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ON ONE APPLICATION OF INFINITE SYSTEMS OF FUNCTIONAL EQUATIONS IN FUNCTION THEORY

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ABSTRACT. The paper presents the investigation of applications of infinite systems of functional equations for modeling functions with complicated local structure that are defined in terms of the nega-$\tilde{Q}$-representation. The infinite systems of functional equations

$$f\left(\hat{\varphi}^k(x)\right)=\widetilde{\beta}_{i_{k+1},k+1}+\widetilde{p}_{i_{k+1},k+1}f\left(\hat{\varphi}^{k+1}(x)\right),$$

where $k=0,1,\dots$, $x=\Delta^{-\widetilde{Q}}_{i_1(x)i_2(x)\dots i_n(x)\dots}$, and $\hat{\varphi}$ is the shift operator of the $\widetilde{Q}$-expansion, are investigated. It is proved that the system has a unique solution in the class of determined and bounded on $[0,1]$ functions. Its analytical presentation is founded. The continuity of the solution is studied. Conditions of its monotonicity and nonmonotonicity, differential, and integral properties are studied. Conditions under which the solution of the system of functional equations is a distribution function of the random variable $\eta=\Delta^{\widetilde{Q}}_{\xi_1\xi_2\dots\xi_n\dots}$ with independent $\widetilde{Q}$-symbols are founded.

1. Introduction

Nowadays, it is well-known that functional equations and systems of functional equations are widely used in mathematics and other sciences. For example, in the information theory, physics, economics, decision theory, etc. [1,2,5,7]. Modeling functions with complicated local structure by systems of functional equations is a shining example of their applications in function theory.

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The class of functions with complicated local structure consists of singular (for example, [13, 22, 23, 30, 31]), continuous nowhere monotonic [10], and nowhere differentiable functions (for example, [3,14,21,25,26,29,30]).

An example of a strictly increasing singular function is the following function, that is called the Salem function

$$f(x) = \beta_{\alpha_1(x)} + \sum_{n=1}^{\infty} \left(\beta_{\alpha_n(x)} \prod_{j=1}^{n-1} p_{\alpha_j(x)} \right),$$

where

$$x = \Delta^2_{\alpha_1\alpha_2\ldots\alpha_n\ldots} \equiv \sum_{n=1}^{\infty} \frac{\alpha_n}{2^n}, \qquad \alpha_n \in \{0,1\},$$

$Q_2 = \{p_0, p_1\}$ is a fixed tuple of integers such that $p_0 + p_1 = 1$, and

$$\beta_{\alpha_n(x)} = \begin{cases} 0 & \text{whenever } \alpha_n(x) = 0 \\ p_0 & \text{whenever } \alpha_n(x) = 1. \end{cases}$$

Properties of the function (including the singularity) were studied by Salem [13] and by other authors [6,8,30]. The Salem function is a distribution function of a random variable with independent identically distributed binary digits and a unique solution of the following system of functional equations in the class of determined and bounded on $[0,1]$ functions:

$$f\left(\frac{x}{2}\right) = p_0 f(x), \qquad f\left(\frac{x+1}{2}\right) = p_0 + p_1 f(x).$$

The system can be written as follows (for functions determined on the segment $[0;1]$):

$$f\left(\Delta^2_{i\alpha_1\alpha_2\ldots\alpha_n\ldots}\right) = \beta_i + p_i f\left(\Delta^2_{\alpha_1\alpha_2\ldots\alpha_n\ldots}\right).$$

In [9], the following generalization of the Salem function is investigated:

$$f(x) = \beta_{\alpha_1(x)} + \sum_{n=2}^{\infty} \left(\beta_{\alpha_n(x)} \prod_{j=1}^{n-1} p_{\alpha_j(x)} \right),$$

where

$$x = \Delta^s_{\alpha_1\alpha_2\ldots\alpha_n\ldots} \equiv \sum_{n=1}^{\infty} \frac{\alpha_n}{s^n}, \qquad \alpha_n \in \{0,1,\ldots,s-1\},$$

$2 < s$ is a fixed positive integer, and

$$\beta_{\alpha_n(x)} = \begin{cases} 0 & \text{whenever } \alpha_n(x) = 0 \\ \sum_{i=0}^{\alpha_n(x)-1} p_i > 0 & \text{whenever } \alpha_n(x) \neq 0. \end{cases}$$

The last-mentioned function is a unique solution of the following system of functional equations in the class of determined and bounded on $[0,1]$ functions:

$$f\left(\frac{i+x}{s}\right) = \beta_i + p_i f(x),$$

where $i = 0, 1, \dots, s-1$, p_i is a real number from $(-1,1)$, and $p_0 + p_1 + \dots + p_{s-1} = 1$.

