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ON FUNCTIONS OF BOUNDED (φ, k)-VARIATION

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ABSTRACT. Given a φ-function φ and $k \in \mathbb{N}$, we introduce and study the concept of (φ, k)-variation in the sense of Riesz of a real function on a compact interval. We show that a function $u: [a, b] \to \mathbb{R}$ has a bounded (φ, k)-variation if and only if $u^{(k-1)}$ is absolutely continuous on $[a, b]$ and $u^{(k)}$ belongs to the Orlicz class $L_\varphi[a, b]$. We also show that the space generated by this class of functions is a Banach space. Our approach simultaneously generalizes the concepts of the Riesz φ-variation, the de la Vallée Poussin second-variation and the Popoviciu kth variation.

1. Introduction

In 1807, J. Fourier ([4]) formulated the following conjecture: Every function (what was meant by function at that time) admits an expansion into what is called today a Fourier series. In 1829, Dirichlet [2] proved the validity of Fourier's conjecture for monotone functions. In 1881, C. Jordan [6], in a critical study of Dirichlet's work, extracted the notion of function of *bounded variation* ($BV[a, b]$) proving that a function $u: [a, b] \to \mathbb{R}$ has a bounded variation if and only if it can be written as a difference of monotone functions. As a consequence, he concluded that for such functions Fourier's conjecture holds. These important facts motivated the generalizations of notion of bounded variation in many ways. For example, in 1910, F. Riesz [15] introduced the notion of *p-bounded variation* $RV_p[a, b]$, for $p \in (1, \infty)$ and proved that a function

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$u\colon [a,b] \to \mathbb{R}$ has a p-bounded variation if and only if u is absolutely continuous on $[a,b]$ ($u \in AC[a,b]$), and $u' \in L_p[a,b]$. Moreover, the formula

$$V_p(u,[a,b]) = \int_a^b |u'(t)|^p \,\mathrm{d}t$$

holds, that is known today as the characterization of Riesz for functions of p--bounded variation.

In 1953, this result was further generalized by Yu. T. Medvedev [10] for the class of φ-variation functions $V^R_{(\varphi,1)}[a,b]$ showing that $u \in V^R_{(\varphi,1)}[a,b]$ if and only if u is absolutely continuous and $u' \in L_\varphi[a,b]$. Also,

$$V^R_{(\varphi,1)}(u) = \int_a^b \varphi\left(|u'(t)|\right) \mathrm{d}t.$$

Previously, in 1908, de la Vallée Poussin [3] introduced the class of functions of bounded second variation $BV_2[a,b]$; here the following results are known:

- u belongs to $BV_2[a,b]$ if and only if u is the difference of two convex functions;
- u belongs to $BV_2[a,b]$ if, only if, u is the indefinite integral of a function of bounded variation.

Combining the notion of p-variation in the sense of Riesz with the second variation in the sense of de la Vallée Poussin, N. Merentes in 1992 [11] obtained a new notion of variation ($RV_{(\varphi,2)}[a,b]$) and showed that $u \in RV_{(\varphi,2)}[a,b]$ if and only if u' is absolutely continuous on $[a,b]$, $u' \in L_p[a,b]$, and

$$V^R_{(\varphi,2)}(u) = \int_a^b \left(|u''(t)|\right)^p \mathrm{d}t.$$

M. T. Popoviciu in 1934 [14] extended the notion of second variation to the case of kth variation for $k > 2$ ($BV^k[a,b]$). Subsequently, this notion has been studied by A. M. Russell [17] in detail, and by M. Wróbel [19].

Recently, in 2010, the authors [12] combined the notion of p-variation ($1 < p < \infty$) in the sense of Riesz with the k-variation in the sense of Popoviciu introducing the new notion of (p,k)-variation in the sense of Riesz-de la Vallée Poussin-Popoviciu. They proved that u has a bounded (p,k)-variation on $[a,b]$ if and only if $u^{(k)}$ is absolutely continuous, $u^{(k)} \in L_p[a,b]$ and

$$V^R_{(p,k)}(u) = \int_a^b \left(\frac{\left|u^{(k)}(t)\right|}{(k-1)!}\right)^p \mathrm{d}t.$$

In the present paper, we combine the notion of φ-variation in Riesz's sense with the k-variation in the sense of Popoviciu to obtain a new general notion called (φ, k)-variation in the sense of Riesz-Popoviciu ($\hat{V}^R_{(\varphi,k)}[a,b]$). In particular, we prove that

if φ is a convex φ-function satisfying the ∞_1 condition and k is a positive integer, then $u \in \hat{V}^R_{(\varphi,k)}[a,b]$ if and only if $u^{(k-1)}$ is absolutely continuous on $[a,b]$,

$$\frac{u^{(k)}}{(k-1)!} \in L_\varphi[a,b]$$

and

$$\hat{V}^R_{(\varphi,k)}\left(u,[a,b]\right) = \int_a^b \varphi\left(\left|\frac{u^{(k)}(t)}{(k-1)!}\right|\right) \mathrm{d}t.$$

This result is stated and proved as Theorem 3.1 below.

2. Some properties of bounded (φ, k)-variation functions

We start with some definitions and known results concerning the Riesz φ-variation, the de la Vallée Poussin second-variation and the Popoviciu kth variation.

By a *φ-function* we mean here a nondecreasing continuous function

$$\varphi\colon\ [0,\infty)\to[0,\infty)$$

such that

$$\big(\varphi(t)=0 \quad\Longleftrightarrow\quad t=0\big) \qquad \text{and} \qquad \lim_{t\to\infty}\varphi(t)=\infty.$$

Remark 2.1 ([18, p. 80])**.** If φ is a convex φ-function, then φ is superadditive and, consequently,

$$\varphi(\lambda t) \le \lambda\varphi(t), \qquad \lambda\in[0,1], \qquad t\ge 0. \tag{1}$$

Definition 2.1. Let φ be a φ-function, $u\colon [a,b]\to\mathbb{R}$ and let $\mathcal{P}\colon a\le t_1<\cdots< t_n\le b$ be a partition of the interval $[a,b]$. Consider the expression

$$\sigma^R_{(\varphi,1)} := \sum_{j=1}^{n-1}\varphi\left(\frac{|u(t_{j+1})-u(t_j)|}{|t_{j+1}-t_j|}\right)|t_{j+1}-t_j|\,.$$

The number

$$V^R_{(\varphi,1)}(u,[a,b]) := \sup_{\mathcal{P}} \sigma^R_{(\varphi,1)},$$

where the supremum is taken over all partitions $\mathcal{P}$ of $[a,b]$, is called the Riesz $(\varphi,1)$-*variation of* u on $[a,b]$. If $V^R_{(\varphi,1)}(u,[a,b]) < \infty$, then we say that the function u has a bounded $(\varphi,1)$-*variation.*

The class of all $(\varphi,1)$-variation functions is denoted by $V^R_{(\varphi,1)}[a,b]$ and the vector space generated by this class is denoted by $RV_{(\varphi,1)}[a,b]$. This vector space $RV_{(\varphi,1)}[a,b]$ equiped with the norm

$$\|u\|^R_{(\varphi,1)} := |u(a)| + \inf\left\{\lambda > 0 : V^R_{(\varphi,1)}\left(\frac{u}{\lambda},[a,b]\right) \leq 1\right\},$$

has a structure of a Banach space.

In [9], it is shown that if φ is a convex φ-function such that $\lim_{t\to\infty} \frac{\varphi(t)}{t}$ is finite, then $RV_{(\varphi,1)}[a,b] = BV[a,b]$. In this way, it is necessary to assume the additional condition for the function φ,

$$\lim_{t\to\infty} \frac{\varphi(t)}{t} = \infty, \tag{2}$$

which we call the ∞_1 condition.

Given a φ-function φ, the set

$$L_\varphi[a,b] = \left\{ u \in \mathbb{R}^{[a,b]} : \int_a^b \varphi(|u(t)|)\mathrm{d}t < \infty \right\}$$

is usually called the Orlicz class defined by φ.

In [10], the following characterization of the class $V^R_{(\varphi,1)}[a,b]$ known in the literature as *Medvedev's lemma* [10] is proved.

Lemma 2.1. *A function u belongs to $V^R_{(\varphi,1)}[a,b]$ if and only if $u \in AC[a,b]$ and $u' \in L_\varphi[a,b]$. Moreover,*

$$V^R_{(\varphi,1)}(u,[a,b]) = \int_a^b \varphi\left(|u'(t)|\right)\mathrm{d}t.$$

In 1908, de la Vallée Poussin [3] introduced the class of bounded second variation functions as follows. Given a function $u\colon [a,b] \to \mathbb{R}$ and a partition $\mathcal{P}$ of $[a,b]$,

$$a \leq t_1 < t_2 \leq t_3 < t_4 \leq \cdots < t_{2n} \leq b, \tag{3}$$

we consider the expression

$$\sigma_2(u,[a,b]) := \sum_{j=1}^{n-1} \left| \frac{u(t_{2(j+1)} - u(t_{2j+1}))}{t_{2(j+1)} - t_{2j+1}} - \frac{u(t_{2j}) - u(t_{2j-1})}{t_{2j} - t_{2j-1}} \right|,$$

and define the variation by

$$V_2(u,[a,b]) = \sup_{\mathcal{P}} \sigma_2(u,\mathcal{P}),$$

where the supremum is taken over all partitions $\mathcal{P}$ of the interval $[a,b]$ of the form (3). The number $V_2(u,[a,b])$ is called the de la Vallée Poussin second variation of u on $[a,b]$. If $V_2(u,[a,b]) < \infty$, then we say that the function u has a bounded second variation on $[a,b]$. In what follows, by $BV_2[a,b]$ we shall denote the class of all functions $u\colon [a,b] \to \mathbb{R}$ of bounded second variation on $[a,b]$.

