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Arrangements of mutually non-attacking chess pieces of mixed type

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Abstract

We present placements of mutually non-attacking chess pieces of mixed type that occupy more than half of the squares of an $m \times n$ board. If both white and black pawns are allowed as separate types, there are arrangements, which we also present, that occupy at least two-thirds of the board squares.

1 Introduction

Many people have searched for ways to place a maximum number of mutually non-attacking chess pieces of a given type on different types of chessboard. For example, the classic *n-queens problem* asks for placements of n mutually non-attacking queens on a chessboard with n rows and n columns [BS09]. There are several variations of the n-queens problem, including different types of board and different types of pieces.

In this paper we consider rectangular boards with m rows and n columns with pieces that are standard, except we allow pawns to be on the first and last row and we ignore rules about check, castling, and *en passant* captures. It is well-known how many non-attacking chess pieces of each of these types can be placed on an $n \times n$ square board [Bro19]. Kathleen Johnson's thesis discusses the extension of most of these results to $m \times n$ rectangular boards [Joh18].

A queen attacks every square on its row, column, and diagonals, so if we do not want the queens to attack any pieces, we can place at most m or n of them, whichever number is smaller.

A rook attacks every square on its row or column, so we can put at most m or n of them, whichever number is smaller.

A bishop attacks each square on the rising diagonal (the set of all squares (i, j) for which $i + j$ equals the sum of the bishop's row and column indices) it is on and on its falling diagonal (the set of all squares (i, j) for which $i - j$ equals the bishop's row index

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minus its column index). There are $m+n-1$ distinct rising diagonals, so we can place at most that many mutually non-attacking bishops. If two corner squares are on the same diagonal, then at most $m+n-2$ mutually non-attacking bishops can be placed.

A king attacks its neighboring squares. Each 2×2 array can hold at most one king without attacks, so we can place at most $\lceil \frac{m}{2} \rceil \lceil \frac{n}{2} \rceil$ kings on the board, where $\lceil a \rceil$ is the smallest integer greater than or equal to a.

We can cover half of the board with knights. Knights move and attack by leaping to a square two squares away vertically or horizontal and one square away in a perpendicular direction. When a knight moves, its destination square has a different color than its origin, so we can place knights on all white squares or on all black squares, and such knights do not attack each other. It is not possible to place more knights on the board without having one attack another.

The squares that a pawn attacks depends on which side it is on. White pawns attack the two squares diagonally forward – e.g. on a large enough board, a pawn on the second row from the bottom and the third column would attack the square in the third row, second column and the square in the third row, fourth column. Black pawns attack the two squares diagonally backward. If we consider pawns of one color, we can cover at least half the squares with pawns (which attack the two squares forward and diagonally adjacent to them) by filling every other row (or every other column) with them, which covers $\max\left(\lceil \frac{m}{2} \rceil n, \lceil \frac{n}{2} \rceil m\right)$ squares.

Recently David Pisa suggested considering arrangements of two or more types of piece. He posed the question [Pis21]:

> What is the maximum number of pieces of mixed types that can fit on a standard chessboard so that no piece threatens any other piece? (The solution must include more than one type of piece, but need not include every type.)

David Pisa found an arrangement of pawns and knights that occupies more than half of the squares of a standard chessboard. Inspired by that pattern and other patterns discovered through the use of the MiniZinc optimization program [NSB07, SFS14], in this paper we show arrangements of non-attacking pieces of mixed type that occupy more than half of the squares of $m \times n$ chessboards. Furthermore, if black pawns are included as a separate piece type, we have mixed type arrangements of non-attacking pieces that fill at least two-thirds of the board.

