Go First Dice for Five Players and Beyond.

Robert Ford, James Grime, Eric Harshbarger and Brian Pollock

Abstract

Before a game begins, the players need to decide the order of play. This order of play is determined by each player rolling a die. Does there exist a set of dice such that draws are excluded and each order of play is equally likely? For four players the solution involves four 12-sided dice, sold commercially as Go First Dice. However, the solution for five players remained an open question. We present two solutions. The first solution has a particular mathematical structure known as binary dice, and results in a set of five 60-sided dice, where every place is equally likely. The second solution is an inductive construction that results in one one 36-sided die; two 48-sided dice; one 54-sided die; and one 20-sided die, where each permutation is equally likely.

Imagine four friends are about to start a game and need to decide who goes first, who goes second, who goes third and who goes fourth.

So the friends decide that each player will roll a die, and order play from highest to lowest. Unfortunately, if two players roll the same number, those players will have to roll again. Potentially this could go on forever.

Is it possible to make a set of dice so that no ties occur, and each player is equally likely to be placed first, second, third and fourth?

In 2012 Robert Ford and Eric Harshbarger solved the problem with a set of four 12-sided dice, [Bel12] [Har18]:

A: [01, 08, 11, 14, 19, 22, 27, 30, 35, 38, 41, 48]
B: [02, 07, 10, 15, 18, 23, 26, 31, 34, 39, 42, 47]
C: [03, 06, 12, 13, 17, 24, 25, 32, 36, 37, 43, 46]
D: [04, 05, 09, 16, 20, 21, 28, 29, 33, 40, 44, 45]

DOI 10.2478/rmm-2023-0004

With these dice each player is equally likely to be placed first, second third and fourth. However, even more remarkably, this property remains true for any subset of dice. So, any subset of three dice is equally likely to be placed first, second and third; and any subset of two dice is equally likely to be placed first and second.

These dice became known as *Go First Dice*. However, a set for five players was an open problem. We present two sets of five Go First Dice, with details of how each set was constructed.

Our first solution is a set of five 60-sided dice found by James Grime and Brian Pollock with a 'binary construction' and computer search.

A: [002, 008, 012, 018, 024, 029, 032, 038, 044, 049, 053, 059, 063, 067, 073, 078, 083, 088, 092, 098, 103, 109, 113, 117, 122, 127, 133, 138, 143, 148, 153, 159, 164, 167, 173, 178, 183, 188, 194, 199, 202, 208, 214, 217, 224, 227, 233, 238, 243, 248, 253, 257, 263, 269, 272, 278, 284, 289, 292, 298]

B: [003, 007, 013, 019, 023, 028, 033, 037, 043, 048, 052, 058, 064, 068, 074, 079, 084, 087, 093, 097, 104, 108, 112, 118, 123, 128, 134, 137, 142, 149, 154, 158, 163, 168, 172, 177, 184, 187, 193, 198, 203, 207, 213, 218, 223, 228, 234, 239, 242, 249, 252, 258, 264, 268, 273, 279, 283, 288, 293, 297]

C: [004, 009, 011, 020, 022, 027, 031, 039, 042, 047, 051, 060, 065, 069, 075, 077, 082, 089, 091, 099, 102, 110, 114, 119, 124, 129, 132, 136, 144, 147, 152, 157, 162, 169, 171, 176, 182, 189, 192, 200, 204, 209, 215, 219, 222, 226, 235, 237, 244, 247, 254, 259, 262, 267, 271, 280, 282, 290, 294, 296]

D: [005, 006, 014, 017, 021, 030, 034, 040, 041, 050, 054, 057, 062, 066, 072, 076, 085, 086, 094, 100, 101, 107, 111, 116, 125, 130, 135, 139, 145, 146, 155, 156, 165, 166, 174, 179, 185, 190, 191, 197, 201, 210, 212, 216, 225, 229, 232, 236, 241, 250, 251, 260, 261, 270, 274, 277, 281, 287, 295, 299]

E: [001, 010, 015, 016, 025, 026, 035, 036, 045, 046, 055, 056, 061, 070, 071, 080, 081, 090, 095, 096, 105, 106, 115, 120, 121, 126, 131, 140, 141, 150, 151, 160, 161, 170, 175, 180, 181, 186, 195, 196, 205, 206, 211, 220, 221, 230, 231, 240, 245, 246, 255, 256, 265, 266, 275, 276, 285, 286, 291, 300]

This set was designed so that the dice are all the same size, and that each place (first, second, third, etc) is equally likely, including all subsets of dice.