In [10], the following system of s functional equations is considered:

$$f\left(\Delta^Q_{i\alpha_1\alpha_2\dots\alpha_n\dots}\right) = \beta_i + p_i f\left(\Delta^Q_{\alpha_1\alpha_2\dots\alpha_n\dots}\right), \qquad i = \overline{0, s-1},$$

where $\max_i |p_i| < 1$, $p_0 + p_1 + \dots + p_{s-1} = 1$, $\beta_0 = 0 < \beta_k = \sum_{i=0}^{k-1} p_i$, and the argument of f is represented in terms of the Q-representation [11, p. 87]. The Q-representation is a generalization of the s-adic representation $\Delta^s_{\alpha_1\alpha_2\dots\alpha_n\dots}$. That is,

$$\Delta^Q_{\alpha_1\alpha_2\dots\alpha_n\dots} \equiv \gamma_{\alpha_1} + \sum_{n=2}^{\infty}\left(\gamma_{\alpha_n}\prod_{j=1}^{n-1} q_{\alpha_j}\right) = x \in [0,1],$$

where $1 < s$ is a fixed positive integer, $Q = \{q_0, q_1, \dots, q_{s-1}\}$ is a fixed tuple of real numbers such that $q_i > 0$ for all $i = \overline{0, s-1}$ and $\sum_{i=0}^{s-1} q_i = 1$. Also, here $\gamma_0 = 0$ and $\gamma_k = \sum_{j=0}^{k-1} q_j$ for $k = 1, 2, \dots, s-1$. The investigations from [10] are generalization of the investigations from [9].

Introducing and investigations of generalizations of the Salem function are new and unknown for real numbers representations $\Delta_{\delta_1\delta_2\dots\delta_n\dots}$ with the removable alphabet. That is, when $\delta_n \in A^0_{t_n} \equiv \{0, 1, \dots, t_n\}$, $t_n \in \mathbb{N}$, and $|A^0_{t_k}| \neq |A^0_{t_l}|$ for $k \neq l$, where $|\cdot|$ is the number of elements of the set. Representations of real numbers by positive [4, 12, 15] and alternating [16, 26] Cantor series, the $\widetilde{Q}$- [11, p. 87–91] and the nega-$\widetilde{Q}$-representation [18, 28] are these representations. For the first time, such investigations were carried out by the author of the present paper for the case of positive Cantor series [17, 19] and presented in April 2014 at the international mathematical conference “Differential equations, computational mathematics, function theory and mathematical methods in mechanics” [17].

In October 2014, the results of the present paper and of the similar research for the case of positive and alternating Cantor series (the papers [17–20, 24, 28]) were presented by the author in reports “Determination of a class of functions, that are represented by Cantor series, by systems of functional equations” and “Polybasic positive and alternating $\widetilde{Q}$-representations and their applications to determination of functions by systems of functional equations” at the seminar on fractal analysis of the Institute of Mathematics of NAS of Ukraine and of the National Pedagogical Dragomanov University (the archive of reports is available at `http://www.imath.kiev.ua/events/index.php?seminarId=21&archiv=1`).

The present paper is devoted to study of one example of applications of systems of functional equations to modeling of functions with complicated local structure. The following new and non-investigated at this moment system of functional equations is studied:

$$f\left(\hat{\varphi}^k(x)\right) = \widetilde{\beta}_{i_{k+1},k+1} + \widetilde{p}_{i_{k+1},k+1} f\left(\hat{\varphi}^{k+1}(x)\right),$$

where $k = 0, 1, \dots$, $\hat{\varphi}$ is the shift operator of the $\widetilde{Q}$-expansion, and

$$x = \Delta^{-\widetilde{Q}}_{i_1(x)i_2(x)\dots i_n(x)\dots} \equiv \Delta^{\widetilde{Q}}_{i_1(x)[m_2-i_2(x)]i_3\dots i_{2k-1}(x)[m_{2k}-i_{2k}(x)]\dots}.$$

The properties of the unique solution to the last-mentioned system in the class of determined and bounded on [0,1] functions

$$F(x) = \beta_{i_1(x),1} + \sum_{k=2}^{\infty}\left[\widetilde{\beta}_{i_k(x),k}\prod_{j=1}^{k-1}\widetilde{p}_{i_j(x),j}\right]$$

are investigated.

