sciendo

TATRA MOUNTAINS
Mathematical Publications
DOI: 10.2478/tmmp-2019-0015
Tatra Mt. Math. Publ. **74** (2019), 7–16

THE ZARISKI TOPOLOGY ON THE GRADED PRIMARY SPECTRUM OVER GRADED COMMUTATIVE RINGS

Khaldoun Al-Zoubi* — Malik Jaradat

Department of Mathematics and Statistics, Jordan University of Science and Technology, Irbid, JORDAN

Department of Mathematics, The international School of Choueifat (MHS-AlDaid), Alain, UAE

ABSTRACT. Let G be a group with identity e and let R be a G-graded ring. A proper graded ideal P of R is called *a graded primary ideal* if whenever $r_g s_h \in P$, we have $r_g \in P$ or $s_h \in Gr(P)$, where $r_g, s_g \in h(R)$. The *graded primary spectrum* $p.Spec_g(R)$ is defined to be the set of all graded primary ideals of R. In this paper, we define a topology on $p.Spec_g(R)$, called Zariski topology, which is analogous to that for $Spec_g(R)$, and investigate several properties of the topology.

1. Introduction

The concept of graded prime ideal was introduced by M. Refai, M. Hailat and S. Obiedat in [10] and studied in [1,8,11].

Zariski topology on the graded prime spectrum of graded commutative rings have been already studied in [7,8,10]. These results will be used in order to obtain the main aims of this paper. The notion of primary spectrum was examined as a generalization of prime spectrum in [6]. They showed that the set of primary ideals can be endowed with a topology called the Zariski topology on primary spectrum of R. Graded primary ideals of a commutative graded ring have been introduced and studied by Refai and Al-Zoubi in [9]. These ideals are generalizations of primary ideals in a graded ring. The set of all graded primary ideals and the set of all primary ideals need not be equal in a graded ring (see [9, Example 1.6]).

2010 Mathematics Subject Classification: 13A02, 16W50.
Keywords: Zariski topology, graded primary spectrum, graded primary ideals.
* Corresponding author.

In this paper, we rely on the graded primary ideals and then, we introduce and study a topology on the graded primary spectrum similar to the one defined in [6], and investigate several properties of the topology.

2. Preliminaries

Convention. Throughout this paper, all the rings are commutative with identity. First, we recall some basic properties of graded rings which will be used in the sequel. We refer to [3], [4] and [5] for these basic properties and more information on graded rings. Let G be a group with identity e. A ring R is called graded (or more precisely, G-graded) if there exists a family of subgroups $\{R_g\}$ of R such that $R = \oplus_{g\in G} R_g$ (as abelian groups) indexed by the elements $g \in G$, and $R_g R_h \subseteq R_{gh}$ for all $g, h \in G$. The summands R_g are called homogeneous components, and elements of these summands are called homogeneous elements. If $r \in R$, then r can be written uniquely $r = \sum_{g\in G} r_g$, where r_g is the component of r in R_g. Also, we write $h(R) = \cup_{g\in G} R_g$. Let

$$R = \underset{g\in G}{\oplus} R_g \quad \text{be a } G\text{-graded ring.}$$

An ideal I of R is said to be a graded ideal if

$$I = \underset{g\in G}{\oplus}(I \cap R_g) := \underset{g\in G}{\oplus} I_g.$$

An ideal of a graded ring need not be graded.

Let R be a G-graded ring. A proper graded ideal I of R is said to be *a graded prime ideal* if whenever $r_g s_h \in I$, we have $r_g \in I$ or $s_h \in I$, where $r_g, s_h \in h(R)$ (see [10]).

Let $Spec_g(R)$ denote the set of all graded prime ideals of R. For each graded ideal I of R, the graded variety of I is the set $V_R^g(I) = \{P \in Spec_g(R) | I \subseteq P\}$. Then, the set $\xi^g(R) = \{V_R^g(I) | I \text{ is a graded ideal of } R\}$ satisfying the axioms for the closed sets of a topology on $Spec_g(R)$ called the Zariski topology on $Spec_g(R)$ (see [7, 8, 10]).

The graded radical of I, denoted by $Gr(I)$, is the set of all $r = \sum_{g\in G} r_g \in R$ such that for each $g \in G$ there exists $n_g \in \mathbb{N}$ with $r_g^{n_g} \in I$. Note that if r is a homogeneous element, then $r \in Gr(I)$ if and only if $r^n \in I$ for some $n \in \mathbb{N}$ (see [10]). In [10], it is shown that $Gr(I)$ is the intersection of all the graded prime ideals of R containing I.

A graded ideal I of R is said to be a graded maximal ideal of R if $I \neq R$ and if J is a graded ideal of R such that $I \subseteq J \subseteq R$, then $I = J$ or $J = R$.

