of these squares are the points in the billiard table whose coordinates are of the form $(n2g, m2g)$ for some integers n, m plus the point (g, g) or the point $(2g, 2g)$.

The number of closed paths

Let r be an integer in the range from 1 to $g-1$, which we use as parameter. Then in Figures 3 and 4 we have given parametric formulas for the boundary points of the closed path going through the point $(r, 0)$. We have also observed that all closed paths in a given billiard table have some boundary point on the bottom a-side. Moreover, a point of the form $(x, 0)$ belongs to the corner paths if x is a multiple of g. The formulas in Figures 3 and 4 then guarantee that each closed path goes through exactly one of the points of the form $(r, 0)$. We deduce that there are $g-1$ closed paths in a given billiard table.

For example, if $g = 2$, then there is only one closed path, which consists of the grid of squares whose corners are the points (x, y) in the billiard table such that $x + y$ is odd.

However, if we count the closed paths up to a symmetry of the billiard table, then their amount is the integer part of $g/2$ because with a symmetry we may replace r by $g-r$. To understand the symmetry visually, consider that a closed path in a given billiard table is determined by the size and position of the corner triangles (whose legs are r and $g-r$), and that a symmetry has the effect of permuting the corner triangles.

Comparing billiard tables

The pattern of a closed path in a billiard table (namely, the structure with the two grids of diagonally oriented squares) only depends on a and b through g. So if we take any two billiard tables with the same g, and we consider the closed path from the point

$(r, 0)$ in both cases, then the two closed paths coincide on the intersection of the two billiard tables. The same holds for the corner paths starting at $(0, 0)$ on various billiard tables having the same g, because the pattern is a grid of squares which only depends on g.

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