In the literature, there are already many studies of attack-free arrangements of more than one type of chess piece [Gar01, Jel05]. For example, the classic Eight Officers Problem asks for a placement of a player's non-pawn pieces (both rooks, both knights, both bishops, the queen, and king) so that no piece attacks any other piece [Gar01]. Other studies have attempted to place set numbers of two or more piece types [Gar01]. George Jelliss has found many mutually non-attacking arrangements of two, three, or more types of chess piece for which the product of the numbers of each type is maximized [Jel05]. However, we have not found any study that seeks to simply maximize the total number of pieces. We begin an approach to that goal in this paper.

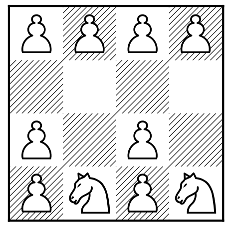

Figure 1: Arrangement of 8 pawns and 2 knights, all mutually non-attacking, on a 4×4 board

2 Results

2.1 Results with white pawns

To look for solutions to Pisa's problem, we first used a MiniZinc model like that given in the Appendix to find maximum non-attacking arrangements of knights, rooks, and pawns. Except for $m = 3$ and the trivial case $m = n = 1$, as we prove in Proposition 1, we get mixed type arrangements of mutually non-attacking pieces with at least $\lceil \frac{n(m+1)}{2} \rceil$ pieces.

Proposition 1. *Let $m \neq 3, n \geqslant 1$, and $mn > 1$. We can place $\lceil \frac{n(m+1)}{2} \rceil$ mutually non-attacking chess pieces of mixed type on an $m \times n$ chess board.*

Proof. We can fill a $1 \times n$ board with $n = \lceil \frac{n(m+1)}{2} \rceil$ mutually non-attacking pieces: just alternate knights and pawns.

For $m = 2$, alternate pawns and knights on the bottom row, and then in the top row place pawns in the same columns that have them in the bottom row (if there is just one column, place a knight instead of a pawn). The total number of pieces on the board is $n + \lceil \frac{n}{2} \rceil = \lceil \frac{n(m+1)}{2} \rceil$, and none attack other pieces.

For even $m = 2k + 2$ with $k > 0$, copy the $m = 2$ arrangement for the bottom two rows. The bottom two rows have $n + \lceil \frac{n}{2} \rceil$ pieces on them. For the other $2k$ rows, starting with the 3rd row from the bottom, alternate between empty rows and rows filled with pawns. See Figure 1 for an example with $m = 4$ rows. The top $2k$ rows have kn pieces, which brings the total number of mutually non-attacking pieces on the board to $(k+1)n + \lceil \frac{n}{2} \rceil = \lceil \frac{n(2k+2+1)}{2} \rceil = \lceil \frac{n(m+1)}{2} \rceil$.

For $m = 6k + 3$ for $k > 0$, repeat the following 6-row pattern k times, starting from the bottom:

> On the bottom row, alternate knights and pawns. On the row above, place pawns in the same columns as the bottom row. Leave the next row empty. On the row above that, alternate pawns and knights. On the row above that, place pawns in the same columns as the previous row. Leave the next row empty. Figure 2 shows this 6-row pattern. These six rows have $3n$ mutually non-attacking pieces on them.

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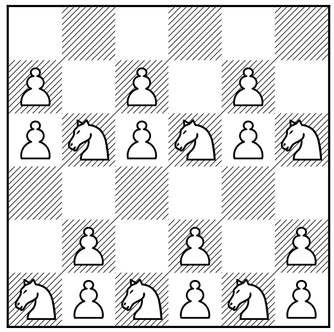

Figure 2: Six-row pattern of mutually non-attacking knights and pawns

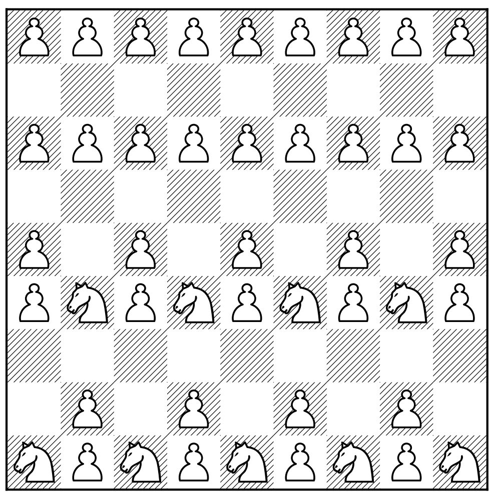

Figure 3: Arrangement of 36 pawns and 9 knights, all mutually non-attacking, on a 9×9 board