A proper graded ideal I of a G-graded ring R is said to be *a graded primary ideal* if whenever $r_g s_h \in I$, we have $r_g \in I$ or $s_h \in Gr(I)$ where $r_g, s_g \in h(R)$ (see [9]). Let $p.Spec_g(R)$ denote the set of all graded primary ideals of R.

3. Results

Definition 3.1. Let R be a G-graded ring and $p.Spec_g(R)$ be the set of all graded primary ideals of R. We define graded primary variety for any subset E of R as $p\text{-}V_R^g(E) = \{q \in p.Spec_g(R) : E \subseteq Gr(q)\}$.

Lemma 3.2 ([9]). *Let Q be a graded primary ideal of a G-graded ring R. Then, $P = Gr(Q)$ is a graded prime ideal of R, and we say that Q is a graded G-P-primary.*

Proposition 3.3. *Let R be a G-graded ring and I and J be two graded ideals of R. Then, the following hold:*

(i) *If $I \subseteq J$, then $p\text{-}V_R^g(J) \subseteq p\text{-}V_R^g(I)$.*

(ii) *If $E \subseteq R$ and I is the graded ideal of R generated by $h(E)$, then $p\text{-}V_R^g(E) = p\text{-}V_R^g(I) = p\text{-}V_R^g\big(Gr(I)\big)$.*

(iii) *$p\text{-}V_R^g(0) = p.Spec_g(R)$ and $p\text{-}V_R^g(R) = \phi$.*

(iv) *Let $\{E_\alpha\}_{\alpha\in\Delta}$ be a family of subsets of R and I_α be graded ideals of R, then $p\text{-}V_R^g(\cup_{\alpha\in\Delta} E_\alpha) = \cap_{\alpha\in\Delta} p\text{-}V_R^g(E_\alpha)$. In particular, $p\text{-}V_R^g(\sum_{\alpha\in\Delta} I_\alpha) = \cap_{\alpha\in\Delta} p\text{-}V_R^g(I_\alpha)$.*

(v) *For every pair I and J of graded ideals of R, we have $p\text{-}V_R^g(I \cap J) = p\text{-}V_R^g(IJ) = p\text{-}V_R^g(I) \cup p\text{-}V_R^g(J)$.*

Proof.

(i) Let I, $J \subseteq R$ with $I \subseteq J$. If $q \in p\text{-}V_R^g(J)$, then $J \subseteq Gr(q)$, and so $I \subseteq Gr(q)$, it follows that $q \in p\text{-}V_R^g(I)$. Hence, $p\text{-}V_R^g(J) \subseteq p\text{-}V_R^g(I)$.

(ii) Let $E \subseteq R$ and I be the graded ideal of R generated by $h(E) \subseteq p\text{-}V_R^g(I) \subseteq p\text{-}V_R^g(E)$. If $q \in p\text{-}V_R^g(E)$, then $E \subseteq Gr(q)$, and so $h(E) \subseteq Gr(q)$, which implies $I \subseteq Gr(q)$, i.e., $q \in p\text{-}V_R^g(I)$, so $p\text{-}V_R^g(E) \subseteq p\text{-}V_R^g(I)$. Thus, $p\text{-}V_R^g(E) = p\text{-}V_R^g(I)$. Now, since $I \subseteq Gr(I)$, by part (i), $p\text{-}V_R^g\big(Gr(I)\big) \subseteq p\text{-}V_R^g(I)$. Now, let $q \in p\text{-}V_R^g\big(Gr(I)\big)$. Then, $I \subseteq Gr(I) \subseteq Gr(q)$, which implies $q \in p\text{-}V_R^g(I)$, so $p\text{-}V_R^g\big(Gr(I)\big) \supseteq p\text{-}V_R^g(I)$. Hence, $p\text{-}V_R^g\big(Gr(I)\big) \subseteq p\text{-}V_R^g(I)$. Thus, $p\text{-}V_R^g(E) = p\text{-}V_R^g(I) = p\text{-}V_R^g\big(Gr(I)\big)$.

(iii) Since $0 \in q \subseteq Gr(q)$ for all graded primary ideals q of R, we have $p\text{-}V_R^g(0) = p.Spec_g(R)$ and $p\text{-}V_R^g(R) = \phi$.