The following result can be found in [3, 14].

Proposition 2.1. *If $u \in BV_2[a,b]$, then $u \in \mathrm{Lip}[a,b]$ and u can be expressed as a difference of two convex functions.*

Now, we are in a position to introduce the following definitions.

Definition 2.2 ([5]). Let $u\colon [a,b] \to \mathbb{R}$ and let $t_1, \ldots, t_n$ be distinct points in $[a,b]$. We define the divided difference of u at points $t_1, \ldots, t_n$ by recurrence:

- $u[t_1] := u(t_1)$,
- $u[t_1, t_2] := \dfrac{u(t_2) - u(t_1)}{t_2 - t_1}$,
- $u[t_1, \ldots, t_n] := \dfrac{u[t_2, \ldots, t_n] - u[t_1, \ldots, t_{n-1}]}{t_n - t_1}$.

Remark 2.2 ([17, p. 548]). Let $x_0, x_1, \ldots, x_k$ be $k+1$ distinct points in $[a,b]$ and suppose that $h_i = x_i - x_0$, $i = 1, \ldots, k$, and $0 < |h_1| < \cdots < |h_k|$. If $f'(x_0)$ exists, then

$$f'(x_0) = k! \lim_{h_k \to 0} \lim_{h_{k-1} \to 0} \ldots \lim_{h_1 \to 0} f[x_0, x_1, \ldots, x_{k+1}]$$

and

$$f'_-(x_0) = k! \lim_{h_k \to 0^-} \lim_{h_{k-1} \to 0^-} \ldots \lim_{h_1 \to 0^-} f[x_0, x_1, \ldots, x_{k+1}],$$

$$f'_+(x_0) = k! \lim_{h_k \to 0^+} \lim_{h_{k-1} \to 0^+} \ldots \lim_{h_1 \to 0^+} f[x_0, x_1, \ldots, x_{k+1}].$$

In the case when two of arguments coincide, we can make the following definition.

Definition 2.3. Let $x_1, \dots, x_{s-1}, \zeta_s, x_s, \dots, x_k$ be $k+1$ distinct points in $[a,b]$. Then, we define

$$[x_1, \dots, x_s, x_s, \dots, x_k] = \lim_{\zeta_s \to x_s} [x_1, \dots, x_{s-1}, \zeta_s, x_s, \dots, x_k],$$

providing this limit exists.

Definition 2.4. Let φ be a φ-function, $k \in I\!N$ and $u\colon [a,b] \to \mathbb{R}$. Given a partition $\mathcal{P} : a \le t_1 < \dots < t_n \le b$ of the interval $[a,b]$ with at least $k+1$ points, we define

$$\sigma^R_{(\varphi,k)}(u,\mathcal{P}) := \sum_{j=1}^{n-k} \varphi\left(\frac{|u[t_{j+1}, \dots, t_{j+k}] - u[t_j, \dots, t_{j+k-1}]|}{|t_{j+k} - t_j|}\right) |t_{j+k} - t_j| \quad (4)$$

and

$$V^R_{(\varphi,k)}(u;[a,b]) = V^R_{(\varphi,k)}(u) := \sup \sigma^R_{(\varphi,k)}(u,\mathcal{P}),$$

where the supremum is taken over all partitions $\mathcal{P}$ of the interval $[a,b]$ with at least $k+1$ points. If $V^R_{(\varphi,k)}(u;[a,b]) < \infty$, we say that the function u has a bounded (φ,k)-variation on $[a,b]$ and the class of such functions is denoted by $V^R_{(\varphi,k)}[a,b]$.

Remark 2.3. If φ is a convex φ-function, then the functional

$$V^R_{(\varphi,k)}(\cdot)\colon \mathbb{R}^{[a,b]} \to [0,\infty) \cup \{\infty\}$$

is also convex and, by (1),

$$V^R_{(\varphi,k)}(\lambda u) \le \lambda V^R_{(\varphi,k)}(u), \qquad \lambda \in [0,1],\ u \in V^R_{(\varphi,k)}[a,b]. \quad (5)$$

Remark 2.4. **(a)** If $k = 1$ and $\varphi(t) = t^p$, $p > 1$, Definition 2.4 coincides with the classical concept of p-variation considered by F. Riesz in 1911 [15]. If $k = 1$ and φ is a convex φ-funtion, this definition coincides with the notion of φ-variation considered by Yu. Medveded [10]. If k is a positive integer and $\varphi(t) = t^p$, $p > 1$, this definition generalizes the concept of p-variation studied by N. Merentes, S. Rivas and J. Sánchez in [12].

(b) Since

$$u[t_j, t_{j+1}, \dots, t_{j+k}] = \frac{u[t_{j+1}, t_{j+2}, \dots, t_{j+k}] - u[t_j, t_{j+1}, \dots, t_{j+k-1}]}{t_{j+k} - t_j},$$

the sum in (4) of Definition 2.4 may be written as

$$\sum_{j=1}^{n-k} \varphi\left(|u[t_j, t_{j+1}, \dots, t_{j+k}]|\right) |t_{j+k} - t_j|.$$

On the other hand, from the properties of k divided differences, we may deduce that, if $u(t) = p(t) = a_k t^k + a_{k-1} t^{k-1} + \dots a_0$, then $u[t_1, t_2, \dots, t_{k+1}] = a_k$, for arbitrary $k+1$ points $t_1,\dots,t_{k+1}$. As a consequence, if u is a polynomial of degree $k-1$, then $V^R_{(\varphi,k)}(u)=0$.

(c) From the definition of the class $V^R_{(\varphi,k)}[a,b]$, we conclude that this class is a symmetric set; if φ is convex, this class is also convex; however, it is not necessarily a linear space. Notice that the space

$$\begin{aligned} RV_{(\varphi,k)}[a,b] &:= \bigcup_{\lambda>0} \lambda V^R_{(\varphi,k)}[a,b] \\ &= \left\{u\in\mathbb{R}^{[a,b]} : \exists\lambda>0, V^R_{(\varphi,k)}\left(\frac{u}{\lambda}\right)<\infty\right\} \end{aligned}$$

forms a vector space.

Indeed, if $u_1,u_2\in RV_{(\varphi,k)}[a,b]$, then $V^R_{(\varphi,k)}(\frac{u_1}{\lambda_1})<\infty$ and $V^R_{(\varphi,k)}(\frac{u_2}{\lambda_2})<\infty$ for some $\lambda_1,\lambda_2>0$. The convexity of functional $V^R_{(\varphi,k)}(\cdot)$ implies

$$V^R_{(\varphi,k)}\Big(\frac{u_1+u_2}{\lambda_1+\lambda_2}\Big)\le\frac{\lambda_1}{\lambda_1+\lambda_2}V^R_{(\varphi,k)}\Big(\frac{u_1}{\lambda_1}\Big)+\frac{\lambda_2}{\lambda_1+\lambda_2}V^R_{(\varphi,k)}\Big(\frac{u_2}{\lambda_2}\Big),$$

thus

$$u_1+u_2\in RV_{(\varphi,k)}[a,b].$$

To prove that $\alpha u_1\in RV_{(\varphi,k)}[a,b]$ for some $\alpha\in\mathbb{R}$, it suffices to observe that

$$V^R_{(\varphi,k)}\left(\frac{u_1}{\lambda_1}\right)=V^R_{(\varphi,k)}\left((-\alpha u_1)\Big(\frac{1}{-\alpha\lambda_1}\Big)\right)\qquad\text{for }\ \alpha<0.$$

Let φ be a convex φ-function and put

$$A:=\left\{u\in\mathbb{R}^{[a,b]} : \exists\lambda>0, V^R_{(\varphi,k)}\Big(\frac{u}{\lambda}\Big)\le 1\right\}.$$

Then, A is balanced as convex and symetric. Moreover, given $u\in RV_{(\varphi,k)}[a,b]$ such that $V^R_{(\varphi,k)}(\frac{u}{\lambda})=k>1$, by (5), we get

$$V^R_{(\varphi,k)}\Big(\frac{u}{k\lambda}\Big)\le\frac{1}{k}V^R_{(\varphi,k)}\Big(\frac{u}{\lambda}\Big)\le 1,$$

so it is also absorbing set. Therefore, the Minkowski functional associated with A given by

$$\mu(u):=\inf\left\{\lambda>0 : V^R_{(\varphi,k)}\Big(\frac{u}{\lambda}\Big)\le 1\right\},\qquad u\in RV_{(\varphi,k)}[a,b],$$

is a seminorm on $RV_{(\varphi,k)}[a,b]$.

Let us recall the following definition given by Popoviciu in [14] and studied by Russell in [17] (cf. also M. Wróbel [19]).