For the final three rows, fill the top row and the third row from the top with pawns and leave the second row empty. Figure 3 is an example of this case. These three rows have $2n$ pieces, so the total number of mutually non-attacking pieces on the board is $3nk + 2n = n(3k+2) = \frac{n(6k+3+1)}{2} = \lceil \frac{n(m+1)}{2} \rceil$.

For $m = 6k+5$ with $k \geqslant 0$, follow the 6-row part of the $6k+3$ pattern $k+1$ times, not including the final empty row. Figure 4 shows this pattern for $m = 11$. The number of mutually non-attacking pieces on the board is $3n(k+1) = \frac{6n(k+1)}{2} = \frac{(m+1)n}{2} = \lceil \frac{n(m+1)}{2} \rceil$.

For $m = 6k+1$ with $k > 0$, follow the 6-row subpattern k times from the bottom, and on the top row alternate knights and pawns, as shown in Figure 5 for $m = 7$. The number of mutually non-attacking pieces on the board is $3nk + n = \frac{n(6k+1+1)}{2} = \lceil \frac{n(m+1)}{2} \rceil$.

The above cases cover all $m \neq 3$. □

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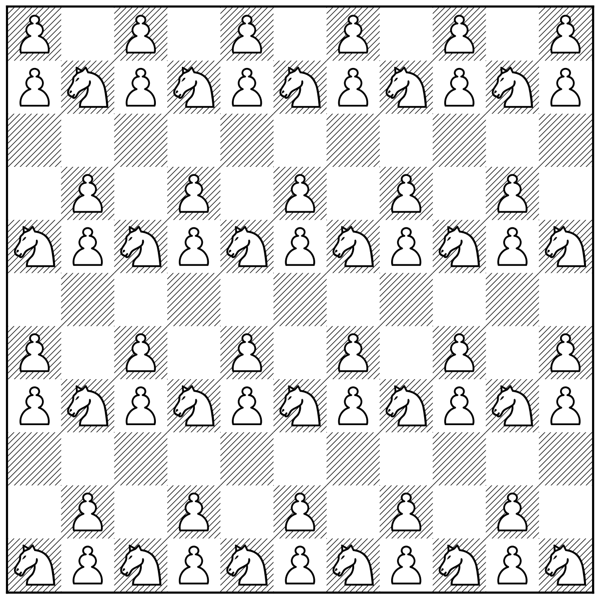

Figure 4: Arrangement of 44 pawns and 22 knights, all mutually non-attacking, on an 11×11 board

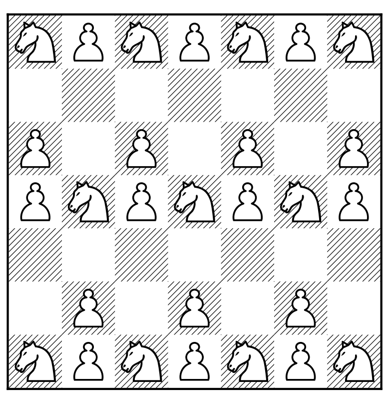

Figure 5: Arrangement of 17 pawns and 11 knights, all mutually non-attacking, on an 7×7 board

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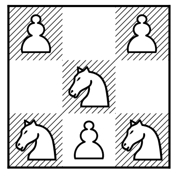

Figure 6: Arrangement of 3 pawns and 3 knights, all mutually non-attacking, on a 3×3 board

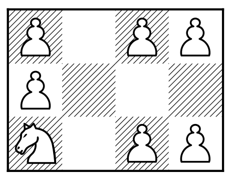

Figure 7: Arrangement of 6 pawns and 1 knight, all mutually non-attacking, on a 3×4 board

On the $3 \times n$ board, we can place $\lceil \frac{n(m+1)}{2} \rceil = 2n$ pawns, n on the top row and n on the bottom row, but that is not a placement of mixed type.