Let $\{E_\alpha : \alpha \in \Delta\}$ be any family of subsets of R. Clearly, $E_\beta \subseteq \cup_{\alpha\in\Delta} E_\alpha$ for all $\beta \in \Delta$ and hence, by Part (i), $p\text{-}V_R^g(\cup_{\alpha\in\Delta} E_\alpha) \subseteq p\text{-}V_R^g(E_\beta)$ for all $\beta \in \Delta$. Thus, $p\text{-}V_R^g(\cup_{\alpha\in\Delta} E_\alpha) \subseteq \cap_{\alpha\in\Delta}\ p\text{-}V_R^g(E_\alpha)$. Conversely, let $q \in \cap_{\alpha\in\Delta}\ p\text{-}V_R^g(E_\alpha)$. Then, $q \in p\text{-}V_R^g(E_\alpha)$ for all $\alpha \in \Delta$, it follows that $E_\alpha \subseteq Gr(q)$ for all $\alpha \in \Delta$. So, $\cup_{\alpha\in\Delta} E_\alpha \subseteq Gr(q)$, i.e., $q \in p\text{-}V_R^g(\cup_{\alpha\in\Delta} E_\alpha)$. Hence, $p\text{-}V_R^g(\cup_{\alpha\in\Delta} E_\alpha) \supseteq \cap_{\alpha\in\Delta}\ p\text{-}V_R^g(E_\alpha)$. Therefore, $p\text{-}V_R^g(\cup_{\alpha\in\Delta} E_i) = \cap_{\alpha\in\Delta} p\text{-}V_R^g(E_\alpha)$.

Let I, J be any two graded ideals of R. Since $IJ \subseteq I \cap J \subseteq I$ and $IJ \subseteq I \cap J \subseteq J$, by part (i), $p\text{-}V_R^g(I) \subseteq p\text{-}V_R^g(I \cap J)$ and $p\text{-}V_R^g(J) \subseteq p\text{-}V_R^g(I \cap J)$. Hence, $p\text{-}V_R^g(I) \cup p\text{-}V_R^g(J) \subseteq p\text{-}V_R^g(I \cap J) \subseteq p\text{-}V_R^g(IJ)$. Let $q \in p\text{-}V_R^g(IJ)$. Then $IJ \subseteq Gr(q)$. By Lemma 3.1, $Gr(q)$ is a graded prime ideal, hence by [10, Proposition 1.2], $I \subseteq Gr(q)$ or $J \subseteq Gr(q)$. Hence $q \in p\text{-}V_R^g(I)$ or $q \in p\text{-}V_R^g(J)$, it follows that $q \in p\text{-}V_R^g(I) \cup p\text{-}V_R^g(J)$. This implies that $p\text{-}V_R^g(IJ) \subseteq p\text{-}V_R^g(I) \cup p\text{-}V_R^g(J)$. Therefore, $p\text{-}V_R^g(I \cap J) = p\text{-}V_R^g(IJ) = p\text{-}V_R^g(I) \cup p\text{-}V_R^g(J)$. □

Definition 3.4. Let R be a G-graded ring. Since $p\text{-}\eta^g(R) = \{\, p\text{-}V_R^g(I) \mid I$ is a graded ideal of $R\}$ is closed under finite union, the family $p\text{-}\eta^g(R)$ satisfies the axioms of topological space for closed sets. So, there exists a topology on $p.Spec_g(M)$ called the Zariski topology and denoted by $p\text{-}\xi^g(R)$.

We note that, since any graded prime ideal is graded primary and equal to its graded radical, the space $Spec_g(R)$ is in fact a subspace of $p.Spec_g(R)$.

Proposition 3.5. *Let R be a G-graded ring. For any homogeneous element r, the set $GX_r^p = p.Spec_g(R) \backslash p\text{-}V_R^g(r)$ is open in $p.Spec_g(R)$ and the family $\{GX_r^p : r \in h(R)\}$ is the basis for the Zariski topology on $p.Spec_g(R)$.*

Proof. Assume that U is any open set in $p.Spec_g(R)$. Thus, $U = p.Spec_g(R) \backslash p\text{-}V_R^g(I)$ for some graded ideal I of R. Notice that $I = \cup_{g \in G} I_g = < h(I) >$. Hence, $p\text{-}V_R^g(I) = p\text{-}V_R^g\big(h(I)\big) = \cap_{r \in h(I)} p\text{-}V_R^g(r)$. So, $U = \cup_{r \in h(I)}\big(p.Spec_g(R) \backslash p\text{-}V_R^g(r)\big) = \cup_{r \in h(I)} GX_r^p$. This implies that $\{GX_r^p : r \in h(R)\}$ is a basis for the Zariski topology on $p.Spec_g(R)$. □

Proposition 3.6. *Let R be a G-graded ring. Then the followings hold for any r, $s \in h(R)$ and the open sets GX_r^p and GX_s^p.*

(i) $Gr(rR) = Gr(sR)$ *if and only if* $GX_r^p = GX_s^p$.

(ii) $GX_{rs}^p = GX_r^p \cap GX_s^p$.

