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GENERALIZED DERIVATIVE AND GENERALIZED CONTINUITY

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ABSTRACT. The main objective of this article is to show that generalized differentiation can be understood as a process of comparing functions and their generalized continuity properties. We show it by working with generalized notions of derivative and continuity. The article covers wide range of types of generalized continuity.

1. Introduction

The fact that a differentiable function is continuous has more topological essence than usually believed. We would like to show this in this article by working with generalized notions of derivative and continuity. We examine generalized differentiation from a topological point of view - as a possibility to compare functions and their continuity properties. We work with the notion of S-continuity [13] that covers many types of weak, generalized continuity.

Let us note that the generalized differentiation is currently developing by several mathematicians in the frame of the theory of arbitrary metric spaces. See, for example, [1], [3], [8], [20]. The differentiation theory in linear topological spaces is a well-known part of analysis. Nonetheless, it seems that there were not any attempts to introduce a differentiation in topological spaces without linear or metric structures before [11] was published.

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2. Generalized derivative

In this paper, when we say "a field", we mean the spaces $\mathbb{R}$ or $\mathbb{C}$ equipped with their natural topologies. But very often we could work with a general "topological field", a field on which the operations of addition, subtraction, multiplication and division would be continuous. Let X be a topological space, Y be a set. A function $g\colon X \to Y$ is discrete on $A \subset X$ at a point $a \in X$, if there is an open neighborhood V of a such that the statement

$$g(a) \notin g\Big(\Big(V \bigcap A\Big)\backslash\{a\}\Big) \quad \textit{holds.}$$

Now, we define the notion of generalized derivative.

Definition 2.1 ([11]). Let X be a topological space, $A \subset X$ and Y be a linear topological space defined over a field F. Let p be a limit point of A and $g : X \to F$ be discrete at p on A. A function $f\colon X \to Y$ has a g-derivative $l \in Y$ at p on A if for every net $\{x_\gamma\}_{\gamma\in\Gamma}$ of points $x_\gamma \in A\backslash\{p\}$ converging to p, the net

$$\left\{\frac{f(x_\gamma) - f(p)}{g(x_\gamma) - g(p)}\right\}_{\gamma\in\Gamma} \quad \text{converges to } l.$$

If l is a g-derivative of f at p on A, then we write

$$l =_{g/A} f'(p) = \lim_{\substack{x\to p,\\ x\in A}} \frac{f(x) - f(p)}{g(x) - g(p)}$$

and

$${}_gf'(p) =_{g/X} f'(p) \quad \text{for} \quad A = X.$$

It is easy to see that, for Hausdorff spaces Y, a g-derivative ${}_gf'(p)$, if it exists, is unique (see [11]).

Remark 2.2. In what follows, if we say "let ${}_{g/A}f'(a)$ exist", we automatically suppose that a is a limit point of A. Moreover, we see that if ${}_{g/A}f'(a)$ exists, then g is discrete on A at the point a. It is easy to see that our generalized derivative is a linear operator. If we put

$$X = A = Y = F = \mathbb{R} \quad \text{and} \quad g(x) \equiv x,$$

we obtain the well-known definition of the derivative of a real valued function. Functions differentiable in this generalized sense need not be continuous. For example, if $g\colon \mathbb{R} \to \mathbb{R}$ is an injective noncontinous function and $f = g$ then, for every a from $\mathbb{R}$, we have ${}_gf'(a) = 1$, so ${}_gf'(a)$ exists. But f is not continuous.

3. S-continuity

The notions defined below were studied, for example by Borsík [4], [5], Neubrunn [22], Njåstad [23] and Piotrowski [25].

Definition 3.1. Let (X,T) be topological space. We say that a set $V \subset X$ is α-open, if and only if there exist an open set $O \in T$ and a nowhere dense set S such that $V = O \setminus S$. The system of all α-open sets in (X,T) is denoted by T_α. It is known that T_α defines a new topology on X.

Let (Y,τ) be a topological space. Let x be from X. We say that a function

$$f\colon (X,T) \to (Y,\tau)$$

is α-continuous at x if for each $W \in \tau$ such that $f(x) \in W$ there exists a $V \in T_\alpha$ such that $x \in V$ and $f(V) \subset W$ is true.