After further computer experimentation with MiniZinc models, we found some patterns of mixed type on $3 \times n$ boards that we believe are best possible. Figure 6 shows a mixed type placement of 6 mutually non-attacking pieces on a 3×3 board. The first two columns of that figure give a mixed type placement of 4 pieces on a 3×2 board.

Proposition 2. *On a $3 \times n$ board with $n \geqslant 4$, we can place $2n-1$ mutually non-attacking pieces of mixed type.*

Proof. In the first column, place a knight in the bottom row and pawns in the other squares. Leave the second column empty. In all other columns, place pawns in the top and bottom row. See Figure 7 for an example. There are $2n-1$ mutually non-attacking pieces on the board. □

We can place more than $\lceil \frac{n(m+1)}{2} \rceil$ in some cases, such as two-column boards with sufficiently many rows.

Proposition 3. *For $m = 4$ or $m \geqslant 6$, on an $m \times 2$ board we can place $m+2$ mutually non-attacking pieces of mixed type.*

Proof. For $m = 4k$, $k \geqslant 1$, repeat the following 4-row block k times, starting from the bottom of the board: Knights on the squares of the bottom row of the block, pawns on the squares of the row above that, and two empty rows above that (so each 4-row block has 4 mutually non-attacking pieces). Then on the top row of the top block, place two

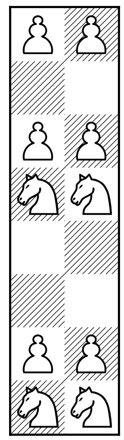

Figure 8: Arrangement of 6 pawns and 4 knights, all mutually non-attacking, on a 8×2 board

pawns. There are $m+2$ total non-attacking pieces on the board. See Figure 8 for an example with $k=2$.

For $m = 4k+1$, $k \geqslant 2$, repeat the 4-row block $k-1$ times (again starting from the bottom of the board), except place a pawn in the second column of the top row of the top block. For the final top five rows, put pawns on the top row, then alternate knights and pawns in the second column for the four rows below the top row. There are $4(k-1)+1+2+4 = 4k+3 = m+2$ total non-attacking pieces on the board. Figure 9 gives an example of this pattern with $k=2$.

For $m = 4k+2$, $k \geqslant 1$, repeat the 4-row block $k+1$ times, except leave out the empty rows of the $(k+1)^{st}$ block. There are $4(k+1) = m+2$ total non-attacking pieces on the board. The bottom six rows of Figure 8 give an example of this pattern for $k=1$.

For $m = 4k+3$, $k \geqslant 1$, repeat the 4-row block k times, except place an additional pawn in the second column of the top row of the top block. Then place a pawn in the second column of the row above the top block, a knight in the second column of the row above that, and two pawns in the row above that. There are $4k+5 = m+2$ total non-attacking pieces on the board. Figure 10 gives an example of this pattern with $k=1$.

The above cases cover all possible values of m □

Proposition 3 gives a higher number than Proposition 1 for all possible values of m.

We can also improve upon the result of Proposition 1 on most boards with 3 columns.