(iii) $GX_r^p = \phi$ *if and only if r is a homogeneous nilpotent.*

(iv) GX_r^p *is quasi compact.*

Proof. (i) Suppose that $GX_r^p = GX_s^p$. Then, $p\text{-}V_R^g(rR) = p\text{-}V_R^g(sR)$. Let q be a graded prime ideal of R such that $rR \subseteq q$. Since q is a graded primary and $rR \subseteq q \subseteq Gr(q)$, we get $q \in p\text{-}V_R^g(rR) = p\text{-}V_R^g(sR)$. Then, $sR \subseteq, Gr(q)$. Since q is graded prime ideal, by [9, Proposition 1.2(4)], we get $Gr(q) = q$. Thus, $sR \subseteq q$. Hence, $Gr(sR) \subseteq Gr(rR)$. Similarly we can show that $Gr(rR) \subseteq Gr(sR)$. Therefore, $Gr(rR) = Gr(sR)$. Conversely, assume that $Gr(rR) = Gr(sR)$. Let $q \in p\text{-}V_R^g(rR)$. Then, $rR \subseteq Gr(q)$. Hence, $sR \subseteq Gr(sR) = Gr(rR) \subseteq Gr(q)$ by [9, Proposition 1.2]. Thus, $q \in p\text{-}V_R^g(sR)$, so $p\text{-}V_R^g(rR) \subseteq p\text{-}V_R^g(sR)$ and hence $GX_s^p \subseteq GX_r^p$. Similarly, we can show that $GX_r^p \subseteq GX_s^p$. Thus, $GX_r^p = GX_s^p$.

(ii) Let $q \in GX^p_r \cap GX^p_s$ for the open sets GX^p_r and GX^p_s. Then, $r \notin Gr(q)$ and $s \notin Gr(q)$. By Lemma 3.1, we get $rs \notin Gr(q)$. It follows that $q \in GX^p_{rs}$. Thus, $GX^p_r \cap GX^p_s \subseteq GX^p_{rs}$. For reverse inclusion, assume that $q \in GX^p_{rs}$. Then, $rs \notin Gr(q)$, namely $r \notin Gr(q)$ and $s \notin Gr(q)$. It follows that $q \in GX^p_r$ and $q \in GX^p_s$So, $GX^p_{rs} \subseteq GX^p_r \cap GX^p_s$.

(iii) Let $r \in h(R)$. Then, $GX^p_r = \phi$ if and only if $p\text{-}V^g_R(r) = p.Spec_g(R)$ if and only if $r \in q$ for all graded primary ideals q of R if and only if r belongs to the intersection of all graded primary ideals if and only if r belongs to the intersection of all graded prime ideals if and only if r belongs to the graded nilradical of R if and only if r is a homogeneous nilpotent.

(iv) Let $r \in h(R)$. Assume that $\{GX^p_{s_\alpha} : \alpha \in \Lambda\}$ is an open cover of GX^p_r, for each $\alpha \in \Lambda$ and $s_\alpha \in h(R)$. Then, $GX^p_r \subseteq \cup_{\alpha\in\Lambda} GX^p_{s_\alpha} = \cup_{\alpha\in\Lambda}\big(p.Spec_g(R)\backslash p\text{-}V^g_R(s_\alpha)\big) = p.Spec_g(R)\backslash \cap_{\alpha\in\Lambda}\, p\text{-}V^g_R(s_\alpha) = p.Spec_g(R)\backslash p\text{-}V^g_R(\cup_{\alpha\in\Lambda} s_\alpha)$, i.e., $p\text{-}V^g_R(\cup_{\alpha\in\Lambda} s_\alpha) \subseteq p\text{-}V^g_R(r) = p\text{-}V^g_R\big(Gr(rR)\big)$. So, $Gr(rR) \subseteq Gr(\cup_{\alpha\in\Lambda}\{s_\alpha\})$. Thus, $r^n \in (\cup_{\alpha\in\Lambda}\{s_\alpha\})$ for some $n \in \mathbb{N}$. There exists a finite subset $\Delta \subseteq \Lambda$ such that $r^n = \sum_{i\in\Delta} t_i s_i$, for any $t_i \in h(R)$ and $i \in \Delta$. Thus, $(rR)^n \subseteq (\{s_i : i \in \Delta\})$, that is, $p\text{-}V^g_R(\{s_i : i \in \Delta\}) \subseteq p\text{-}V^g_R(r^n) = p\text{-}V^g_R(r)$. Hence, $p\text{-}V^g_R\big(\sum_{i\in\Delta}(s_i)\big) = \cap_{i\in\Delta} p\text{-}V^g_R(s_i) \subseteq p\text{-}V^g_R(r)$. So, $p.Spec_g(R) - p\text{-}V^g_R(r) \subseteq p.Spec_g(R) - \cap_{i\in\Delta} p\text{-}V^g_R(s_i) = \cup_{i\in\Delta}\big(p.Spec_g(R) - p\text{-}V^g_R(s_i)\big) = \cup_{i\in\Delta} GX^p_{s_i}$. Thus, $GX^p_r \subseteq \cup_{i\in\Delta} GX^p_{s_i}$. Since Δ is finite, GX^p_r is a quasi compact. □

Corollary 3.7. *Let R be a G-graded ring. Then, $p.Spec_g(R)$ is quasi-compact.*

Proof. It can be seen directly from Proposition 3.6(iv). □

Definition 3.8. Let R be a G-graded ring. A family of graded ideals $\{P_\alpha\}_{\alpha\in\Lambda}$ satisfies condition (A) if for each $r_g \in h(R)$, there is $n \in \mathbb{N}$ such that for all $\alpha \in \Lambda$, if $r_g \in Gr(P_\alpha)$, then $r_g^n \in P_\alpha$.