Definition 2.5. Let $k \geq 1$ be an integer. For a given partition $\mathcal{P} : a \leq t_1 < \cdots \cdots < t_n \leq b$, with $n \geq k+1$, and a function $u\colon [a,b] \to \mathbb{R}$, we define

$$\sigma_k(u, \mathcal{P}) := \sum_{n=1}^{n-k} |u[t_{j+1}, \ldots, t_{j+k}] - u[t_j, \ldots, t_{j+k-1}]| , \tag{6}$$

and

$$V_k(u; [a,b]) = V_k(u) := \sup_{\mathcal{P}} \sigma_k(u, \mathcal{P}),$$

where the supremum is taken over all the partitions $\mathcal{P}$ of the interval $[a,b]$ with at least $k+1$ points. If $V_k(u; [a,b]) < \infty$, we say that the function u has a bounded k-variation on the interval $[a,b]$, and the vector space of such functions is denoted by $BV_k[a,b]$.

Modifying the sum (6) slightly, we can consider the following similar but different definition. If k is a positive integer, $u\colon [a,b] \to \mathbb{R}$ and

$$\mathcal{P} : a \leq t_1 < \cdots < t_k \leq t_{k+1} < \cdots < t_{2k} \leq t_{2k+1} < \cdots < t_{kn} \leq b$$

is a partition of the interval $[a,b]$, with at least kn points, we define

$$\hat{\sigma}_k(u, [a,b]) = \hat{\sigma}_k(u) := \sum_{j=1}^{n-1} \left| u[t_{jk+1}, \ldots, t_{(j+1)k}] - u[t_{(j-1)k+1}, \ldots, t_{jk}] \right| ,$$

and

$$\hat{V}_k(u; [a,b]) = \hat{V}_k(u) := \sup \mathcal{P} \hat{\sigma}_k(u, \mathcal{P}),$$

where the supremum is taken over all partitions $\mathcal{P}$ of the interval $[a,b]$ with at least kn points.

We define the vector space

$$B\hat{V}_k[a,b] = \Big\{ u\colon [a,b] \to \mathbb{R} : \hat{V}_k(u) < \infty \Big\}.$$

From this definition, it can be inferred that if $a < c < b$, then

$$\hat{V}_k[a,b] \geq \hat{V}_k[a,c] + \hat{V}_k[c,b].$$

Theorem 2.1. *For k being a positive integer, we have the estimates*

$$\hat{V}_k(u, [a,b]) \leq V_k(u, [a,b]) \leq 3k \hat{V}_k(u, [a,b])$$

and, therefore, $BV_k[a,b] = B\hat{V}_k[a,b]$.

Proof. Let $u \in B\hat{V}_k[a,b]$ and $a \leq t_1 < \cdots < t_{k+1} \leq b$. Consider numbers $b_1, \ldots b_k, c_1, \ldots, c_k$ such that

$$t_1 < b_1 < \cdots < b_k = t_2, \qquad t_k < c_1 < \cdots < c_k = t_{k+1}.$$

Then,

$$
\begin{aligned}
&|u[t_2, \ldots, t_{k+1}] - u[t_1, \ldots, t_k]| \\
&\leq |u[t_2, \ldots, t_{k+1}] - u[b_1, \ldots, b_k]| \\
&\quad + |u[b_1, \ldots, b_k] - u[c_1, \ldots, c_k]| \\
&\quad + |u[c_1, \ldots, c_k] - u[t_1, \ldots, t_k]| \\
&\leq 3\hat{V}_k(u, [t_1, t_{k+1}]).
\end{aligned}
$$

In this way, if $\mathcal{P}\colon a \leq t_1 < \ldots < t_n \leq b$ is a partition of $[a,b]$ and, without lost of generality, $n = lk$ for some natural number $l \geq 2$, then

$$
\begin{aligned}
&\sum_{j=1}^{(l-1)k} \left| u[t_{j+1,\ldots,} t_{j+k}] - u[t_j, \ldots, t_{t_{j+k-1}}] \right| \\
&\leq \sum_{j=1}^{(l-1)k} 3\hat{V}_k \left(u, [t_j, t_{j+k}]\right) \\
&\leq 3 \Big(\hat{V} \left(u, [t_1, t_{1+k}]\right) + \hat{V}_k \left(u, [t_{k+1}, t_{2k+1}]\right) + \cdots + \hat{V}_k \left(u, [t_{(l-2)k+1}, t_{(l-1)k+1}]\right) \\
&\quad + \hat{V}_k \left(u, [t_2, t_{2+k}]\right) + \hat{V}_k \left(u, [t_{2+k}, t_{2k+2}]\right) + \cdots + \hat{V}_k \left(u, [t_{(l-2)k+2}, t_{(l-1)k+2}]\right) \\
&\quad + \cdots + \hat{V}_k \left(u, [t_k, t_{2k}]\right) + \hat{V}_k \left(u, [t_{2k}, t_{3k}]\right) + \cdots + \hat{V}_k \left(u, [t_{(l-1)k}, t_{lk}]\right) \Big) \\
&\leq 3k\hat{V}_k \left(u, [a,b]\right).
\end{aligned}
$$

Hence,

$$
V_k(u, [a,b]) \leq 3k\hat{V}_k \left(u, [a,b]\right)
$$

and so, $u \in BV_k[a,b]$. Therefore, we conclude that $\hat{BV}_k[a,b] \subset BV_k[a,b]$. On the other hand, if $u \in BV_k[a,b]$ and

$$
\mathcal{P}\colon a \leq t_1 < \cdots < t_k \leq t_{k+1} \cdots < t_{2k} \leq t_{2k+1} < \cdots < t_{nk} \leq b
$$

is a partition of the interval $[a,b]$, then from the triangular inequality we obtain

$$
\begin{aligned}
&\left| u[t_{jk+1}, \cdots, t_{(j+1)k}] - u[t_{(j-1)k+1}, \cdots, t_{jk}] \right| \\
&\leq \left| u[t_{jk+1}, \cdots, t_{(j+1)k}] - u[t_{jk}, t_{jk+1} \cdots, t_{jk+k-1}] \right| \\
&\quad + |u[t_{jk}, t_{jk+1} \ldots, t_{jk+k-1}] - u[t_{jk-1}, t_{jk}, t_{jk+1}, \ldots, t_{jk+k-2}]| \\
&\quad + \ldots + \left| u[t_{jk-k+2}, t_{jk-k+3} \ldots, t_{jk,} t_{jk+1}] - u[t_{(j-1)k+1}, \ldots, t_{jk}] \right| \\
&= \sum_{i=0}^{k-1} \left| u[t_{jk-i+1}, \ldots, t_{(j+1)k-i}] - u[t_{jk-i}, \ldots, t_{(j+1)k-i-1}] \right|.
\end{aligned}
$$

From here, putting $l = jk - 1$, we get:

$$\begin{aligned}
&\sum_{j=1}^{n-1} \big|u[t_{jk+1},\dots,t_{(j+1)k}] - u[t_{(j-1)k+1},\dots,t_{jk}]\big| \\
&\le \sum_{j=1}^{n-1} \sum_{l=jk-k+1}^{jk} |u[t_{l+1},\dots,t_{l+k}] - u[t_l,\dots,t_{l+k-1}]| \\
&= \sum_{l=1}^{(n-1)k} |u[t_{l+1},\dots,t_{l+k}] - u[t_l,\dots,t_{l+k-1}]| = V_k(u,[a,b]).
\end{aligned}$$

Finally, we get $\hat{V}_k(u,[a,b]) \le V_k(u,[a,b])$, so, $u \in B\hat{V}_k[a,b]$. As a consequence, the inclusion $BV_k(u,[a,b]) \subset B\hat{V}_k(u,[a,b])$ holds as well. □

From this result, we deduce that all properties of the space $BV_k[a,b]$ are shared by the space $B\hat{V}_k[a,b]$ as well. We summarize some of these properties in the following

Theorem 2.2 ([17]). *If k is a positive integer, then*

(i) *If $V_k(u,[a,b]) < \infty$, then $u[t_1,\dots,t_k]$ is bounded for all $t_1,\dots,t_k \in [a,b]$.*

(ii) *$BV_{k+1}[a,b] \subset BV_k[a,b]$.*

(iii) *If $u \in BV_2[a,b]$, then u is absolutely continuous on $[a,b]$, u'_+ exists on $[a,b)$, u'_- exists on $(a,b]$ and u has finite derivatives exept for at most countably many points. If $u \in BV_k[a,b]$, $k \ge 3$, then $u^{(r)}$, $r = 1,\dots,k-2$, exist and belong to $BV_{k-r}[a,b]$ and $u^{(k-1)}$ exists a.e. in $[a,b]$.*

(iv) *If $u \in BV_k[a,b]$, then $u = u_{r1} - u_{r2}$, where u_{r1}, u_{r2} are r-convex functions ($r = 1,\dots,k$), which means that*

$$u_{ri}[t_1,\dots,t_{r+1}] \ge 0, \quad \textit{for} \quad i = 1,2, \quad r = 1,\dots,k; \quad t_1,\dots,t_{r+1} \in [a,b].$$

In the following theorem, we present a relation between the class $V^R_{(\varphi,k)}[a,b]$ and the space $BV_k[a,b]$.

Theorem 2.3. *If φ is a convex φ-function and k a positive integer, then $V^R_{(\varphi,k)}[a,b] \subset BV_k[a,b]$ and*

$$V_k(u) \le k(b-a) + \frac{1}{\varphi(1)} V^R_{(\varphi,k)}(u), \qquad u \in V^R_{(\varphi,k)}[a,b].$$

Also, if the ∞_1 condition does not hold, then

$$V^R_{(\varphi,k)}[a,b] = BV_k[a,b].$$

Moreover, the above relations are also true if we replace $V_k(u)$ by $\hat{V}_k(u)$ and $BV_k[a,b]$ with $B\hat{V}_k[a,b]$.