Proposition 4. *For $m \geqslant 3$, on an $m \times 3$ board we can place $2m-1$ mutually non-attacking pieces of mixed type.*

Proof. Given an $m \times 3$ board with $m \geqslant 3$, place knights on the first and last squares of the bottom row, a rook in the second square of the row immediately above the bottom

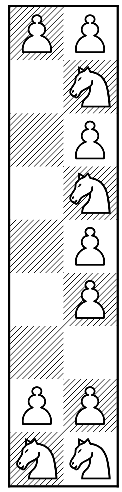

Figure 9: Arrangement of 7 pawns and 4 knights, all mutually non-attacking, on a 9×2 board

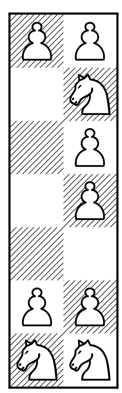

Figure 10: Arrangement of 6 pawns and 3 knights, all mutually non-attacking, on a 7×2 board

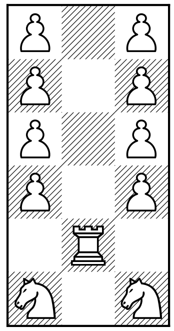

Figure 11: Arrangement of 8 pawns, 2 knights, and 1 rook, all mutually non-attacking, on a 6×3 board

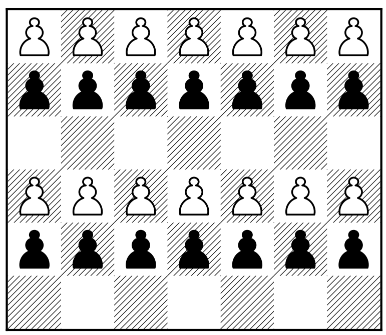

Figure 12: Arrangement of 28 mutually non-attacking black and white pawns on a 6×7 board

row, and pawns on the first and last squares of the remaining rows. (Figure 11 gives an example with $m = 6$.) There are $2m - 1$ mutually non-attacking pieces on the board. □

For $m \geqslant 7$, the result in Proposition 4 is higher than that of Proposition 1.

2.2 Results involving white and black pawns

If we allow both white and black pawns as separate types, then we can occupy at least two-thirds of the board. Consider the pattern, noted in 2016 by Dave Barlow according to [Fri22], where the bottom row is empty, the next row is filled with black pawns, the row above is filled with white pawns, and this 3-row pattern repeats for the rest of the board. An example of this pattern is shown in Figure 12. All squares in roughly two-thirds of the rows are occupied, and none of those pawns attack each other.

If the board has $3k$ rows and n columns, then the pattern covers $2kn$ squares out of $3kn$ squares, which is exactly two-thirds of the board. If the board has $3k - 1$ rows and n columns, skip the bottom empty row to produce a pattern covering $2kn$ out of $3kn - n$

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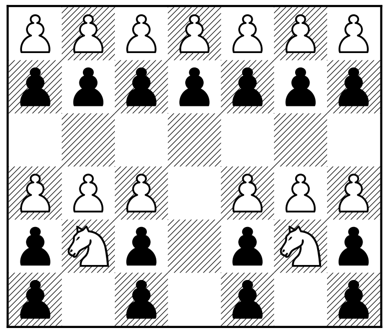

Figure 13: Arrangement of 13 white pawns, 15 black pawns, and 2 knights, all mutually non-attacking, on a 6×7 board

squares, which is more than two-thirds. If the board has $3k+1$ rows and n columns, start with a bottom row of white pawns and then follow the $3k$-row pattern for the remaining rows. We get $(2k+1)n$ out of $(3k+1)n$ squares occupied, which is more than two-thirds of the board.

In many cases we can find other interesting patterns of non-attacking pieces occupying at least two-thirds of the board.

Proposition 5. *Let $k \geqslant 1$ and $n \geqslant 4$. Then there is a placement of mutually non-attacking white pawns, black pawns, and knights on a $3k \times n$ board that fills at least two-thirds of the board.*

Proof. On the top $3(k-1)$ rows of the board, alternate rows of white pawns, black pawns, and empty squares. In the third row from the bottom, leave every fourth square empty and fill the rest with white pawns. In the second row from the bottom, leave every fourth square empty, place a knight in the second square and every fourth subsequent square (i.e., squares $2, 6, 10, \ldots$), and black pawns in the other squares. Finally in the bottom row place black pawns in the same columns as placed in the second row. Figure 13 illustrates an example of this pattern.