2. $\widetilde{Q}$-representation, its partial cases and the shift operator

Let $\widetilde{Q} = \|q_{i,n}\|$ be a fixed matrix, where $i = \overline{0, m_n}$, $m_n \in N^0_\infty = \mathbb{N} \cup \{0, \infty\}$, $n = 1, 2, \dots$, and the following system of properties is true for elements $q_{i,n}$ of the last-mentioned matrix

1°. $q_{i,n} > 0$;

2°. for all $n \in \mathbb{N} : \sum_{i=0}^{m_n} q_{i,n} = 1$;

3°. for any $(i_n), i_n \in \mathbb{N} \cup \{0\} : \prod_{n=1}^{\infty} q_{i_n,n} = 0$.

Definition 1. An expansion of a number x from $[0, 1)$ by the following positive series

$$a_{i_1(x),1} + \sum_{n=2}^{\infty}\left[a_{i_n(x),n}\prod_{j=1}^{n-1} q_{i_j(x),j}\right], \qquad (1)$$

where

$$a_{i_n,n} = \begin{cases} \sum\limits_{i=0}^{i_n-1} q_{i,n} & \text{whenever } i_n \neq 0 \\ 0 & \text{whenever } i_n = 0, \end{cases}$$

is called *the $\widetilde{Q}$-expansion of a number x* [11, p. 89]. Defining an arbitrary number $x \in [0, 1)$ by expansion (1) is denoted by $x = \Delta^{\widetilde{Q}}_{i_1 i_2 \dots i_n \dots}$ and the last-mentioned notation is called *the $\widetilde{Q}$-representation of x.*

Definition 2. A number $x \in [0,1)$ which has the period (0) in its own $\widetilde{Q}$--representation is called $\widetilde{Q}$-rational, i.e.,

$$\Delta^{\widetilde{Q}}_{i_1 i_2 \ldots i_{n-1} i_n (0)}.$$

The other numbers in $[0,1]$ are called $\widetilde{Q}$-irrational.

Proposition 1. *If the condition $m_n < \infty$ holds for all $n > n_0$, where n_0 is a fixed integer, then each $\widetilde{Q}$-rational number has two different $\widetilde{Q}$-representations, i.e.,*

$$\Delta^{\widetilde{Q}}_{i_1 i_2 \ldots i_{n-1} i_n (0)} = \Delta^{\widetilde{Q}}_{i_1 i_2 \ldots i_{n-1} [i_n - 1] m_{n+1} m_{n+2} \ldots}.$$

The $\widetilde{Q}$-representation of real numbers is:

- the Q^*-representation (or Q^*_s-representation) when the equality $m_n = s-1$ is true for all $n \in \mathbb{N}$, where $\mathbb{N} \ni s = \text{const} > 1$;
- the Q-representation (or Q_s-representation) when the equality $m_n = s-1$ is true for an arbitrary $n \in \mathbb{N}$, where $\mathbb{N} \ni s = const > 1$ and $q_{i,n} = q_i$ for all $n \in \mathbb{N}$;
- the Q^*_∞-representation whenever the condition $m_n = \infty$ holds for any $n \in \mathbb{N}$;
- the Q_∞-representation whenever the conditions $m_n = \infty$ and $q_{i,n} = q_i$ hold for all $n \in \mathbb{N}$;
- the representation by a positive Cantor series whenever the conditions $m_n = d_n - 1$ and $q_{i,n} = \frac{1}{d_n}$, $i = \overline{0, d_n - 1}$, hold for any $n \in \mathbb{N}$, where (d_n) is a fixed sequence of positive integers and $d_n > 1$;
- the s-adic representation whenever the equalities $m_n = s-1$ and $q_{i,n} = q_i = \frac{1}{s}$ are true for all $n \in \mathbb{N}$, where $s > 1$ is a fixed positive integer.

Definition 3. The mapping defined by

$$\hat{\varphi}(x) = \hat{\varphi}\left(\Delta^{\widetilde{Q}}_{i_1 i_2 \ldots i_n \ldots}\right) = a_{i_2,2} + \sum_{k=3}^{\infty}\left[a_{i_k,k}\prod_{j=2}^{k-1} q_{i_j,j}\right]$$

is called *the shift operator $\hat{\varphi}$ of the $\widetilde{Q}$-expansion of a number x.*

The following mapping

$$\hat{\varphi}^k(x) = a_{i_{k+1},k+1} + \sum_{n=k+2}^{\infty}\left[a_{i_n,n}\prod_{j=k+1}^{n-1} q_{i_j,j}\right]$$

is called *the shift operator of rank k of the $\widetilde{Q}$-expansion of a number x.*

The last-mentioned and the following definitions of $\hat{\varphi}^k$ are equivalent.