Lemma 3.9. *Let R be a G-graded ring. Then, the following statements are equivalent:*

(i) *A family $\{P_\alpha\}_{\alpha\in\Lambda}$ of graded ideals in R satisfies condition (A).*

(ii) *For each (countable) subset $\Delta \subset \Lambda$, $Gr(\cap_{\alpha\in\Delta} P_\alpha) = \cap_{\alpha\in\Delta} Gr(P_\alpha)$.*

Proof.
(i)⇒(ii) Assume that $\{P_\alpha\}_{\alpha\in\Lambda}$ satisfies condition (A) and $r_g \in \cap_{\alpha\in\Delta} Gr(P_\alpha) \cap h(R)$, then there exists $n \in \mathbb{N}$ such that $r_g^n \in P_\alpha$ for each $\alpha \in \Delta$. So, $r_g \in Gr(\cap_{\alpha\in\Delta} P_\alpha)$. This yields that $\cap_{\alpha\in\Delta} Gr(P_\alpha) \subseteq Gr(\cap_{\alpha\in\Delta} P_\alpha)$. The other inclusion always holds.

(ii)⇒(i) Given $r_g \in h(R)$, let $\Delta = \{\alpha \in \Lambda : r_g \in Gr(P_\alpha)\}$. Then, the displayed equation says that there is n such that $r_g^n \in P_\alpha$ for all $\alpha \in \Delta$. Finally, we note that if the displayed equation fails for some subset $\Delta \subset \Lambda$, then it fails for a countable subset of Δ. Indeed, if $r_g \notin Gr(\cap_{\alpha\in\Delta} P_\alpha)$, then for each n there exists $\alpha_n \in \Delta$ such that $r_g^n \notin P_{\alpha_n}$, hence $r_g \notin Gr(\cap_{n=1}^{\infty} P_{\alpha_n})$. □

Lemma 3.10 ([5]). *Let R be a G-graded ring and M a graded R-module. Then, R has at least one graded maximal ideal. In particular, if I is a proper graded ideal of R, then there exists a graded maximal ideal Q of R with $I \subseteq Q$.*

Recall that the dimension of a graded ring R denoted by $dim_g(R)$ is defined to be : $\sup\{n \in \{0, 1, 2, \ldots\}$: there exists a strict chain of graded prime ideals of R of the length n $\}$.

Lemma 3.11. *Let R be a G-graded ring. Then, the following statements are equivalent:*

(i) $dim_g(R) = 0$.

(ii) $Rr_g + \cup_{n=1}^{\infty}(0 :_R r_g^n) = R$ *for every* $r_g \in h(R)$.

(iii) *For every* $r_g \in h(R)$, *there exists* $n \in \mathbb{N}$ *such that* $Rr_g^n = Rr_g^{n+1}$.

Proof.
(i)⇒ (ii) Assume that $dim_g(R) = 0$ and $Rr_g + \cup_{n=1}^{\infty}(0 :_R r_g^n)$ is a proper graded ideal of R. Then, it is contained in a graded prime ideal P of R by Lemma 3.10. Look at the multiplicative system $U = \{r_g^n s_h : n = 0, 1, 2, \ldots$ and $s_h \in h(R)\backslash P\}$. Now, $0 \notin U$ because $s_h \notin P$ and $\cup_{n=1}^{\infty}(0 :_R r_g^n) \subseteq P$. So, there is a graded prime ideal P' of R that misses U. Since $U \supseteq h(R)\backslash P$, we have $P' \subseteq P$. Moreover, $r_g \in P\backslash P'$, so $P' \neq P$. Thus, P' is not graded maximal, i.e., $dim_g(R) \neq 0$, which is a contradiction.

(ii)⇒ (i) We prove the contrapositive of the statement. Suppose there exist distinct graded prime ideals $P' \subset P$ in R and let $r_g \in P \cap h(R)\backslash P'$. Then, $\cup_{n=1}^{\infty}(0 :_R r_g^n) \subset P'$ and $Rr_g \subset P$. So, $Rr_g + \cup_{n=1}^{\infty}(0 :_R r_g^n) \subset P$, i.e., $Rr_g + \cup_{n=1}^{\infty}(0 :_R r_g^n)$ is a proper graded ideal of R.