None of the pieces attack any other piece. In the bottom three rows, at least two-thirds of the squares are occupied. In the $3(k-1)$ top rows, two thirds of the rows are filled with pawns. So the pattern occupies at least two-thirds of the entire board. □

Proposition 6. *Let $k \geqslant 1$ and $n \geqslant 2$. Then there is a placement of mutually non-attacking white pawns and black pawns on a $(3k+1) \times n$ board that fills at least two-thirds of the board.*

Proof. On the top $3(k-1)$ rows of the board, alternate rows of white pawns, black pawns, and empty squares. In the third row from the bottom, leave every fourth square empty and fill the rest with white pawns. Fill the fourth row from the bottom with white pawns

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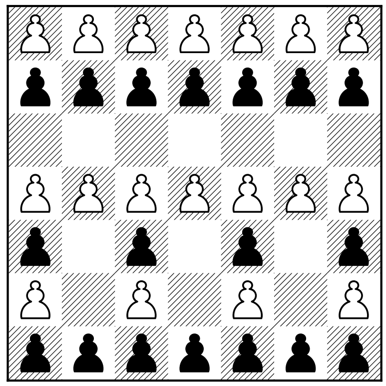

Figure 14: Arrangement of 18 white and 18 black pawns, all mutually non-attacking, on a 7×7 board

and the bottom row with black pawns. In the third row from the bottom, place black pawns in the first, third, and every second subsequent square. In the second row from the bottom, place white pawns in the first, third, and every second subsequent square. Figure 14 illustrates an example of such a placement.

None of the pawns attack any other pawn. At least three-quarters of the squares in the bottom four rows are occupied (two complete rows and two half-rows), and two-thirds of the $3(k-1)$ top rows are filled with pawns, so this pattern occupies more than two-thirds of the board. □

Proposition 7. *Let $k \geqslant 1$ and $n \geqslant 2$. Then there is a placement of mutually non-attacking white pawns, black pawns, and knights on a $(3k+2) \times n$ board that fills more than two-thirds of the board.*

Proof. On the top $3k$ rows alternate rows of white pawns, black pawns, and empty squares. On the second row from the bottom, alternate white pawns and empty squares. In the bottom row, alternate white pawns and knights. Figure 15 illustrates this pattern on an 8×8 board.

None of the pieces attack any other piece. Two-thirds of the top $3k$ rows are filled with pawns, and at least three-quarters of the bottom two rows are filled. So more than two-thirds of the entire board is occupied by the pattern. □

3 Conclusions and Open Problems

We have shown that more than half of a rectangular board can be occupied by mutually non-attacking pieces of mixed type. If white and black pawns are allowed as

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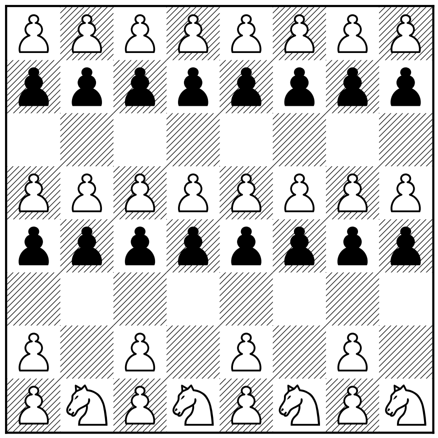

Figure 15: Arrangement of 24 white pawns, 16 black pawns, and 4 knights, all mutually non-attacking, on a 8×8 board

separate types, then there are placements of mutually non-attacking pieces of mixed type that occupy at least two-thirds of the board.

These results provoke many open questions, some of which we list here:

1. We have not shown that our patterns are the best possible. For small values of m and n, based on MiniZinc model solutions, Tables 1 and 2 indicate the maximum number of pieces in a mixed type arrangement of mutually non-attacking chess pieces without black pawns and with black pawns, respectively. These tables also indicate the maximum number of mutually non-attacking pieces without the "mixed-type" condition.