The second solution was found by Eric Harshbarger using an inductive method.

A: [002, 015, 016, 018, 019, 032, 038, 051, 052, 054, 055, 068, 070, 083, 084, 086, 087, 100, 107, 120, 121, 123, 124, 137, 139, 152, 153, 155, 156, 169, 175, 188, 189, 191, 192, 205]

DOI 10.2478/rmm-2023-0004

B: [005, 006, 010, 014, 022, 023, 025, 031, 041, 042, 046, 050, 058, 059, 061, 067, 073, 074, 078, 082, 090, 091, 093, 099, 110, 111, 115, 119, 127, 128, 130, 136, 142, 143, 147, 151, 159, 160, 162, 168, 178, 179, 183, 187, 195, 196, 198, 204]
C: [003, 009, 011, 012, 020, 024, 028, 029, 039, 045, 047, 048, 056, 060, 064, 065, 071, 077, 079, 080, 088, 092, 096, 097, 108, 114, 116, 117, 125, 129, 133, 134, 140, 146, 148, 149, 157, 161, 165, 166, 176, 182, 184, 185, 193, 197, 201, 202]
D: [004, 007, 008, 013, 017, 021, 026, 027, 030, 040, 043, 044, 049, 053, 057, 062, 063, 066, 072, 075, 076, 081, 085, 089, 094, 095, 098, 109, 112, 113, 118, 122, 126, 131, 132, 135, 141, 144, 145, 150, 154, 158, 163, 164, 167, 177, 180, 181, 186, 190, 194, 199, 200, 203]
E: [001, 033, 034, 035, 036, 037, 069, 101, 102, 103, 104, 105, 106, 138, 170, 171, 172, 173, 174, 206]

This set uses dice of different sizes including one 36-sided die; two 48-sided dice; one 54-sided die; and one 20-sided die; and is designed with the stronger property that each permutation is equally likely, including subsets.

We will now proceed to describe how each set was found or constructed.

1 Binary construction with search

Initially, a computer search seemed like a good option for finding a set of five Go First Dice. Although it was immediately clear that this was going to be a large task.

In particular, we knew the number of sides was going to be some multiple of 30. That's because, if we start with five m-sided dice, we will have m^5 equally likely outcomes. If each place is to be equally likely, this number must be divisible by 5 and, since 5 is prime, m is also divisible by 5. Similarly, because we want all subsets of dice to be fair, m must also be divisible by 2 and 3.

So we decided to restrict our search to sets of dice where the wins and losses were particularly neat.

Let's start by assuming we have n dice with m sides. If we write our dice in separate rows, then the sides will form m columns. Fill the dice with the values 1 to nm, column by column, so that the last die contains the highest or lowest value of each column. The following is an example of a valid set of Go First Dice made in this style:

A: [02, 05, 07, 12, 14, 17]
B: [03, 04, 08, 11, 13, 18]
C: [01, 06, 09, 10, 15, 16]

With this construction, a player will automatically win if they roll the higher side, or if they roll the same side and have a higher value.

To emphasise the high and low values, we can rewrite the dice so that each successive die contains only two values, one that is higher and one that is lower than all previous values, while maintaining the order of each column.

A: [2, 2, 2, 2, 2, 2]
B: [3, 1, 3, 1, 1, 3]
C: [0, 4, 4, 0, 4, 0]

Notice that A is essentially a dummy die, containing only one value, while B is constructed so that half the values are higher than A and half lower than A. This will ensure A and B are equally likely to be placed first and second in a two-player game.

Next, C is written with one value higher and one value lower than all previous values, in this case we have used the values 0 and 4. If half the values are high values then each place is equally likely when C plays two-player games against A or B.

A three-player game is trickier, as the position of the high values matter. To find out where to place our high values, let's consider which of the three dice will win the roll.