(ii)⇒(iii) If r_g satisfies (ii), then $t_h r_g + s_\lambda = 1$ where $s_\lambda r_g^n = 0$ and $t_h, s_\lambda \in h(R)$. So, $t_h r_g^{n+1} = r_g^n$. This yields that $Rr_g^n = Rr_g^{n+1}$.

(iii)⇒(ii) If r_g satisfies (iii), then there exists $t_h \in h(R)$ such that $r_g^n = t_h r_g^{n+1}$, hence $r_g^n(1 - t_h r_g) = 0$. It follows that $1 - t_h r_g \in (0 : r_g^n)$ and hence $1 \in Rr_g + \cup_{n=1}^{\infty}(0 :_R r_g^n)$. Thus, $R = Rr_g + \cup_{n=1}^{\infty}(0 :_R r_g^n)$. □

Proposition 3.12. *Let R be a G-graded ring. Then, the following statements are equivalent:*

(i) $dim_g(R) = 0$.

(ii) *Condition* (A) *holds for the family of all graded ideals of* R.

(iii) *Condition* (A) *holds for the family of all graded primary ideals of* R.

P r o o f.
(i)⇒ (ii) Suppose that $dim_g(R) = 0$ and $r_g \in h(R)$. By Lemma 3.11, there exists $n \in \mathbb{N}$ such that $Rr_g^n = Rr_g^{n+1}$. If I is any graded ideal and $r_g \in Gr(I)$, then $r_g^m \in I$ for some m, hence $r_g^n \in I$ also. Thus, (A) holds for the family of all graded ideals of R.

(ii)⇒ (iii) Clear.

(iii)⇒ (i) Suppose that $dim_g(R) > 0$. Let I be a graded prime ideal of R that is not graded maximal. The graded primary ideals of R/I are in one-to-one correspondence with the graded primary ideals of R that contain I, and this correspondence respects intersections and graded radicals, so we may assume that R is a graded integral domain. Let r_g be a nonzero nonunit of $h(R)$ and let P be a graded minimal prime of the graded ideal Rr_g. For each positive integer n, let $P'_n = R_P r_g^n \cap R = \{r \in R : s_h r \in Rr_g^n$ for some $s_h \in h(R)\backslash P\}$. Each P'_n is graded P-primary, hence $\cap_{n=1}^{\infty} Gr(P'_n) = P$. To show that $r_g \notin Gr(\cap_{n=1}^{\infty} P'_n)$, it suffices to show that $r_g^{n-1} \notin P'_n$. Suppose, by way of contradiction, that $r_g^{n-1} \in P'_n$. Then, $s_h r_g^{n-1} \in Rr_g^n$ for some $s_h \in h(R)\backslash P$, so $s_h \in Rr_g \subset P$ because R is a graded integral domain, a contradiction. Thus, $r_g \notin Gr(\cap_{n=1}^{\infty} P'_n)$. This yields that $Gr(\cap_{n=1}^{\infty} P'_n) \neq Gr(\cap_{n=1}^{\infty} P'_n)$. By Lemma 3.9, we get a contradiction. Therefore, $dim_g(R) = 0$. □

Let R be a G-graded ring and let $p.Spec_g(R)$ be endowed with the Zariski-topology. Let W be a subset of $p.Spec_g(R)$. We will denote $\cap_{P \in Y} P$ by $\Im(W)$ and the closure of W in $p.Spec_g(R)$ by $Cl(W)$.

Proposition 3.13. *Let R be a G-graded ring with $dim_g(R) = 0$ and $Y \subseteq p.Spec_g(R)$. Then, $p\text{-}V_R^g\big(\Im(W)\big) = Cl(W)$. Hence, W is closed if and only if $p\text{-}V_R^g\big(\Im(W)\big) = W$.*

P r o o f. Let $q \in W$. Then, $\Im(W) \subseteq q \subseteq Gr(q)$, it follows that $q \in p\text{-}V_R^g\big(\Im(W)\big)$. Thus, $Y \subseteq p\text{-}V_R^g\big(\Im(W)\big)$. This yields that $Cl(W) \subseteq p\text{-}V_R^g\big(\Im(W)\big)$. For the reverse inclusion, let $p\text{-}V_R^g(I)$ be a closed subset of $p.Spec_g(R)$ including W. Hence, $I \subseteq Gr(q)$ for all $q \in W$. Then, $I \subseteq Gr\big(\Im(W)\big)$ by Proposition 3.12. Let $q' \in p\text{-}V_R^g\big(\Im(W)\big)$. It follows that $\Im(W) \subseteq Gr(q')$. Hence, $I \subseteq Gr\big(\Im(W)\big) \subseteq Gr(q')$, and so $q' \in p\text{-}V_R^g(I)$, that is, $p\text{-}V_R^g\big(\Im(W)\big)$ is the smallest closed subset of $p.Spec_g(R)$ which includes W. □

Proposition 3.14. *Let R be a G-graded ring and $q \in p.Spec_g(R)$. Then, the followings hold:*