Proof. Let $u \in V^R_{(\varphi,k)}[a,b]$ and $\mathcal{P}\colon a \le t_0 < t_1 < \cdots < t_n \le b$ be a partition of $[a,b]$ containing at least $k+1$ points. Consider the set

$$\Gamma = \left\{ j \in \{0,\dots,n-k\} : \frac{|u[t_{j+1},t_{j+2},\dots,t_{j+k}] - u[t_j,t_{j+1},\dots,t_{j+k-1}]|}{|t_{j+k}-t_j|} \le 1 \right\}.$$

Since φ is convex and $\varphi(0)=0$, we have $\varphi(t) \ge t\varphi(1)$ for $t \ge 1$ and

$$\begin{aligned}
&\sum_{j=0}^{n-k} |u[t_{j+1},t_{j+2},\dots,t_{j+k}] - u[t_j,t_{j+1},\dots,t_{j+k-1}]| \\
&= \sum_{j=0}^{n-k} \frac{|u[t_{j+1},t_{j+2},\dots,t_{j+k}] - u[t_j,t_{j+1},\dots,t_{j+k-1}]|}{|t_{j+k}-t_j|} |t_{j+k}-t_j| \\
&= \sum_{j\in\Gamma} |t_{j+k}-t_j| \\
&\quad + \frac{1}{\varphi(1)} \sum_{j\notin\Gamma} \varphi\left(\frac{|u[t_{j+1},t_{j+2},\dots,t_{j+k}] - u[t_j,t_{j+1},\dots,t_{j+k-1}]|}{|t_{j+k}-t_j|} \right) |t_{j+k}-t_j| \\
&\le \sum_{j=0}^{n-k} |t_{j+k}-t_j| + \frac{1}{\varphi(1)} V^R_{(\varphi,k)}(u) \\
&\le \sum_{j=0}^{n-k} (t_{j+k} - t_{j+k-1} + t_{j+k-1} - t_{j+k-2} + \ldots + t_{j+1} - t_j) + \frac{1}{\varphi(1)} V^R_{(\varphi,k)}(u) \\
&\le k(b-a) + \frac{1}{\varphi(1)} V^R_{(\varphi,k)}(u).
\end{aligned}$$

Therefore, we get that $u \in BV_k[a,b]$ and the indicated relation is verified. The proof of the second part of the theorem is the counterpart of the proof of theorem in [9] saying that if the ∞_1 condition does not hold, then

$$BV[a,b] = RV_{(\varphi,1)}[a,b].$$

□

Remark 2.5. As a consequence of Theorem 2.1 and Theorem 2.3, we have that Theorem 2.3 is also true if we replace $BV_k[a,b]$ with $B\hat{V}_k[a,b]$ and $V_k(u)$ with $\hat{V}_k(u)$. In view of this result, from now on, we will assume that φ is a convex φ-function that verifies the ∞_1 condition. Also, functions of bounded (φ,k)-variation in the sense of Riesz share all properties with the functions of bounded k-variation.

Proposition 2.2. *Let φ be a φ-function and k a positive integer. Then,*

$$V^R_{(\varphi,k+1)}[a,b] \subset V^R_{(\varphi,,k)}[a,b].$$

P r o o f. Let $u \in V^R_{(\varphi,,k+1)}[a,b]$. Then, Theorem 2.3 implies that $u \in BV_{k+1}[a,b]$. As a consequence, there is a constant K such that $|u[s_1,\ldots,s_{k+1}]| \le K$ for any choice of $s_1,\ldots,s_k \in [a,b]$. Consider a partition $\mathcal{P}\colon a \le t_0 < t_1 < \cdots < t_n \le b$ of the interval $[a,b]$ containing at least $k+1$ points. Then,

$$\sum_{j=0}^{n-k} \varphi\left(\frac{|u[t_{j+1},t_{j+2},\ldots,t_{j+k}] - u[t_j,t_{j+1},\ldots,t_{j+k-1}]|}{|t_{j+k}-t_j|}\right)|t_{j+k}-t_j|$$
$$= \sum_{j=0}^{n-k} \varphi\left(|u[t_j,t_{j+1},\ldots,t_{j+k}]|\right)|t_{j+k}-t_j|$$
$$\le \varphi(K)\sum_{j=0}^{n-k} \varphi\,|t_{j+k}-t_j| \le \varphi(K)\,k\,(b-a).$$

From this, we conclude that $V^R_{(\varphi,k)}(u,[a,b]) \le \varphi(K)\,k\,(b-a)$. □

Corollary 2.1. *Let k be a positive integer and φ a φ-function. Then,*

$$RV_{(\varphi,k+1)}[a,b] \subset RV_{(\varphi,k)}[a,b].$$

Using notation of Definition 2.3, we have the following lemma.

Lemma 2.2 ([17, Theorem 8]). *If $u\colon [a,b] \to \mathbb{R}$ has a derivative in $t_1,t_2,\ldots,t_k \in [a,b]$, then*

$$u'[t_1,t_2,\ldots,t_k] = u[t_1,t_1,t_2,\ldots,t_k] + \ldots + u[t_1,t_2,\ldots,t_{k-1},t_k,t_k].$$

Proposition 2.3. *Let φ be a convex φ-function, $k \ge 2$ an integer, and $u \in V^R_{(\varphi,k)}[a,b]$. If u' exists, then $\frac{u'}{k} \in V^R_{(\varphi,,k-1)}[a,b]$ and*

$$V^R_{(\varphi,k-1)}\left(\frac{u'}{k}\right) \le V^R_{(\varphi,k)}(u).$$

P r o o f. Let $\mathcal{P} : a \le t_0 < t_1 < \cdots < t_n \le b$ be a partition of $[a,b]$. From the above lemma and the convexity of φ, it follows that

$$\sum_{j=0}^{n-(k-1)} \varphi\left(\frac{1}{k}\,|u'[t_j,t_{j+1},\ldots,t_{j+k-1}]|\right)|t_{j+k-1}-t_j|$$
$$= \sum_{j=0}^{n-(k-1)} \varphi\left(\sum_{h=j}^{j+(k-1)} \frac{|u[t_j,t_{j+1},\ldots t_{h-1},t_h,t_h,t_{h+1},\ldots,t_{j+k-1}]|}{k}\right)|t_{j+k-1}-t_j|$$
$$\le \frac{1}{k}\sum_{j=0}^{n-(k-1)}\left(\sum_{h=j}^{j+(k-1)} \varphi\left(|u[t_j,t_{j+1},\ldots,t_h,t_h,\ldots,t_{j+k-1}]|\right)\right)|t_{j+k-1}-t_j|$$
$$\le V^R_{(\varphi,k)}(u).$$
□

Corollary 2.2. *Let φ be a convex φ-function, $k \geq 3$ a positive integer, and $u \in RV_{(\varphi,k)}[a,b]$. Then, $u' \in RV_{(\varphi,k-1)}[a,b]$.*

Comparing with the definition of $\hat{V}_k[a,b]$ and modifying the sum in (4), we can consider the following definition. Given a positive integer k, a φ-function φ, a function $u\colon [a,b] \to \mathbb{R}$, and a partition

$$\mathcal{P}\colon a \leq t_1 < \cdots < t_k \leq t_{k+1} < \cdots < t_{2k} \leq t_{2k+1} < \cdots < t_{nk} \leq b$$

of the interval $[a,b]$, let us consider the expression

$$\begin{aligned}&\hat{\sigma}^R_{(\varphi,k)}(u,\mathcal{P})\\ &:= \sum_{j=1}^{n-1} \varphi\left(\frac{\left|u[t_{jk+1},\ldots,t_{(j+1)k}] - u[t_{(j-1)k+1},\ldots,t_{jk}]\right|}{\left|t_{(j+1)k} - t_{(j-1)k+1}\right|}\right)\left|t_{(j+1)k} - t_{(j-1)k+1}\right|\end{aligned}$$

and put

$$\hat{V}^R_{(\varphi,k)}(u;[a,b]) = \hat{V}^R_{(\varphi,k)}(u) := \sup_{\mathcal{P}} \hat{\sigma}^R_{(\varphi,k)}(u,\mathcal{P}),$$

where the supremum is taken over all partitions $\mathcal{P}$ of the interval $[a,b]$ containing at least $2k-1$ points. Let

$$\hat{V}^R_{(\varphi,k)}[a,b] = \left\{u\colon [a,b] \to \mathbb{R} : \hat{V}^R_{(\varphi,k)}(u) < \infty\right\}.$$

From this definition, it immediately follows that $\hat{V}^R_{(\varphi,k)}[a,b]$ is a symmetric set and, if φ is convex, this set is convex.

Consequently, for convex φ,

$$\begin{aligned}R\hat{V}_{(\varphi,k)}[a,b] &= \left\{u\colon [a,b] \to \mathbb{R}\ : \exists\lambda > 0,\ \hat{V}^R_{(\varphi,k)}\left(\frac{u}{\lambda}\right) < \infty\right\}\\ &= \bigcup_{\lambda>0} \lambda \hat{V}^R_{(\varphi,k)}[a,b]\end{aligned}$$

is the vector space generated by $\hat{V}^R_{(\varphi,k)}[a,b]$.