   If we trust the results in Tables 1 and 2, we can generate upper bounds.

   Proposition 8. *If at most* 8 *mutually non-attacking pieces (queens, rooks, bishops, knights, and white and black pawns) can be placed on a* 3×4 *board, then for all* $\epsilon > 0$ *and* m, n *sufficiently large, the maximum number of mutually non-attacking pieces on an* $m \times n$ *board is at most* $\frac{2}{3} + \epsilon$ *of the board.*

   Proof Sketch: Let $m = 3a + i$ and $n = 4b + j$, where a, b, i, j are non-negative integers with $a, b \geqslant 1, 0 \leqslant i \leqslant 2$, and $0 \leqslant j \leqslant 3$. Take an $m \times n$ board and divide the lower-left $3a \times 4b$ corner into an array of ab rectangular blocks of size 3×4. Each such block has a capacity of 8. So, at most we can fill $8ab + 4bi + 3aj + ij$ of the available $mn = 12ab + 4bi + 3aj + ij$ squares. The fraction of the board that is occupied is

   $$\frac{8ab + 4bi + 3aj + ij}{12ab + 4bi + 3aj + ij}$$

   which approaches $\frac{2}{3}$ as m and n approach infinity. □

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$m \backslash n$	1	2	3	4	5	6	7	8
1	- (1)	2	3	4	5	6	7	8
2	2	4	5	6	8	9	11	12
3	3	4	6	7 (8)	9 (10)	11 (12)	13 (14)	15 (16)
4	4	6	8	10	13	15	18	20
5	5	6	9 (10)	12	15	18	21	24
6	6	8	11 (12)	14	18	21	25	28
7	7	9	13 (14)	16	20 (21)	24	28	32
8	8	10	15 (16)	18	23 (24)	27	32	36

Table 1: Maximum number of pieces in a non-attacking arrangement of mixed type on an $m \times n$ board, where the allowed types are queen, king, bishop, knight, rook, and (white) pawn. Numbers in parentheses are maximum numbers when the "mixed type" constraint is removed and all pieces can be of the same type, in those cases where removing the constraint changes the maximum number.

Using similar arguments and assuming the correctness of the 8×4 entry in Table 1, we can show the following upper bound.

Proposition 9. *If at most* 18 *mutually non-attacking pieces (queens, rooks, bishops, knights, and white pawns) can be placed on a* 8×4 *board, then for all* $\epsilon > 0$ *and* m, n *sufficiently large, the maximum number of mutually non-attacking pieces on an* $m \times n$ *board is at most* $\frac{9}{16} + \epsilon$ *of the board.*

However, even with these upper bounds, the general questions remain open: What is the maximum number of mutually non-attacking pieces that can be placed in an arrangement, or in a mixed-type arrangement, on a rectangular chessboard?

2. How many non-attacking pieces can be placed on a different type of board, such as a cylinder, torus, or three-dimensional board?

   We can make a few comments about the problem on a cylinder or torus board. First note that a placement of mutually non-attacking pieces on a cylinder or torus board formed from an $m \times n$ rectangular board remains a placement of mutually non-attacking pieces if we make the cuts needed to turn the board back into an $m \times n$ rectangle. Therefore, the maximum number of mutually non-attacking pieces on a cylinder or torus board is no greater than the maximum number for the corresponding rectangle.

   Also we note that the patterns of Propositions 1 and 7 work for the corresponding cylinder boards if the number of columns is even and the pattern of Figure 12 works for all corresponding cylinder boards.