In this construction, a die automatically wins if it rolls a higher side. For example, when $m = 6$, each die automatically wins 55 outcomes and loses 55 outcomes, leaving 51 outcomes in contention. These are the outcomes when two or more dice roll the same side, and who wins depends on the placement of the high and low values. If the dice are to be fair, C must win $1/3$ of the contended outcomes.

If we number the sides from 1 to 6, then a high value on side s of die C will increase its wins by $2s - 1$. These are the outcomes where either A or B roll the same side as C and the other die rolls a lower side, or where A, B and C all roll the same side. In our example, the high values of C are found on sides 2, 3, and 5, which increases the number of wins by 17, one-third of the contended outcomes as required.

In general, for m-sided dice, a high value on side s of the third die contributes $2s - 1$ wins, and these wins must add up to $m(3m - 1)/6$.

However, since half of the values on C must be high values, we can tidy-up the previous condition to say that the sides with high value must add up to $m(3m + 2)/12$.

So when $m = 6$, we need 3 high values on dice B and C, and for C we need sides with high values to add up to 10, which is true in our example.

We must stress that these are only necessary conditions, it is still possible to satisfy the summation conditions and still not be a valid set of Go First Dice. However, we may quickly eliminate any sets that do not satisfy these conditions.

1.1 For more than three players

We may now generalise the ideas found for three players to sets of 4, 5 and potentially n dice.

As before, the number of outcomes in contention are when two or more dice roll the same side. And because the last die in our set only uses values that are higher or lower than all other dice, counting the wins is relatively easy.

In general, if we have k dice, then a high value on side s of the k^{th} die contributes $s^{k-1} - (s-1)^{k-1}$ wins. For the dice to be fair, these wins must add up to $1/k$ of the contended outcomes. The number of contended outcomes is $m^k - k \sum_{i=0}^{m-1} i^{k-1}$. And the k^{th} die must satisfy all conditions for sets of k dice and fewer.

Defining each die by a list of its high sides, s_i, results in the following conditions:

1. First die is an m-sided die;
2. Second die: All of the above and $\sum s_i^0 = m/2$;
3. Third die: All of the above and $\sum s_i = m(3m+2)/12$;
4. Fourth die: All of the above and $\sum s_i^2 = (m^2(m+1))/6$;
5. Fifth die: All of the above and $\sum s_i^3 = m(5m^2(3m+4) - 4)/120$.

These four conditions are remarkably compact and only require us to perform various sums with the sides that have high values.

We can also derive some column-wise conditions. For any subset of three dice, the second die must win half of the contended outcomes not won by the third die. So let's consider columns where the second die has a high value, and the third die has a low value. If both dice satisfy condition 3, high-low values will occur in 1/3 of the columns. Note that this is not necessary if one of the dice does not satisfy condition 3.

For example, here is a set of four 12-sided dice:

A: [3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3]
B: [4, 2, 2, 4, 2, 4, 2, 4, 2, 4, 4, 2]
C: [5, 1, 5, 1, 1, 5, 1, 5, 5, 1, 5, 1]
D: [6, 0, 0, 6, 6, 0, 6, 0, 0, 6, 6, 0]

This set satisfies all necessary summation conditions, with C and D satisfying the column-wise condition. It is equivalent to the original set of Ford and Harshbarger.

1.2 Five players

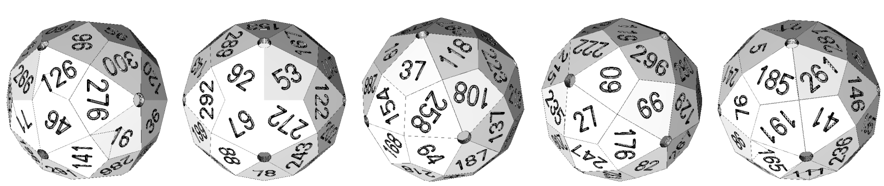
We were now ready to find a solution for five players, armed with a few conditions that allowed us to quickly reject any incorrect solutions.

We knew the numbers of sides was to be some multiple of 30. Unfortunately, when $m = 30$, there are no solutions to conditions $1 - 5$ above, meaning a binary construction of five 30-sided dice was impossible. So we continued our search with $m = 60$.