Theorem 2.4. *Let k be a positive integer and φ a convex φ-function.*

(i) *If $u \in \hat{V}^R_{(\varphi,k)}[a,b]$, then $V^R_{(\varphi,k)}(\frac{u}{k},[a,b]) \leq k\hat{V}^R_{(\varphi,k)}(u,[a,b])$.*

(ii) *If $u \in V^R_{(\varphi,k)}[a,b]$, then $\hat{V}^R_{(\varphi,k)}(\frac{u}{k},[a,b]) \leq \frac{1}{k}V^R_{(\varphi,k)}(u,[a,b])$.*

(iii) $R\hat{V}_{(\varphi,k)}[a,b] = RV_{(\varphi,k)}[a,b]$.

Proof. Let $u \in \hat{V}^R_{(\varphi,k)}[a,b]$ and $a \leq t_1 < \ldots < t_{k+1} \leq b$. Consider numbers $a_1,\ldots,a_k,b_1,\ldots,b_k$, such that

$$t_1 = a_1 < \ldots < a_k \leq t_2, \qquad t_k \leq b_1 < \ldots < b_k = t_{k+1}.$$

Now, we proceed similarly as in the proof of Theorem 2.1. Using the triangular inequality and the convexity of φ, we have

$$\begin{aligned}
&\varphi\Big(\frac{|u[t_2,\dots,t_{k+1}]-u[t_1,\dots,t_k]|}{3\,|t_{k+1}-t_1|}\Big)\,|t_{k+1}-t_1| \\
&\le \frac{1}{3}\bigg[\varphi\Big(\frac{|u[t_2,\dots,t_{k+1}]-u[a_1,\dots,a_k]|}{|t_{k+1}-a_1|}\Big)\,|t_{k+1}-a_1| \\
&\quad+\varphi\Big(\frac{|u[a_1,\dots,a_k]-u[b_1,\dots,b_k]|}{|b_k-a_1|}\Big)\,|b_k-a_1| \\
&\quad+\varphi\,\Big(\frac{|u[b_1,\dots,b_k]-u[t_1,\dots,t_k]|}{|b_k-t_1|}\Big)\,|b_k-t_1|\bigg] \\
&\le \hat{V}^R_{(\varphi,k)}(u,[t_1,t_{k+1}]).
\end{aligned}$$

Therefore, if $\mathcal{P}\colon a\le t_1<\dots<t_n\le b$ is a partition of $[a,b]$, proceeding as in the proof of Theorem 2.1, we obtain

$$\begin{aligned}
&\sum_{j=1}^{n-k}\varphi\left(\frac{|u[t_{j+1,\dots,}t_{j+k}]-u[t_j,\dots,t_{j+k-1}]|}{3\,|t_{j+k}-t_j|}\right)|t_{j+k}-t_j| \\
&\le kV^R_{(\varphi,k)}\,(u,[a,b])\,,
\end{aligned}$$

hence,

$$V^R_{(\varphi,k)}\Big(\frac{u}{3},[a,b]\Big)\le k\hat{V}^R_{(\varphi k)}\,(u,[a,b])\,.$$

Therefore, $\hat{RV}_{(\varphi,k)}[a,b]\subset RV_{(\varphi,k)}[a,b]$.

If we consider $u\in V^R_{(\varphi,k)}[a,b]$ and a partition

$$\mathcal{P}\colon a\le t_1<\dots<t_k\le t_{k+1}<\dots<t_{2k}\le t_{2k+1}<\dots<t_{nk}\le b$$

of the interval $[a,b]$, then proceeding as in the proof of Theorem 2.1 and using the triangular inequality and the convexity of φ, we get

$$\begin{aligned}
&\varphi\left(\frac{\big|u[t_{jk+1},\dots,t_{(j+1)k}]-u[t_{(j-1)k+1},\dots,t_{jk}]\big|}{k\,\big|t_{(j+1)k}-t_{(j-1)k+1}\big|}\right)\times\big|t_{(j+1)k}-t_{(j-1)k+1}\big| \\
&\le\frac{1}{k}\sum_{i=0}^{k-1}\varphi\left(\frac{\big|u[t_{jk-i+1},\dots,t_{(j+1)k-i}]-u[t_{jk-i},\dots,t_{(j+1)k-i-1}]\big|}{\big|t_{(j+1)k-i}-t_{jk-i}\big|}\right) \\
&\qquad\qquad\times\big|t_{(j+1)k-i}-t_{jk-i}\big|.
\end{aligned}$$

From here, we get in turn

$$\sum_{j=1}^{n-1}\varphi\left(\frac{\left|u[t_{jk+1},\ldots,t_{(j+1)k}]-u[t_{(j-1)k+1},\ldots,t_{jk}]\right|}{k\left|t_{(j+1)k}-t_{(j-1)k+1}\right|}\right)\times\left|t_{(j+1)k}-t_{(j-1)k+1}\right|$$
$$\leq\frac{1}{k}V^R_{(\varphi,k)}(u,[a,b]).$$

Thus, we conclude that $\hat{V}^R_{(\varphi,k)}(\frac{u}{k},[a,b])\leq\frac{1}{k}V^R_{(\varphi,k)}(u,[a,b])$ and, as a consequence, $RV_{(\varphi,k)}[a,b]\subset\hat{RV}_{(\phi,k)}[a,b]$, □

In case $\varphi(t)=t^p$, $p>1$, the class $V_{(\varphi,k)}[a,b]$ coincides with the vector space $RV_{(p,k)}[a,b]$ which has been studied in [12]. So, we get the following

Corollary 2.3. *Let k be a positive integer. Then, $RV_{(p,k)}[a,b]=\hat{R}V_{(p,k)}[a,b]$.*

Moreover, proceeding in a similar way as in the proof of Theorem 2.3, the following theorem can be proved.

Theorem 2.5. *Let φ be a convex φ-function and k a positive integer. Then, $\hat{V}^R_{(\varphi,k)}[a,b]\subset BV_k[a,b]$ and*

$$\hat{V}_k(u)\leq b-a+\frac{1}{\varphi(1)}\hat{V}^R_{(\varphi,k)}(u),\qquad u\in\hat{V}^R_{(\varphi,k)}[a,b].$$

Moreover, if the ∞_1 condition does not hold, then

$$\hat{V}^R_{(\varphi,k)}[a,b]=BV_k[a,b].$$

Theorem 2.6. *Let φ be a convex φ-function, $k\geq 2$ be a positive integer, and $u\in\hat{V}^R_{(\varphi,k)}[a,b]$. Then, $u^{(k-1)}$ exists and is continuous on $[a,b]$.*

P r o o f. Fix arbitrarilly a natural number $k\geq 2$ and take $u\in\hat{V}^R_{(\varphi,k)}[a,b]$. Applying Theorem 2.4, we get $\frac{u}{3}\in BV_k[a,b]$ and, by [17, Theorem 12], it follows that $u^{(k-2)}\in BV_2[a,b]$. Consequently, $u^{(k-2)}$ is continuous and can be expressed as a difference of two convex functions. As a consequence, the unilateral derivatives $u^{(k-1)}_+$(t) on $[a,b)$ and $u^{(k-1)}_-(t)$ on $(a,b]$ exist and are continuous. In addition, the set E of all points $t\in[a,b]$, where $u^{(k-1)}(t)$ does not exist, is countable and $u^{(k-1)}$ is continuous on $[a,b]\setminus E$ (Theorem 2.2).

Suppose that there exists $x_0\in(a,b)$, where $u^{(k-1)}$ does not exist. Then,

$$\alpha_{x_0}=\left|u^{(k-1)}_+(x_0)-u^{(k-1)}_-(x_0)\right|>0. \tag{7}$$

Consider $3(k+2)$ distinct points $t_1,\ldots,t_{k-2},\alpha_1,\ldots,\alpha_{k-2},s_1,\ldots,s_{k-2}\in(a,b)$ and $h>0$ such that

$$a\leq x_0-h<t_1<\ldots<t_{k-2}<x_0<\alpha_1<\ldots<\alpha_k<s_1<\ldots<s_{k-2}<x_0+h\leq b.$$

By Remark 2.2, we have, for sufficiently small $h > 0$,

$$u_+^{(k-1)}(x_0) = (k-2)! \lim_{h\to 0^+}\left(\lim_{\substack{s_1\to x_0+h\\ \alpha_{k-2}\to x_0}} \frac{u[s_1,\dots,s_{k-2},x_0+h]-u[x_0,\alpha_1,\dots,\alpha_{k-2}]}{h}\right)$$

and

$$u_-^{(k-1)}(x_0) = (k-2)! \lim_{h\to 0^+}\left(\lim_{\substack{\alpha_{k-2}\to x_0\\ t_{k-2}\to x_0-h}} \frac{u[x_0,\alpha_1,\dots,\alpha_{k-2}]-u[x_0-h,t_1,\dots,t_{k-2}]}{h}\right).$$