(i) *$Cl(\{q\}) = p\text{-}V_R^g(q)$.*

(ii) *$I \in Cl(\{q\})$ if and only if $q \subseteq Gr(I)$ for any $I \in p.Spec_g(R)$.*

P r o o f.
(i) Let $W = \{q\}$. Then $Cl(\{q\}) = p\text{-}V_R^g(q)$ by Proposition 3.13.
(ii) It is an immediate consequence of (i). $\square$

Proposition 3.15. *Let R be a G-graded ring. Then, $p.Spec_g(R)$ is a T_0-space if and only if for any two graded ideals q_1 and q_2 in $p.Spec_g(R)$, $p\text{-}V_R^g(q_1) = p\text{-}V_R^g(q_2)$ implies that $q_1 = q_2$.*

P r o o f. Let q_1, $q_2 \in p.Spec_g(R)$. By Proposition 3.14, $Cl(\{q_1\}) = Cl(\{q_2\})$ if and only if $p\text{-}V_R^g(q_1) = p\text{-}V_R^g(q_2)$ if and only if $q_1 = q_2$. Now, by the fact that a topological space is a T_0-space if and only if the closures of distinct points are distinct, we conclude that for any graded R-module M, $p.Spec_g(R)$ is a T_0-space. $\square$

Lemma 3.16. *Let R be a G-graded ring. If every graded prime ideal is graded maximal, that is, $dim_g(R) = 0$, then $Spec_g(R)$ is a T_2-space.*

P r o o f. If $\mid Spec_g(R) \mid = 1$ or $\mid Spec_g(R) \mid = 2$, then $Spec_g(R)$ is a T_2-space. Now, assume that $\mid Spec_g(R) \mid > 2$. Then, we can take three distinct elements in $Spec_g(R)$, say p_1, p_2, and p_3. Since every graded prime ideal is graded maximal,

$$V_R^g(p_1) = \{p_1\}, \quad V_R^g(p_3) = \{p_3\},$$

$$V_R^g(p_1p_3) = V_R^g(p_1) \cup V_R^g(p_3) = \{p_1, p_3\} = Spec_g(R) - V_R^g(p_2),$$

$$V_R^g(p_2p_3) = V_R^g(p_2) \cup V_R^g(p_3) = \{p_2, p_3\} = Spec_g(R) - V_R^g(p_1)$$

and

$$V_R^g(p_2) = \{p_2\} = Spec_g(R) - V_R^g(p_1p_3)$$

are open sets in $Spec_g(R)$. This implies that

$$p_1 \in V_R^g(p_1p_3) \quad \text{and} \quad p_2 \in V_R^g(p_2).$$

Moreover,

$$V_R^g(p_1p_3) \cap V_R^g(p_2) = \phi.$$

$\square$

Proposition 3.17. *Let R be a G-graded ring. Then, the following statements are equivalent:*

(i) *Every graded primary ideal is a graded maximal ideal in R.*

(ii) *$p.Spec_g(R)$ is a T_2-space.*

(iii) *$p.Spec_g(R)$ is a T_1-space.*

(iv) *$p.Spec_g(R)$ is a T_0-space.*

Proof.

(i)⇒ (ii) Assume that every graded primary ideal is a graded maximal ideal in R. Since every graded maximal ideal is graded prime, we get $Spec_g(R)$ coincides with $p.Spec_g(R)$. Since, R is a graded zero dimensional ring. By Lemma 3.16, $Spec_g(R)$ is a T_2-space, and so is $p.Spec_g(R)$.

(ii)⇒(iii)⇒(iv) Clear.

(iv)⇒(i) Let $p.Spec_g(R)$ be a T_0- space and $q \in p.Spec_g(R)$. Then, we have $Cl(\{q\}) = p\text{-}V_R^g(q) = p\text{-}V_R^g\big(Gr(q)\big) = Cl\big(\{Gr(q)\}\big)$. Then, $q = Gr(q)$ by Proposition 3.15. Hence q is a graded prime ideal by Lemma 3.1. □

Definition 3.18. Let R be a G-graded ring. The graded Zariski primary radical of a graded ideal I of R, denoted by $Zp\text{-}Gr(I)$, is the intersection of all members of $p\text{-}V_R^g(I)$ for the Zariski topology, that is, $Zp\text{-}Gr(I) = \cap_{q \in p\text{-}V_R^g(I)} q = \cap\{q \in p.Spec_g(R) \mid I \subseteq Gr(q)\}$. We say, a graded ideal I is a Z_P-radical ideal if $I = Zp\text{-}Gr(I)$.

A topological space X is said to be Noetherian if the open subsets of X satisfy the ascending chain condition (or if every descending chain of closed subsets is stationary (see [2]).