Using a computer search, our method was to start with the potential fifth dice that satisfied all necessary summation conditions. We would then add a fourth dice that satisfied both sum conditions and column-wise conditions, and continued to be build backwards in this way. Sets that were not immediately eliminated could then be tested to see if each place was equally likely.

Since this was still a very large task, we tried to speed up the search with a few hunches. These hunches included making the third and fourth dice interchangeable, and making the fifth die a bit-inverted palindrome.

We found one set that satisfies all necessary conditions, and checks out as a valid set of five Go First Dice. It is the set of five 60-sided dice given in the introduction.



This method was investigated by James Grime and Brian Pollock, and was the first set of five Go First Dice found.

DOI 10.2478/rmm-2023-0004

It should be noted that while this set has the property that each *place* is equally likely, not every *permutation* is equally likely. For example, while A and E are both equally likely to be first, the ordering ABCDE is not as likely as EDCBA. See [Enr19] for an online tool that checks dice for place-fairness and permutation-fairness

Next, we describe a construction that is not only place-fair but also permutation-fair.

2 Inductive construction

Originally, Go First Dice were imagined with all dice having the same number of sides. However, this does not need to be the case.

For example, the following is a set of three permutation-fair dice, consisting of one 2-sided die, one 4-sided die and one 6-sided die.

A: [03, 10]
B: [02, 04, 09, 11]
C: [01, 05, 06, 07, 08, 12]

These dice have 48 possible outcomes, with each ordering of A, B and C appearing equally frequently.

We can denote which value appears on which dice with a simple two-row notation:

1	2	3	4	5	6	7	8	9	10	11	12
C	B	A	B	C	C	C	C	B	A	B	C

Or, more simply, in one-line notation as follows:

CBABCCCCBABC

We will now describe how to construct a set of $n+1$ permutation-fair dice from a known set of n permutation-fair dice. This method is believed to have been first used by Paul Vanetti [Kno12], although no details were given. The method was then reverse-engineered by Robert Ford, and more fully investigated by Eric Harshbarger.

We'll start with the previous set of three permutation-fair dice. To create a fourth die, D, we will need to create some gaps in the numbering of A, B and C. Let's repeat the values on the dice, thus making each die larger, with suitably large gaps between the copied values.

For example, we could copy each die three times, adding 100 to the values of the first copy, 200 to the values of the second copy, and 300 to the values of the third copy:

$\mathcal{A}$: [103, 110, 203, 210, 303, 310]
$\mathcal{B}$: [102, 104, 109, 111, 202, 204, 209, 211, 302, 304, 309, 311]
$\mathcal{C}$: [101, 105, 106, 107, 108, 112, 201, 205, 206, 207, 208, 212, 301, 305, 306, 307, 308, 312]

Importantly, if the original set are permutation-fair then so is the expanded set. This may feel intuitively true, but let's check an example. Since dice A, B, C are permutation-fair, then $P(\mathrm{A} > \mathrm{B} > \mathrm{C}) = 1/6$. We will show that the same is true for $P(\mathcal{A} > \mathcal{B} > \mathcal{C})$.

When we roll a die from the expanded set, the face it lands on might be from the first copy, the second copy or the third copy. If we want $\mathcal{A} > \mathcal{B} > \mathcal{C}$ then the dice may land on the following copies:

$$\mathcal{ABC} = 333, 332, 331, 322, 321, 311, 222, 221, 211, 111$$

In other words, these are the weakly decreasing sequences of the values 1, 2 and 3.

If three dice occupy the same copy, such as 333, 222 and 111 then they will automatically inherit permutation-fairness. If two dice occupy the same copy, such as 332, 331, 322, 311, 221, 211 then the subset of two dice inherit permutation-fairness. If each die land on a distinct copy, such as 321, then permutation-fairness isn't a concern since we already know that $\mathcal{A} > \mathcal{B} > \mathcal{C}$.

So, for each possibility on our list, we can calculate the probability that $\mathcal{A} > \mathcal{B} > \mathcal{C}$. Since we are equally likely to land on any of the three copies, the total probability can be calculated as follows:

$$\begin{aligned} P(\mathcal{A} > \mathcal{B} > \mathcal{C}) &= \left(\frac{1}{3}\right)^3 \left(\frac{1}{3!} + \frac{1}{2!1!} + \frac{1}{2!1!} + \frac{1}{1!2!} + \frac{1}{1!1!1!} + \frac{1}{1!2!} + \frac{1}{3!} + \frac{1}{2!1!} + \frac{1}{1!2!} + \frac{1}{3!}\right) \\ &= \frac{1}{6} \end{aligned}$$

So our example has inherited permutation-fairness.