Hence, putting

$$\begin{aligned} A_{x_0}^h &= A_{x_0}^h(t_1,\dots,t_{k-2},\alpha_1,\dots,\alpha_{k-2},s_1,\dots,s_{k-2}) \\ &:= \frac{u[s_1,\dots,s_{k-2},x_0+h]-u[x_0,\alpha_1,\dots,\alpha_{k-2}]}{h} \\ &\quad - \frac{u[x_0,\alpha_1,\dots,\alpha_{k-2}]-u[x_0-h,t_1,\dots,t_{k-2}]}{h}, \end{aligned}$$

by (7), we get $A_{x_0}^h(t_1,\dots,t_{k-2},\alpha_1,\dots,\alpha_{k-2},s_1,\dots,s_{k-2}) \neq 0$ and

$$\alpha_{x_0} = (k-2)! \left| \lim_{h\to 0}\left(\lim_{\substack{s_1\to x_0+h\\ \alpha_{k-2}\to x_0\\ t_{k-2}\to x_0-h}} A_{x_0}^h(t_1,\dots,t_{k-2},\alpha_1,\dots,\alpha_{k-2},s_1,\dots,s_{k-2})\right)\right|. \tag{8}$$

Since

$$\begin{aligned} V_{(\varphi,k)}^R(u) &\geq \varphi\left(\frac{\left|A_{x_0}^h(t_1,\dots,t_{k-2},\alpha_1,\dots,\alpha_{k-2},s_1,\dots,s_{k-2})\right|}{2h}\right) 2h \\ &= \varphi\left(\frac{\left|A_{x_0}^h\right|}{2h}\right)\left(\frac{\left|A_{x_0}^h\right|}{2h}\right)^{-1}\left|A_{x_0}^h\right|, \end{aligned}$$

applying (8), the continuity of φ and the ∞_1 condition, we have

$$V_{(\varphi,k)}^R(u) \geq \lim_{h\to 0}\left(\lim_{\substack{s_1\to x_0+h\\ \alpha_{k-2}\to x_0\\ t_{k-2}\to x_0-h}} \varphi\left(\frac{\left|A_{x_0}^h\right|}{2h}\right)\left(\frac{\left|A_{x_0}^h\right|}{2h}\right)^{-1}\right)\frac{\alpha_{x_0}}{(k-2)!} = \infty,$$

which contradicts the fact that $\hat{V}_{(\varphi,k)}^R(u) < \infty$. Hence, $u^{(k-1)}$ exists on $[a,b]$. Since $E = \emptyset$, we conclude that $u^{(k-1)}$ is continuous on the whole interval $[a,b]$. □

3. Main result

Now, we will need the following proposition.

Proposition 3.1 ([5], [1]). *Let $n \geq 1$ and let $t_1, \ldots, t_{n+1}$ be points of the interval $[a,b] \subset \mathbb{R}$. If $u \in C^n[a,b]$, then there exist*

$$\xi \in [\min\{t_1, \ldots, t_{n+1}\}, \max\{t_1, \ldots, t_{n+1}\}] \ \text{ such that } u[t_1, \ldots, t_{n+1}] = \frac{u^{(n)}(\xi)}{n!}.$$

In particular, if the function u is n-times continuously differentiable in the neighbourhood of t, then

$$u[\underbrace{t, \ldots, t}_{n+1\ times}] = \frac{u^{(n)}(t)}{n!}.$$

The main result reads as follows.

Theorem 3.1. *Let φ be a convex φ-function that satisfies the ∞_1 condition and k a positive integer. Then, $u \in \hat{V}^R_{(\varphi,k)}[a,b]$ if and only if $u^{(k-1)}$ is absolutely continuous on $[a,b]$, $\frac{u^{(k)}}{(k-1)!} \in L_\varphi[a,b]$, and the following equality holds*

$$\hat{V}^R_{(\varphi,k)}\left(u[a,b]\right) = \int_a^b \varphi\left(\left|\frac{u^{(k)}(t)}{(k-1)!}\right|\right) \mathrm{d}t.$$

P r o o f. Fix $u \in \hat{V}^R_{(\varphi,k)}[a,b]$ and consider a partition $\mathcal{P}\colon a \leq t_1 < \cdots < t_n \leq b$ of the interval $[a,b]$ with points $s_1, \ldots, s_{nk} \in [a,b]$ such that

$$\begin{aligned} t_1 = s_1 < \cdots < s_k < t_2 = s_{k+1} < \cdots < s_{2k} < t_3 = s_{2k+1} \\ < \cdots < t_{n-1} = s_{(n-2)k+1} < \cdots < s_{(n-1)k} < s_{(n-1)k+1} \\ < \cdots < s_{n_k} = t_n. \end{aligned}$$

By Proposition 3.1, there exist intermediate points

$$\xi_j \in (s_{(j-1)k+1}, s_{jk}), \qquad j = 1, \ldots, n,$$

such that

$$\frac{u^{(k-1)}(\xi_j)}{(k-1)!} = u[s_{(j-1)k+1}, \ldots, s_{jk}], \qquad j = 1, \ldots, n.$$

Since $u \in \hat{V}^R_{(\varphi,k)}[a,b]$, we have

$$\sum_{j=1}^{n-1} \varphi\left(\frac{\left|u^{(k-1)}(\xi_{j+1}) - u^{(k-1)}(\xi_j)\right|}{(k-1)!\left|s_{(j+1)k} - s_{(j-1)k+1}\right|}\right)\left|s_{(j+1)k} - s_{(j-1)k+1}\right|$$

$$= \sum_{j=1}^{n-1} \varphi\left(\frac{\left|u[s_{jk+1},\dots,s_{(j+1)k}] - u[s_{(j-1)k+1},\dots,s_{jk}]\right|}{\left|s_{(j+1)k} - s_{(j-1)k+1}\right|}\right)\left|s_{(j+1)k} - s_{(j-1)k+1}\right|$$

$$\leq \hat{V}^R_{(\varphi,k)}(u,[a,b]).$$

Passing to the limit as $s_{jk} \to s_{(j-1)k+1} = t_j$, $j = 1,\cdots,n-1$, and $s_{(n-1)k+1} \to s_{nk} = t_n$, we get $\xi_j \to t_j$, $j = 1,\dots,n$. Since $u^{(k-1)}$ is continuous on $[a,b]$, we obtain

$$\sum_{j=1}^{n-1} \varphi\left(\frac{\left|u^{(k-1)}(t_{j+1}) - u^{(k-1)}(t_j)\right|}{(k-1)!\,|t_{j+1} - t_j|}\right)|t_{j+1} - t_j| \leq \hat{V}^R_{(\varphi,k)}(u,[a,b]).$$

So, $\frac{u^{(k-1)}}{(k-1)!} \in V^R_{(\varphi,1)}[a,b]$ and $V^R_{(\varphi,1)}(\frac{u^{(k-1)}}{(k-1)!},[a,b]) \leq \hat{V}^R_{(\varphi,k)}(u,[a,b])$. By Medvedev's lemma [10], we have $\frac{u^{(k-1)}}{(k-1)!} \in AC[a,b]$, $\frac{u^{(k)}}{(k-1)!} \in L_\varphi[a,b]$ and

$$V^R_{(\varphi,1)}\left(\frac{u^{(k-1)}}{(k-1)!},[a,b]\right) = \int_a^b \varphi\left(\frac{\left|u^{(k)}(t)\right|}{(k-1)!}\right)\mathrm{d}t.$$

As a consequence, $u^{(k-1)} \in AC[a,b]$, $\frac{u^{(k)}}{(k-1)!} \in L_\varphi[a,b]$, and

$$\int_a^b \varphi\left(\frac{\left|u^{(k)}(t)\right|}{(k-1)!}\right)\mathrm{d}t \leq \hat{V}_{(\varphi,k)}(u,[a,b]).$$

Conversely, let us suppose that $u^{(k-1)}$ is absolutely continuous on $[a,b]$ and $\frac{u^{(k)}}{(k-1)!} \in L_\varphi[a,b]$. Consider any partition

$$\mathcal{P}\colon a \leq t_1 < \ldots < t_k \leq t_{k+1} < \ldots < t_{2k} \leq t_{2k+1} < \ldots < t_{nk} \leq b$$

of the interval $[a,b]$. By Proposition 3.1, there exist intermediate points

$$\xi_j \in \left(t_{(j-1)k+1}, t_{j_k}\right), \qquad j = 1,\dots,n,$$

such that

$$\frac{u^{(k-1)}(\xi_j)}{(k-1)!} = u[t_{(j-1)k+1},\dots,t_{jk}], \qquad j = 1,\dots,n.$$

Using the properties of the φ-function φ, the absolute continuity of $u^{(k-1)}$ on $[a,b]$, and Jensen's inequality, we obtain

$$\varphi\left(\frac{\left|u[t_{jk+1},\dots,t_{(j+1)k}]-u[t_{(j-1)k+1},\dots,t_{jk}]\right|}{\left|t_{(j+1)k}-t_{(j-1)k+1}\right|}\right)\left|t_{(j+1)k}-t_{(j-1)k+1}\right|$$

$$=\int_{t_{(j-1)k+1}}^{t_{(j+1)k}}\varphi\left(\frac{\left|u^{(k-1)}(\xi_{j+1})-u^{(k-1)}(\xi_j)\right|}{(k-1)!\left|t_{(j+1)k}-t_{(j-1)k+1}\right|}\right)\mathrm{d}\xi$$

$$=\int_{t_{(j-1)k+1}}^{t_{(j+1)k}}\varphi\left(\int_{\xi_j}^{\xi_{j+1}}\frac{\left|u^{(k)}(t)\right|}{(k-1)!\left|t_{(j+1)k}-t_{(j-1)k+1}\right|}\mathrm{d}t\right)\mathrm{d}\xi$$

$$\leq\int_{t_{(j-1)k+1}}^{t_{(j+1)k}}\varphi\left(\int_{t_{(j-1)k+1}}^{t_{(j+1)k}}\frac{\left|u^{(k)}(t)\right|}{(k-1)!\left|t_{(j+1)k}-t_{(j-1)k+1}\right|}\mathrm{d}t\right)\mathrm{d}\xi$$

$$\leq\int_{t_{(j-1)k+1}}^{t_{(j+1)k}}\frac{1}{\left|t_{(j+1)k}-t_{(j-1)k+1}\right|}\int_{t_{(j-1)k+1}}^{t_{(j+1)k}}\varphi\left(\frac{\left|u^{(k)}(t)\right|}{(k-1)!}\right)\mathrm{d}t\mathrm{d}\xi$$

$$=\int_{t_{(j-1)k+1}}^{t_{(j+1)k}}\varphi\left(\frac{\left|u^{(k)}(t)\right|}{(k-1)!}\right)\mathrm{d}t.$$