Let X, Y be topological spaces. A function $f\colon X \to Y$ is said to be quasicontinuous at x from X if and only if, for any open set V such that $f(x) \in V$ and any open set U such that $x \in U$, there exists a nonempty open set $O \subset U$ such that $f(O) \subset V$.

Let $X = \mathbb{R}$, let Y be a topological space. A function $f\colon X \to Y$ is said to be left (right) hand sided quasicontinuous at a point x from $\mathbb{R}$ if for every $\delta > 0$ and for every open neighborhood V of $f(x)$ there exists an open nonempty set $W \subset (x-\delta, x)$ ($W \subset (x, x+\delta)$) such that $f(W) \subset V$. A function f is bilaterally quasicontinuous at x if it is both left and right hand sided quasicontinuous at this point.

The generalized type of continuity mentioned in the following definition was called "$\mathbb{A}$-continuity" in [13]. This definition has roots in some works of M. Matejdes, see, e.g., [18]. But since the name "$\mathbb{A}$-continuity" was used in another meaning in [17], we prefer - and introduce it here - the name "$\mathcal{S}$-continuity" for this notion.

Definition 3.2 (according to [13])**.** Let X, Y be topological spaces. Let $\mathcal{S}$ be a system of nonempty subsets of X. A function $f\colon X \to Y$ is said to be $\mathcal{S}$-continuous at x from X if and only if, for any open set V such that $f(x) \in V$ and any open set U such that $x \in U$, there exists a set S from $\mathcal{S}$ such that $S \subset U$ and $f(S) \subset V$.

Remark 3.3. Because of the generality of $\mathcal{S}$-continuity, we have just characterized a wide range of systems of functions. Concretely, if X, Y are topological spaces, $\mathcal{S}$ a system of nonempty subsets of X, and if a function

$$f\colon X \to Y \quad \text{is } \mathcal{S}\text{-continuous at a point } x,$$

then it is

(1) continuous at x if $S = \{U; U \text{ is open in } X \text{ and } x \in U\}$,

(2) α-continuous at x if $S = \{O; O \text{ is } \alpha\text{-open in } X \text{ and } x \in O\}$,

(3) quasicontinuous at x if $S = \{U; U \text{ is open in } X\}$.

Suppose, moreover, that $X = \mathbb{R}$. Then, f is

(4) left (right) hand sided quasicontinuous at x if

$$S = \big\{V; V = (a,b) \quad \text{and} \quad a < b < x\big\}$$
$$\big(S = \{V; V = (a,b) \quad \text{and} \quad x < a < b\}\big),$$

(5) bilaterally quasicontinuous at x if

$$S = \big\{V; V = (a,b) \cup (c,d) \quad \text{and} \quad a < b < x < c < d\big\}.$$

Now, we introduce two notions of similarity of functions (defined in [12]). Here, we are using the terminology used in [13].

Definition 3.4. Let X, Y, Z be topological spaces, let $g\colon X \to Y$, $f\colon X \to Z$ be functions. Let x be from X. We say that f is g-continuous at x if, for every net $\{x_\gamma\}_{\gamma\in\Gamma}$ of elements from X converging to x, the following holds.

(*) If the net

$$\big\{g(x_\gamma)\big\}_{\gamma\in\Gamma} \quad \text{converges in } Y,$$

then the net

$$\big\{f(x_\gamma)\big\}_{\gamma\in\Gamma} \quad \text{converges in } Z.$$

Let A be a subset of X. We say that f is g-continuous on A if, for every x from A, f is g-continuous at x. If $A = X$, we say that f is g-continuous. (The g-continuity of f does not guarantee that f will automatically have all nice properties of g. It must be always examined, whether a property of g is inherited also by f.)

4. Generalized derivative and topological properties of functions

In what follows, we will use the following theorem proved in [13].

Theorem 4.1. *Let X, Y, Z be Hausdorff topological spaces. Let $g\colon X \to Y$, $f\colon X \to Z$ be functions. Let x be a point from X. Let $\mathcal{S}$ be a system of nonempty subsets of X and let g be $\mathcal{S}$-continuous at x. If f is g-continuous at x, then f is $\mathcal{S}$-continuous at x, too.*

The following theorem works with the notions and results presented above and it shows how the properties of general differentiability and general continuity are connected.