3. The problem of finding a maximal arrangement of non-attacking pieces of a single type is often interpreted as a graph theory problem [BS09, Joh18, Wat04]: each

$m\backslash n$	1	2	3	4	5	6	7	8
1	- (1)	2	3	4	5	6	7	8
2	2	4	6	8	10	12	14	16
3	3	6	8	8	11	13	16	16
4	4	6	10	12	16	18	22	24
5	5	8	12	16	20	24	28	32
6	6	10	14	16	22	25	30	32
7	7	12	16	20	26	30	36	40
8	8	12	18	24	30	36	42	48

Table 2: Maximum number of pieces in an arrangement of non-attacking pieces of mixed type on an $m \times n$ board, where the allowed types are queen, rook, bishop, knight, white pawn, and black pawn. Numbers in parentheses are maximum numbers when the "mixed type" constraint is removed and all pieces can be of the same type, in those cases where removing the constraint changes the maximum number.

square is a vertex of the graph, each possible move of the piece from one square to another is an edge of that graph, and the problem becomes the problem of finding a maximum set of vertices for which no pair of vertices share an edge (i.e., a *maximum independent set*). Is there a useful graph-theoretic interpretation of finding maximum arrangements of non-attacking pieces of mixed type?

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Appendix: MiniZinc code

Here is a MiniZinc model for arrangements of the maximum number of mutually nonattacking knights, rooks, and pawns.

```
include "globals.mzn";

int: m;
int: n;
int: xguess;

array[0..m-1,0..n-1] of var 0..3: board;
% board[i,j] = 0 if (i,j) is empty, 1 if (i,j) is Knight, 2 if Rook,
% and 3 if Pawn

array[1..4] of string: sy;

sy=[".","N","R","P"];

constraint count(board,1)+count(board,2)+count(board,3)>=xguess;
% don't accept any board with fewer than xguess pieces

constraint forall(i in 2..m-1,j in 1..n-1)(board[i,j]!=1 \/board[i-2,j-1]==0);
constraint forall(i in 2..m-1,j in 0..n-2)(board[i,j]!=1 \/board[i-2,j+1]==0);
constraint forall(i in 1..m-1,j in 2..n-1)(board[i,j]!=1 \/board[i-1,j-2]==0);
constraint forall(i in 1..m-1,j in 0..n-3)(board[i,j]!=1 \/board[i-1,j+2]==0);
constraint forall(i in 0..m-2,j in 2..n-1)(board[i,j]!=1 \/board[i+1,j-2]==0);
constraint forall(i in 0..m-2,j in 0..n-3)(board[i,j]!=1 \/board[i+1,j+2]==0);
```

```
constraint forall(i in 0..m-3,j in 1..n-1)(board[i,j]!=1 \/board[i+2,j-1]==0);
constraint forall(i in 0..m-3,j in 0..n-2)(board[i,j]!=1 \/board[i+2,j+1]==0);
% if a square has a knight, all the attacked squares are empty

constraint forall(i in 0..m-1,j in 0..n-1)(board[i,j]!=2 \/ sum(j2 in 0..n-1
where j2!=j)(board[i,j2]) + sum(i2 in 0..m-1 where i2!=i)(board[i2,j])==0);
% if a square has a rook, all attacked squares are empty

constraint forall(i in 1..m-1,j in 1..n-1)(board[i,j]!=3 \/board[i-1,j-1]==0);
constraint forall(i in 1..m-1,j in 0..n-2)(board[i,j]!=3 \/board[i-1,j+1]==0);
% if a square has a pawn, both attacked squares are empty

solve maximize count(board,1)+count(board,2)+count(board,3);
%maximize the number of pieces on the board

output[sy[fix(board[i,j]+1)] ++ if j==n-1 then "\n" else "" endif ++ if
i+j==m+n-2 then "\n"++show(count(board,1)+count(board,2)+count(board,3))
else "" endif |i in 0..m-1,j in 0..n-1];
```

To look for exclusively mixed type arrangements, we can add the following code before the **solve** statement:

```
constraint count(board,2)+count(board,3)>0;% at least one piece is not a knight
constraint count(board,1)+count(board,3)>0;% at least one piece is not a rook
constraint count(board,1)+count(board,2)>0;% at least one piece is not a pawn
```

We can expand the model to allow for placements of queens, kings, and bishops.