Therefore,

$$\sum_{j=1}^{n-1}\varphi\left(\frac{\left|u[t_{jk+1},\dots,t_{(j+1)k}]-u[t_{(j-1)k+1},\dots,t_{jk}]\right|}{\left|t_{(j+1)k}-t_{(j-1)k+1}\right|}\right)\left|t_{(j+1)k}-t_{(j-1)k+1}\right|$$

$$\leq\int_a^b\varphi\left(\frac{\left|u^{(k)}(t)\right|}{(k-1)!}\right)\mathrm{d}t.$$

Thus, we get

$$\hat{V}^R_{(\varphi,k)}(u,[a,b])\leq\int_a^b\varphi\left(\frac{\left|u^{(k)}(t)\right|}{(k-1)!}\right)\mathrm{d}t,$$

which completes the proof. □

Corollary 3.1. *Let φ be a convex φ-function satisfying the ∞_1 condition and k a positive integer. Then,*

(i) *$u \in \hat{V}^R_{(\varphi,k)}[a,b]$ if and only if $\frac{u^{(k-r)}}{(k-r)!} \in \hat{V}^R_{(\varphi,r)}[a,b]$, $r = 1,\dots,k-1$, and*

$$\hat{V}^R_{(\varphi,r)}\left(\frac{(r-1)!}{(k-1)!}u^{(k-r)}\right) = \hat{V}^R_{(\varphi,k)}(u).$$

(ii) *$u \in RV_{(\varphi,k)}[a,b]$ if and only if $u^{(k-r)} \in RV_{(\varphi,r)}[a,b]$, $\quad r = 1,\dots k-1$.*

Corollary 3.2. *Let φ be a convex φ-function satisfying the ∞_1 condition and k a positive integer. Then,*

(i) $\bigcap_{k=1}^{\infty} RV_{(\varphi,k)}[a,b] = C_\infty[a,b]$,
(ii) $\bigcup_{k=1}^{\infty} RV_{(\varphi,k)}[a,b] = RV_{(\varphi,1)}[a,b]$.

Proposition 3.2 ([7], [8])**.** *Let φ_1, φ_2 be φ-functions. Then,*

(i) *$L_{\varphi_1}[a,b]$ is a vector space if and only if φ_1 satisfies the condition $\triangle_2(\infty)$, that is, there exist numbers $\eta > 0$, $t_0 \geq 0$ such that*

$$\varphi(2t) \leq \eta\varphi(t), \qquad t \geq t_0. \tag{9}$$

(ii) *$L_{\varphi_1}[a,b] \subset L_{\varphi_2}[a,b]$ if and only if there exist numbers η, $t_0 > 0$ such that*

$$\varphi_2(t) \leq \eta\varphi_1(t), \qquad t \geq t_0.$$

Lemma 3.1. *Let φ_1 and φ_2 be φ-functions. Then, the following result holds.*

(i) $\limsup\limits_{t\to\infty} \frac{\varphi_1(t)}{\varphi_2(t)}$ *is finite if there are $\eta > 0, t_0 > 0$, such that $\varphi_1(t) \leq \eta\varphi_2(t)$, $t \geq t_0$.*
(ii) $\limsup\limits_{t\to\infty} \frac{\varphi_1(t)}{\varphi_2(t)} = \infty$, *then there are $\eta > 0, t_0 > 0$, such that $\varphi_2(t) \leq \eta\varphi_1(t)$, $t \geq t_0$.*

Theorem 3.2. *Let k be a positive integer, and let φ_1 and φ_2 be convex φ-functions. Then,*

$$\hat{V}^R_{(\varphi_1,k)}[a,b] \subset \hat{V}^R_{(\varphi_2,k)}[a,b]$$

if and only if there are $\eta > 0$, $t_0 > 0$, such that

$$\varphi_2(t) \leq \eta\varphi_1(t), \qquad t \geq t_0.$$

P r o o f. Suppose first that there are $\eta > 0$, $t_0 > 0$, such that $\varphi_2(t) \leq \eta\varphi_1(t)$, $t \geq t_0$. By Proposition 3.2, we know that $L_{\varphi_1}[a, b] \subset L_{\varphi_2}[a, b]$. If $u \in \hat{V}^R_{(\varphi_1,k)}[a, b]$, then by Theorem 3.1 we have

$$u^{(k-1)} \in AC[a, b] \quad \text{and} \quad \frac{u^{(k)}}{(k-1)!} \in L_{\varphi_1}[a, b] \subset L_{\varphi_2}[a, b].$$

Using Theorem 3.1 again, we conclude that $u \in \hat{V}^R_{(\varphi_2,k)}[a, b]$.

On the other hand, if $\hat{V}^R_{(\varphi_1,k)}[a, b] \subset \hat{V}^R_{(\varphi_2,k)}[a, b]$, then there are $\eta > 0$, $t_0 > 0$, such that $\varphi_1(t) \leq \eta$, $\varphi_2(t)$, $t \geq t_0$. Applying reasoning as in the first part of this proof, we obtain the inclusion $\hat{V}^R_{(\varphi_2,k)}[a, b] \subset \hat{V}^R_{(\varphi_1,k)}[a, b]$, which is a contradiction. Therefore, the claimed inequality is established. $\square$

Corollary 3.3. *Let k be a positive integer, and let φ_1 and φ_2 be φ-functions. Then, $RV_{(\varphi_1,k)}[a, b] \subset RV_{(\varphi_2,k)}[a, b]$ if and only if the relation (8) is satisfied.*

Theorem 3.3. *Let k be a positive integer, and let φ be a φ-function. Then, $\hat{V}^R_{(\varphi,k)}[a, b]$ is a vector space if and only if φ satisfies the $\Delta_2(\infty)$ condition.*

P r o o f. Suppose that φ satisfies the $\triangle_2(\infty)$ condition. Then, by Proposition 3.2, $L_\varphi[a, b]$ is a vector space. If $u, v \in \hat{V}^R_{(\varphi_1,k)}[a, b]$, $\alpha, \beta \in \mathbb{R}$, then by Theorem 3.1,

$$u^{(k-1)}, \quad v^{(k-1)} \in AC[a, b] \quad \text{and} \quad \frac{u^{(k)}}{(k-1)!}, \frac{v^{(k)}}{(k-1)!} \in L_\varphi[a, b].$$

Since $AC[a, b]$ and $L_\varphi[a, b]$ are vector spaces, we get that

$$(\alpha u + \beta v)^{(k-1)} \in AC[a, b] \quad \text{and} \quad \frac{(\alpha u + \beta v)^{(k-1)}}{(k-1)!} \in L_\varphi[a, b].$$

So, by Theorem 3.1, we get the result.

Conversely, suppose that $V_\varphi[a, b]$ is a vector space. Then, if we consider the convex φ-function $\varphi_1(t) = \varphi(2t)$, $t \geq 0$, we obtain the inclusion

$$V_\varphi[a, b] \subset V_{\varphi_1}[a, b].$$

From Theorem 3.2, we conclude that φ satisfies the $\triangle_2(\infty)$ condition. $\square$

Since the set

$$A = \left\{u \in RV_{(\varphi,k)}[a, b] \ : \ \exists\lambda > 0 \colon \hat{V}^R_{(\varphi,k)}(\frac{u}{\lambda}) \leq 1\right\}$$

is balanced and absorbing, the Minkowski functional associated with A given by the formula

$$\mu_A(u) = \inf\left\{\lambda > 0 : \hat{V}^R_{(\varphi,k)}\left(\frac{u}{\lambda}\right) \leq 1\right\} = \inf\left\{\lambda > 0 : \int\limits_a^b \varphi\left(\frac{|u^{(k)}(t)|}{\lambda(k-1)!}\mathrm{d}t\right) < 1\right\}$$

is a seminorm on $R\hat{V}_{(\varphi,k)}[a, b]$, and, by Theorem 2.4 (iii), is a seminorm on $RV_{(\varphi,k)}[a, b]$.