Definition 4. Let $(\widetilde{Q}_k)$ be a sequence of the following matrixes $\widetilde{Q}_k$:

$$\widetilde{Q}_k = \begin{pmatrix} q_{0,k+1} & q_{0,k+2} & \dots & q_{0,k+j} & \dots \\ q_{1,k+1} & q_{1,k+2} & \dots & q_{1,k+j} & \dots \\ \vdots & \vdots & \ddots & \vdots & \dots \\ q_{m_{k+1}-1,k+1} & q_{m_{k+2}-2,k+2} & \dots & q_{m_{k+j},k+j} & \dots \\ q_{m_{k+1},k+1} & q_{m_{k+2}-1,k+2} & \dots & & \dots \\ & q_{m_{k+2},k+2} & \dots & & \dots \end{pmatrix},$$

where $k = 0, 1, 2, \dots,\ j = 1, 2, 3, \dots$

Let $\left(\mathcal{F}^{\widetilde{Q}_k}_{[0;1)}\right)$ be a sequence of sets $\mathcal{F}^{\widetilde{Q}_k}_{[0,1)}$ of all possible $\widetilde{Q}_k$-representations (of numbers from $[0,1)$) generated by the matrix $\widetilde{Q}_k$, where $\widetilde{Q}_0 \equiv \widetilde{Q}$.

The mapping $\pi\left(\hat{\varphi}^k(x, \widetilde{Q})\right)$ such that

$$\varphi^k\colon [0,1)\times\widetilde{Q}\to[0,1)\times\widetilde{Q}_k;\ \left((i_1, i_2, \dots, i_k, \dots), \widetilde{Q}\right)\to\left((i_{k+1}, i_{k+2}, i_{k+j}, \dots), \widetilde{Q}_k\right),$$

$$\pi : [0,1)\times\widetilde{Q}_k\to[0,1); \qquad (x', \widetilde{Q}_k) \to x'$$

is called *the shift operator $\hat{\varphi}^k$ of rank k of the $\widetilde{Q}$-representation of a number $x \in [0,1)$*.

Remark 1. For the compactness of the presentation of the paper, we will use the notation $\hat{\varphi}^k$ instead of the notation $\pi\left(\hat{\varphi}^k(x, \widetilde{Q})\right)$ in this paper.

Since $x = a_{i_1,1} + q_{i_1,1}\hat{\varphi}(x)$, we have

$$\hat{\varphi}(x) = \frac{x - a_{i_1,1}}{q_{i_1,1}}.$$

It is easy to see that the following expressions are true.

$$\hat{\varphi}^k(x) = \frac{1}{q_{0,1}q_{0,2}\cdots q_{0,k}}\Delta^{\widetilde{Q}}_{\underbrace{0\dots0}_{k}i_{k+1}i_{k+2}\dots}.$$

$$\hat{\varphi}^{k-1}(x) = a_{i_k,k} + q_{i_k,k}\hat{\varphi}^k(x) = \frac{1}{q_{0,1}q_{0,2}\cdots q_{0,k-1}}\Delta^{\widetilde{Q}}_{\underbrace{0\dots0}_{k-1}i_k i_{k+1}\dots}. \tag{2}$$

Remark. Throughout the paper, we will consider that $m_n < \infty$ for all $n \in \mathbb{N}$.

3. Nega-$\widetilde{Q}$-representation of real numbers from $[0,1]$

Theorem 1 ([18, 24]). *For an arbitrary $x \in [0,1)$, there exists a sequence (i_n), $i_n \in N^0_{m_n}$, such that*

$$x = \sum_{i=0}^{i_1-1} q_{i,1} + \sum_{n=2}^{\infty}\left[(-1)^{n-1}\widetilde{\delta}_{i_n,n}\prod_{j=1}^{n-1}\widetilde{q}_{i_j,j}\right] + \sum_{n=1}^{\infty}\left(\prod_{j=1}^{2n-1}\widetilde{q}_{i_j,j}\right), \qquad (3)$$

where

$$\widetilde{\delta}_{i_n,n} = \begin{cases} 1 & \text{if } n \text{ is even and } i_n = m_n \\ \sum\limits_{i=m_n-i_n}^{m_n} q_{i,n} & \text{if } n \text{ is even and } i_n \neq m_n \\ 0 & \text{if } n \text{ is odd and } i_n = 0 \\ \sum\limits_{i=0}^{i_n-1} q_{i,n} & \text{if } n \text{ is odd and } i_n \neq 0, \end{cases}$$

and the first sum in expression (3) *equals* 0 *whenever $i_1 = 0$.*

Definition 5. An expansion of a number x by series (3) is called *the nega-$\widetilde{Q}$-expansion of x* and is denoted by $\Delta^{-\widetilde{Q}}_{i_1i_2\ldots i_n\ldots}$. The last-mentioned notation is called *the nega-$\widetilde{Q}$-representation of a number x* ([18]).