In the general case, imagine we roll n dice, $X_1, X_2, \cdots, X_n$. Make a sequence of the copies they land on, and let's say n_i dice occupying the ith copy. If the sequence is weakly decreasing then the probability of $X_1 > X_2 > \cdots > X_n$ is $\prod_i (1/n_i!)$ and 0 otherwise.

The number of sequences where n_i dice occupying the ith copy is $n!/\prod_i (1/n_i!)$, but only one of these sequences is weakly decreasing. So, for each multiplicity, the probability

DOI 10.2478/rmm-2023-0004

of $X_1 > X_2 > \cdots > X_n$ is $1/n!$.

Finally, summing over all multiplicities, the total probability of $X_1 > X_2 > \cdots > X_n$ is $1/n!$ as required. Similarly, all other orderings of $X_1, X_2, \cdots, X_n$ will be equally likely, as well as their subsets.

So, expanded sets do inherit permutation-fairness. We will now create a fourth die, D, using values that fit in the gaps of the expanded set, while maintaining permutation-fairness.

2.1 Constructing a new die

Let's return to our previous example of three permutation-fair dice. We made three copies of these dice to make the expanded set $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$. We now want to create a new die, D, by insert values for D between the gaps of the expanded set. Our problem is to find how many values we need to insert while maintaining permutation-fairness.

For simplicity, let's refer to the expanded dice as A, B and C. Then, in one line notation, we want to create something that looks like this

$$d_0 \quad \text{CBABCCCCBABC} \quad d_1 \quad \text{CBABCCCCBABC} \quad d_2 \quad \text{CBABCCCCBABC} \quad d_3$$

where the d_i are placeholders for values of D that are to be inserted between the gaps.

Let k_i denoted the number of values inserted in each position d_i. This means die D will have $K = \sum_i k_i$ faces. To find the values of k_i, imagine rolling the four dice together, and calculating the probability of various permutations.

Let's start by calculating the probability of order ABCD. In that case, A, B and C have all landed in copies with larger values than D. For example, if D lands on d_1, then the other dice have landed in the second and third copies. This occurs with probability $(2/3)^3$, and the probability that A > B > C is 1/6, due to the expanded set inheriting permutation-fairness.

In full, if D lands on d_0, d_1, d_2 or d_3, then the probability A, B, C have landed on copies with larger values will be 1, $(2/3)^3$, $(1/3)^3$ and 0, respectively, and in each case the probability that A > B > C is 1/6. We can now calculate the probability of order ABCD to be:

$$
\begin{aligned}
&P(ABCD)\\
&=\sum_{d_i} P(D \in d_i)P(A,B,C > D)P(A > B > C)\\
&=\left(\frac{k_0}{K}\right)(1)\left(\frac{1}{6}\right)+\left(\frac{k_1}{K}\right)\left(\frac{2}{3}\right)^3\left(\frac{1}{6}\right)+\left(\frac{k_2}{K}\right)\left(\frac{1}{3}\right)^3\left(\frac{1}{6}\right)+\left(\frac{k_3}{K}\right)(0)\left(\frac{1}{6}\right)
\end{aligned}
$$

Calculating the probability of order ABDC is a little trickier, because D splits the order into two parts. Namely, values that are higher than D and values that are lower than D. Therefore, for each d_i, the probability of A and B landing on two higher copies and C landing on a lower copy will be (1)(0), $(2/3)^2(1/3)$, $(1/3)^2(2/3)$ and (0)(1), respectively. And in each case, the probability of A $>$ B is 1/2. So the probability of order ABDC will be:

$$
\begin{aligned}
&P(ABDC)\\
&=\sum_{d_i} P(D \in d_i)P(A,B > D)P(D > C)P(A > B)\\
&=\left(\frac{k_0}{K}\right)(1)(0)\left(\frac{1}{2}\right)+\left(\frac{k_1}{K}\right)\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)\left(\frac{1}{2}\right)+\left(\frac{k_2}{K}\right)\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)\left(\frac{1}{2}\right)+\left(\frac{k_3}{K}\right)(0)(1)\left(\frac{1}{2}\right)
\end{aligned}
$$