Theorem 4.2. *Let X be a Hausdorff topological space. Let F be a field and Y be a linear topological space defined over the field F. Let $f\colon X \to Y$, $g\colon X \to F$ be functions. Let $p \in X$, let g be S-continuous at p. If f has a g-derivative at p, then f is S-continuous at p, too.*

Proof. Let $\{x_\gamma\}_{\gamma\in\Gamma}$ be a net of elements from X converging to p such that the net $\{g(x_\gamma)\}_{\gamma\in\Gamma}$ converges in F. Then for each $\gamma \in \Gamma$ the following holds

$$\begin{aligned} f(x_\gamma) &= \big(f(x_\gamma) - f(p)\big) + f(p) \\ &= \frac{f(x_\gamma) - f(p)}{g(x_\gamma) - g(p)}\big(g(x_\gamma) - g(p)\big) + f(p). \end{aligned} \tag{1}$$

Since the net

$$\frac{f(x_\gamma) - f(p)}{g(x_\gamma) - g(p)} \quad \text{converges to} \quad {}_g f'(p)$$

and the net

$$g(x_\gamma) \quad \text{converges to an element} \quad z \in F,$$

then (because of (1)) the net

$$\big\{f(x_\gamma)\big\}_{\gamma\in\Gamma} \quad \text{converges to} \quad {}_g f'(p)\big(z - g(p)\big) + f(p).$$

We have just proved that f is g-continuous at p. Since g is S-continuous at p, according to the preceding theorem, we have just proved that f is S-continuous at p, too. □

As the previous theorem showed, the existence of the generalized derivative ${}_g f'(p)$ implies that f is g-continuous at p. We will use this fact and obtain some new results concerning the functions from the first Baire class (see, e.g., [15]) and the functions from the classes B_1^* and B_1^{**} (see [21], [24])). We say that a function $f\colon \mathbb{R} \to \mathbb{R}$ belongs to the class B_1^* if for every closed set C there is an open interval

$$I = (a, b) \quad \text{with} \quad I \bigcap C \neq \emptyset$$

such that the restriction f_C is continuous on I ([21]). We say that a function $f\colon X \to Y$ (where X and Y are topological spaces) belongs to the class B_1^{**} if either $D(f) = \emptyset$ or $f_{D(f)}$ is a continuous function - where $D(f)$ denotes the set of all discontinuity points of f [24].

In what follows, we will use the assertions of the following two theorems proven in [14].

Theorem 4.3. *Let X, Y, Z be complete separable metric spaces, let $g\colon X \to Y$, $f\colon X \to Z$ be functions. If g is of the first Baire class and f is g-continuous on X, then f is of the first Baire class, too.*

Theorem 4.4. *Let X, Y, Z be topological spaces. Let $g\colon X \to Y$ belong to the class B_1^{**} (let $g\colon \mathbb{R} \to \mathbb{R}$ belong to the class B_1^*) and let $f\colon X \to Z$ ($f\colon \mathbb{R} \to \mathbb{R}$) be g-continuous on X. Then, f belongs to the class B_1^{**} (then f belongs to the class B_1^*), too.*

Our final two theorems show how the topological property "being from certain class" is transferred from a function to a function, if these functions are connected by the existence of a generalized derivative.

Theorem 4.5. *Let X be a Hausdorff topological space, Y be a Banach space. Let $f\colon X \to Y$, $g\colon X \to R$ be functions. Let g be of the first Baire class. If for each x from X there exists a generalized derivative ${}_gf'(x)$, then f is of the first Baire class.*

Proof. This is a direct consequence of Theorem 4.3 and of the fact that f is g-continuous (see the proof of Theorem 4.2). □

In a similar way, when we combine Theorem 4.4 and a part of the proof of Theorem 4.2, we obtain this assertion concerning the classes B_1^* and B_1^{**}:

Theorem 4.6.

(a) *Let X be a Hausdorff topological space. Let F be a field and Y be a linear topological space defined over the field F. Let $f\colon X \to Y$, $g\colon X \to F$ be functions. Let g be of the class B_1^{**}. If f has a g-derivative at each x from $\mathbb{R}$, then f belongs to the class B_1^{**}, too.*

(b) *Let $f\colon \mathbb{R} \to \mathbb{R}$, $g\colon \mathbb{R} \to \mathbb{R}$ be functions, let g be of the class B_1^*. If f has a g-derivative at each x from $\mathbb{R}$, then f belongs to the class B_1^*, too.*

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