Proposition 3.19. *Let R be a G-graded ring with $dim_g(R) = 0$. Then, R has Noetherian graded primary spectrum if and only if the ACC for the graded Zariski primary radical ideals of R holds.*

Proof. Suppose the ACC holds for the graded Zariski primary radical ideals of R.Let $p\text{-}V_R^g(I_1) \supseteq p\text{-}V_R^g(I_2) \supseteq \dots$ be a descending chain of closed subsets $p\text{-}V_R^g(I_i)$ of $p.Spec_g(R)$, where I_i is a graded ideal of R. Then, $\Im\big(p\text{-}V_R^g(I_1)\big) = Zp\text{-}Gr(I_1) \subseteq \Im\big(p\text{-}V_R^g(I_2)\big) = Zp\text{-}Gr(I_2) \subseteq \dots$ is an ascending chain of graded Zariski primary radical ideals of R. So, by assumption, there exists $n \in \mathbb{N}$ such that for all $i \in \mathbb{N}$, $Zp\text{-}Gr(I_n) = Zp\text{-}Gr(I_{n+i})$. Now, by Proposition 3.13, $p\text{-}V_R^g(I_n) = p\text{-}V_R^g\big(Zp\text{-}Gr(I_n)\big) = V_R^g\big(Zp\text{-}Gr(I_{n+i})\big) = p\text{-}V_R^g(I_{n+i})$. Thus, R has Noetherian graded primary spectrum. Conversely, suppose that R has a Noetherian graded primary spectrum. Let $I_1 \subseteq I_2 \subseteq \dots$ be an ascending chain of the graded Zariski primary radical ideals of R.Thus, $p\text{-}V_R^g(I_1) \supseteq p\text{-}V_R^g(I_1) \supseteq \dots$ is a descending chain of closed subsets $p\text{-}V_R^g(I_i)$ of $p.Spec_g(R)$. By assumption, there is $n \in \mathbb{N}$ such that for all $i \in \mathbb{N}$, $p\text{-}V_R^g(I_n) = p\text{-}V_R^g(I_{n+i})$. Therefore, $I_n = Zp\text{-}Gr(I_n) = \Im\big(p\text{-}V_R^g(I_n)\big) = \Im\big(p\text{-}V_R^g(I_{n+i})\big) = Zp\text{-}Gr(I_{n+i}) = I_{n+i}$. Therefore, the ACC for the graded Zariski primary radical ideals of R holds. □

REFERENCES

[1] AL-ZOUBI, K.—QARQAZ, F.: *An intersection condition for graded prime ideals*, Boll. Unione Mat. Ital. **11** (2018), 483–488.

[2] MUNKRES, J. R.: *Topology*. A a first course, Prentice-Hall, Inc. XVI, Englewood Cliffs, New Jersey, 1975.

[3] NĂSTĂSESCU, C.—VAN OYSTAEYEN, F.: *Graded and Filtered Rings and Modules*. In: *Lecture Notes in Math., Vol. 758*, Springer-Verlag, Berlin, 1979.

[4] ______ *Graded Ring Theory*. In: textslMathematical Library, Vol. 28, North-Holand Publishing Co., Amsterdam-New York, 1982.

[5] ______ *Methods of Graded Rings*. In: *Lecture Notes in Math., Vol. 1836*, Springer-Verlag, Berlin, 2004.

[6] ÖZKIRIŞCI, N. A.—KILIÇ, Z.—KOÇ, S.: *A note on primary spectrum over commutative rings*, An. Ştiinţ. Univ. Al. I. Cuza Iaşi. Mat. (N.S.) **64** (2018), no. 1, 111–119.

[7] OZKIRISCI, N. A.—ORAL, K. H.—TEKIR, U.: *Graded prime spectrum of a graded module*, Iran J. Sci. Technol. **37A3** (2013), 411–420.

[8] REFAI, M.: *On properties of G-spec*(R), Sci. Math. Jpn. **53** (2001), no. 3,411–415.

[9] REFAI, M.—AL-ZOUBI, K.: *On graded primary ideals*, Turkish. J. Math. **28** (2004), no. 3, 217–229.

[10] REFAI, M.—HAILAT, M.—OBIEDAT, S.: *Graded radicals on graded prime spectra*, Far East J. of Math. Sci. (FJMS) Spec. Vol., Part I, (2000), 59–73.

[11] UREGEN, R. N.—TEKIR, U.—ORAL, K. H.: *On the union of graded prime ideals*, **14**, no. 1, 114–118; https://doi.org/10.1515/phys-2016-0011.

Received February 28, 2019

Khaldoun Al-Zoubi
Department of Mathematics and Statistics
Jordan University of Science and Technology
P.O.Box 3030
Irbid 22110
JORDAN
E-mail: kfzoubi@just.edu.jo

Malik Jaradat
Department of Mathematics
The international School of Choueifat
(MHS-AlDaid)
P.O.Box 66973
Alain
UAE
E-mail: mjaradat@mhs.sabis.net