Similarly we have:

$$
\begin{aligned}
&P(ADBC)\\
&=\sum_{d_i} P(D \in d_i)P(A > D)P(D > B,C)P(B > C)\\
&=\left(\frac{k_0}{K}\right)(1)(0)\left(\frac{1}{2}\right)+\left(\frac{k_1}{K}\right)\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^2\left(\frac{1}{2}\right)+\left(\frac{k_2}{K}\right)\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^2\left(\frac{1}{2}\right)+\left(\frac{k_3}{K}\right)(0)(1)\left(\frac{1}{2}\right)
\end{aligned}
$$

$$
\begin{aligned}
&P(DABC)\\
&=\sum_{d_i} P(D \in d_i)P(D > A,B,C)P(A > B > C)\\
&=\left(\frac{k_0}{K}\right)(0)\left(\frac{1}{6}\right)+\left(\frac{k_1}{K}\right)\left(\frac{1}{3}\right)^3\left(\frac{1}{6}\right)+\left(\frac{k_2}{K}\right)\left(\frac{2}{3}\right)^3\left(\frac{1}{6}\right)+\left(\frac{k_3}{K}\right)(1)\left(\frac{1}{6}\right)
\end{aligned}
$$

If each permutation is to be equally likely, we must set these probabilities to 1/24. Finally, solving for k_i gives $k_0 = 1$, $k_2 = 3$, $k_3 = 3$ and $k_3 = 1$ with $K = 8$.

We have now derived a fourth die, D. In one-line notation the set will look like this:

D CBABCCCCBABC DDD CBABCCCCBABC DDD CBABCCCCBABC D

Closing the gaps, so the dice use consecutive integers, results in this set:

A: [04, 11, 19, 26, 34, 41]
B: [03, 05, 10, 12, 18, 20, 25, 27, 33, 35, 40, 42]
C: [02, 06, 07, 08, 09, 13, 17, 21, 22, 23, 24, 28, 32, 36, 37, 38, 39, 43]
D: [01, 14, 15, 16, 29, 30, 31, 44]

This is a set of four permutation-fair dice. Establishing all orders of the full set are equally likely is enough to ensure the orders of all subsets are also equally likely.

In general, if we start with a set of permutation-fair dice $X_1, \cdots, X_n$ and an expanded set made from r copies. Then, for all orders of $X_1, \cdots, X_n$ and new die X_{n+1} to be equally likely, we must find solutions, k_i, to the following set of equations:

$$\frac{1}{(n+1)!} = \frac{1}{l!(n-l)!} \sum_{i=0}^{r} \frac{k_i}{K} \left(\frac{i}{r}\right)^l \left(\frac{r-i}{r}\right)^{n-l}, \quad \text{for } l = 0, \cdots, n$$

This can also be applied to all subsets of $X_1, \cdots, X_n$. In particular, when $n = l$ we get the following neat set of equations:

$$\frac{1}{l+1} = \sum_{i=0}^{r} \frac{k_i}{K} \left(\frac{i}{r}\right)^l, \quad \text{for } l = 0, \cdots, n$$

2.2 Five or more players

Inductive construction can be used to construct permutation-fair dice from a known set of permutation-fair dice, or build them from scratch. For example, the set of three dice above was constructed from the trivial 1-sided die.

In general, if we start with n dice and make r copies, we get the following table of results for the new die. Table entries are written in the form $(k_0, k_1, \cdots, k_r), K$:

n\r	1	2	3	4	5	6
1	(1, 1), K = 2	(0, 1, 0), K = 1	(0, 1, 1, 0), K = 2	(0, 0, 1, 0, 0), K = 1	(0, 0, 1, 1, 0, 0), K = 2	(0, 0, 0, 1, 0, 0, 0), K = 1
2	-	(1, 4, 1), K = 6	(1, 3, 3, 1), K = 8	(1, 0, 4, 0, 1), K = 6	(0, 11, 1, 1, 11, 0), K = 24	(1, 0, 0, 4, 0, 0, 1), K = 6
3	-	(1, 4, 1), K = 6	(1, 3, 3, 1), K = 8	(1, 0, 4, 0, 1), K = 6	(0, 11, 1, 1, 11, 0), K = 24	(1, 0, 0, 4, 0, 0, 1), K = 6
4	-	-	-	(7, 32, 12, 32, 7), K = 90	(19, 75, 50, 50, 75, 19), K = 288	(1, 5, 1, 6, 1, 5, 1), K = 20
5	-	-	-	(7, 32, 12, 32, 7), K = 90	(19, 75, 50, 50, 75, 19), K = 288	(1, 5, 1, 6, 1, 5, 1), K = 20
6	-	-	-	-	-	(41, 216, 27, 272, 27, 216, 41), K = 840

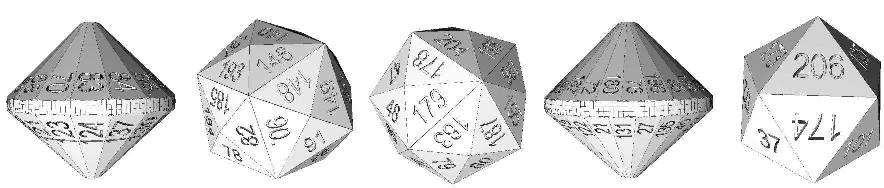
We could have made a smaller set of four dice in our previous example. Starting with the set of three dice, and taking two copies instead of three, would have resulted in a 4-sided die A, an 8-sided die B, a 12-sided die C and a 6-sided die D.

If we then made six copies of this set, we will have a 24-sided dice A, a 48-sided die B, a 72-sided die C, a 36-sided die D and a 20-sided die E. This gives a set with a total of 200 sides, and is the smallest five player set of permutation fair dice known to date, when smallest means fewest number of faces in the set.

However, this set does include one large 72-sided die. Alternatively, if we want each individual die to be smaller, we could start with the following four player set;

ACDBBDDCBCCDBAADAACDBBCBDDCCDBA

We may now continue as before, by taking six copies and creating a 20-sided die, E. This results in one 36-sided die, two 48-sided dice, one 54-sided die and a 20-sided die. This is 206 faces in total, but the individual dice are generally smaller and more usable. This our preferred set, and is the set given in the introduction.



One important point to note about this set is that the initial set of four dice was not found by the inductive construction. Instead, the set was discovered using a computer

DOI 10.2478/rmm-2023-0004

program written by Landon Kryger - which has been extraordinarily helpful in searching for permutation-fair sets, though it has yet to find a five player set on its own.

Finally, what if we wanted our dice to all be the same size? One set of permutation-fair dice, with dice the same size, is based on the following set of four 18-sided dice:

DCBCDBBCDAAACAAADBBBDDAAACCCBDDCDCBB-
-BBCDCDDBCCCAAADDBBBDAAACAAADCBBDCBCD

By making ten copies of this set we may construct a fifth die with 36 sides, giving four 180-sided dice and one 36-sided die. However, making five copies of the fifth dice will result in a set of five 180-sided permutation-fair dice.

The two solutions given in the introduction should be compared and contrasted. Both binary construction and induction were designed to make the search or construction of Go First Dice easier, and both have nice features. Still, neither method is comprehensive and other solutions, including smaller solutions, may still exist.

References

[Bel12] A. Bellos, *Puzzler develops game-changing Go First dice*, The Guardian, 2012. `http://www.guardian.co.uk/science/alexs-adventures-in-numberland/2012/sep/18/puzzler-go-first-dice`

[Enr19] B. Enright, *Dice checker*, 2019. `http://www.brandonenright.net/cgi-bin/dice_perm_util.pl`

[Kno12] B. Knoll, *Five Player Go First Dice*, 2012. `http://byronknoll.blogspot.se/2012/10/five-player-go-first-dice.html`

[Har18] E. Harshbarger, *Go First Dice*, 2018. `http://www.ericharshbarger.org/dice/go_first_dice.html`

[Har12] E. Harshbarger, "Who Goes First", *Games Magazine*, Vol. 36, No. 10, Issue 286, Dec 2012.