4. A Banach space of functions of bounded (φ, k)-variation

Lemma 4.1. *Let φ be a φ-function, k a positive integer, and $u\colon [a,b] \to X$. Then, the following holds.*

(i) *If $\mu_A(u) \neq 0$, then $\hat{V}^R_{(\varphi,k)}\left(\frac{u}{\mu_A(u)}\right) \leq 1$.*

(ii) *For $\lambda > 0$, we have: $\lambda \geq \mu_A(u)$ if, and only if, $\hat{V}^R_{(\varphi,k)}\left(\frac{u}{\lambda}\right) \leq 1$.*

(iii) *If $0 \leq \mu_A(u) \leq 1$, then $\hat{V}^R_{(\varphi,k)}(u) \leq \mu_A(u)$.*

Proof. (i) Let $\lambda_n > 0$ be such that $\lambda_n > \lambda = \mu_A(u)$, $\hat{V}^R_{(\varphi,k)}\left(\frac{u}{\lambda_n}\right) \leq 1$, $n \in \mathbb{N}$, and λ_n converges to $\mu_A(u)$ as $n \to \infty$. Since $\frac{u}{\lambda_n}$ pointwise converges to $\mu_A(u)$ as $n \to \infty$, by the lower semicontinuity of the functional $\hat{V}^R_{(\varphi,k)}(\cdot)$, we get

$$1 \geq \lim_{n\to\infty} \hat{V}^R_{(\varphi,k)}\left(\frac{u}{\lambda_n}\right) = \hat{V}^R_{(\varphi,k)}\left(\frac{u}{\mu_A(u)}\right).$$

(ii) It is sufficient to show that if $0 < \mu_A(u) < \lambda$, then $\mu_A\left(\frac{u}{\lambda}\right) < 1$. By the convexity of functional $\mu_A(\cdot)$ and of part (i), we have

$$\hat{V}^R_{(\varphi,k)}\left(\frac{u}{\lambda}\right) \leq \frac{\mu_A(u)}{\lambda}\hat{V}^R_{(\varphi,k)}\left(\frac{u}{\mu_A(u)}\right) \leq \frac{\mu_A(u)}{\lambda} < 1,$$

which completes the proof.

(iii) For $0 < \lambda < 1$, by the convexity of functional $\mu_A(\cdot)$, we have

$$\frac{1}{\lambda}\hat{V}^R_{(\varphi,k)}(u) \leq \hat{V}^R_{(\varphi,k)}\left(\frac{u}{\lambda}\right) \leq 1.$$

Thus, $\hat{V}^R_{(\varphi,k)}(u) \leq \lambda$, and therefore $\hat{V}^R_{(\varphi,k)}(u)$ is a lower bound of

$$\left\{\lambda : \hat{V}^R_{(\varphi,k)}\left(\frac{u}{\lambda}\right) \leq 1\right\}.$$

As a consequence, we have $\hat{V}^R_{(\varphi,k)}(u) \leq \mu_A(u)$.

□

Corollary 4.1. *Let φ be a φ-function, k a positive integer, and $u\colon [a,b] \to \mathbb{R}$ such that $\mu_A(u) = 0$. Then, $u^{(k-1)}$ is constant.*

Proof. By property (iii) of Lemma 4.1, we see that $\hat{V}^R_{(\varphi,k)}(u) = 0$, so

$$\int_a^b \varphi\left(\frac{|u^{(k)}(t)|}{(k-1)!}\right)\mathrm{d}t = 0,$$

and therefore,

$$\varphi\left(\frac{u^{(k)}(t)}{(k-1)!}\right)=0$$

almost everywhere. This shows that $u^{(k)}(t)=0$ almost everywhere. Since $u^{(k-1)}$ is absolutely continuous, $u^{(k-1)}$ is constant. □

Consider the function $\|\,.\,\|: RV_{(\varphi,k)}[a,b]\to\mathbb{R}$ defined by

$$\|u\|^{R}_{(\varphi,k)}:=|u(a)|+|u'(a)|+\ldots+\left|u^{(k-1)}(a)\right|+\inf\left\{\lambda>0:\hat{V}^{R}_{(\varphi,k)}\left(\frac{u}{\lambda}\right)\le 1\right\}.$$

By standard properties of the Minkowski functional and the above corollary, we get that $\|\,.\,\|^{R}_{(\varphi,k)}$ is a norm on $RV_{(\varphi,k)}[a,b]$.

Theorem 4.1. *The space* $\left(RV_{(\varphi,k)}[a,b],\|\,.\,\|^{R}_{(\varphi,k)}\right)$ *is a Banach space.*

Proof. Let $\{u_n\}_{n\ge1}$ be a Cauchy sequence in $\left(RV_{(\varphi,k)}[a,b],\|\,.\,\|^{R}_{(\varphi,k)}\right)$. Thus, for every $\epsilon>0$, we can choose $N>0$ such that

$$|(u_n-u_m)(a)|\le\varepsilon,\qquad \left|(u_n^{(i)}-u_m^{(i)})(a)\right|\le\varepsilon,\qquad i=1\ldots,k-1,\tag{10}$$

and

$$\mu_A(u_n-u_m)\le\varepsilon,\qquad n,m\ge N.\tag{11}$$

By (10), it follows that

$$\{u_n(a)\}_{n\ge1}\quad\text{and}\quad\left\{u_n^{(i)}(a)\right\}_{n\ge1},\quad i=1\ldots,k-1,$$

are Cauchy sequences in $\mathbb{R}$, because they converge. Applying 4.1 (ii) and Corollary 3.1 (i), by (11), we have

$$\hat{V}^{R}_{(\varphi,1)}\left(\frac{u_n^{(k-1)}-u_m^{(k-1)}}{(k-1)!\quad\varepsilon}\right)=\hat{V}^{R}_{(\varphi,k)}\left(\frac{u_n-u_m}{\varepsilon}\right)\le 1,\tag{12}$$

for $n,m>N$, whence

$$\varphi\left(\frac{\left|(u_n^{(k-1)}-u_m^{(k-1)})(t)-(u_n^{(k-1)}-u_m^{(k-1)})(a)\right|}{\varepsilon(k-1)!\,|t-a|}\right)|t-a|\le 1,$$

for all $n,m>N$ and $t\in(a,b]$.

Since φ satisfies ∞_1 condition, by (2),

$$\lim_{s\to0+}s\varphi^{-1}\left(\frac{1}{s}\right)=\lim_{r\to\infty}\frac{r}{\varphi(r)}=0,$$

therefore, there exists $M > 0$ such that

$$|t-a|\,\varphi^{-1}\left(\frac{1}{|t-a|}\right) \leq M, \qquad t \in (a,b],$$

and, consequently, by (10), for $i = k-1$,

$$\left|u_n^{(k-1)}(t) - u_m^{(k-1)}(t)\right| \leq (1+M)(k-1)!\,\varepsilon, \qquad n,m > N, \quad t \in [a,b].$$

Thus,

$$\left\{u_n^{(k-1)}\right\}_{n\geq 1}$$

satisfies a uniform Cauchy condition which together with convergence

$$\left\{u_n^{(k-2)}(a)\right\}_{n\geq 1}$$

gives a uniform convergence of

$$\left\{u_n^{(k-2)}\right\}_{n\geq 1}.$$

Repeating this procedure, after $(k-1)$ steps, we get the existence of $(k-1)$-continuously differentiable function $u\colon [a,b] \to \mathbb{R}$ such that

$$u_n \rightrightarrows u \quad \text{and} \quad u_n^{(i)} \rightrightarrows u^{(i)}, \quad i = 1\ldots,k-1.$$

Therefore, taking into account (12), and by the lower semicontinuity of functional $\hat{V}^R_{(\varphi,1)}(\cdot)$, we get

$$\hat{V}^R_{(\varphi,1)}\left(\frac{u_n^{(k-1)} - u^{(k-1)}}{(k-1)! \quad \varepsilon}\right) \leq \lim_{m\to\infty} \hat{V}^R_{(\varphi,1)}\left(\frac{u_n^{(k-1)} - u_m^{(k-1)}}{(k-1)! \quad \varepsilon}\right) \leq 1,$$

for all $n,m > N$ and, consequently, by Corollary 3.1 (i),

$$\hat{V}^R_{(\varphi,k)}\left(\frac{u_n - u}{\varepsilon}\right) \leq 1, \quad n > N.$$

Hence, $u_n - u \in RV_{(\varphi,k)}[a,b]$, $n \in \mathbb{N}$, and, by Lemma 4.1 (ii),

$$\mu_A\left(u_n - u\right) \leq \varepsilon. \tag{13}$$

Thus, $u \in RV_{(\varphi,k)}[a,b]$, as $RV_{(\varphi,k)}[a,b]$ is a vector space, and by (10), uniform convergence $\{u_n\}_{n\geq 1}$ to u and (13), the sequence $\{u_n\}_{n\geq 1}$ converges to u in the norm $\|\cdot\|^R_{(\varphi,k)}$. $\square$

Theorem 4.2. *The space $RV_{(\varphi,k)}[a,b]$ is an algebra.*

P r o o f. For $u, v \in RV_{(\varphi,k)}[a,b]$, we have

$$(uv)^{(k-1)} = \sum_{j=0}^{k-1} \binom{k-1}{j} u^{(k-1-j)} v^{(j)}$$

and

$$u^{(k-1-j)}, v^{(j)} \in RV_{(\varphi,1)}[a,b], \qquad j = 0, \dots, k-1,$$

by Corollary 3.1. Since $RV_{(\varphi,1)}[a,b]$ is an algebra, $(uv)^{(k-1)} \in RV_{(\varphi,1)}[a,b]$, and so,

$$uv \in RV_{(\varphi,k)}[a,b].$$

□

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