The numbers from a countable subset of $[0,1]$ have two different nega-$\widetilde{Q}$--representations, i.e.,

$$\Delta^{-\widetilde{Q}}_{i_1i_2\ldots i_{n-1}i_n m_{n+1}0m_{n+3}0m_{n+5}\ldots} = \Delta^{-\widetilde{Q}}_{i_1i_2\ldots i_{n-1}[i_n-1]0m_{n+2}0m_{n+4}\ldots}, \; i_n \neq 0.$$

These numbers are called *nega-$\widetilde{Q}$-rational* and the rest of the numbers from $[0,1]$ are called *nega-$\widetilde{Q}$-irrational*.

Definition 6. The set of all numbers from $[0,1]$ such that the first n digits $i_1, i_2, \ldots, i_n$ of the nega-$\widetilde{Q}$-representation of the numbers are equal to c_1, $c_2, \ldots, c_n$, respectively, is called *a cylinder $\Delta^{-\widetilde{Q}}_{c_1c_2\ldots c_n}$ of rank n with the base* $c_1c_2\ldots c_n$. That is,

$$\Delta^{-\widetilde{Q}}_{c_1c_2\ldots c_n} \equiv \left\{x : [0,1] \ni x = \Delta^{-\widetilde{Q}}_{c_1c_2\ldots c_n i_{n+1}i_{n+2}\ldots i_{n+k}\ldots}, i_{n+k} \in N^0_{m_{n+k}}, k \in \mathbb{N}\right\},$$

where $c_1, c_2, \ldots, c_n$ is a fixed tuple of symbols from $N^0_{m_1}, N^0_{m_2}, \ldots, N^0_{m_n}$, respectively.

By ([18]), it follows that the following statements are true.

Lemma 1. *Cylinders* $\Delta^{-\widetilde{Q}}_{c_1c_2\ldots c_n}$ *have the following properties:*

(1) *a cylinder* $\Delta^{-\widetilde{Q}}_{c_1c_2\ldots c_n}$ *is the following closed interval:*

$$\Delta^{-\widetilde{Q}}_{c_1c_2\ldots c_n} = \left[\Delta^{-\widetilde{Q}}_{c_1c_2\ldots c_n m_{n+1}0m_{n+3}0m_{n+5}\ldots}, \quad \Delta^{-\widetilde{Q}}_{c_1c_2\ldots c_n 0m_{n+2}0m_{n+4}0m_{n+6}\ldots}\right] \quad \textit{if } n \textit{ is odd,}$$

$$\Delta^{-\widetilde{Q}}_{c_1c_2\ldots c_n} = \left[\Delta^{-\widetilde{Q}}_{c_1c_2\ldots c_n 0m_{n+2}0m_{n+4}0m_{n+6}\ldots}, \quad \Delta^{-\widetilde{Q}}_{c_1c_2\ldots c_n m_{n+1}0m_{n+3}0m_{n+5}\ldots}\right] \quad \textit{if } n \textit{ is even;}$$

(2) *for any* $n \in \mathbb{N}$,

$$\Delta^{-\widetilde{Q}}_{c_1c_2\ldots c_n c} \subset \Delta^{-\widetilde{Q}}_{c_1c_2\ldots c_n};$$

(3) *for all* $n \in \mathbb{N}$,

$$\Delta^{-\widetilde{Q}}_{c_1c_2\ldots c_n} = \bigcup_{c=0}^{m_{n+1}} \Delta^{-\widetilde{Q}}_{c_1c_2\ldots c_n c};$$

(4)

$$\sup \Delta^{-\widetilde{Q}}_{c_1c_2\ldots c_{n-1}c} = \inf \Delta^{-\widetilde{Q}}_{c_1c_2\ldots c_{n-1}[c+1]} \quad \textit{if } n \textit{ is odd},$$

$$\sup \Delta^{-\widetilde{Q}}_{c_1c_2\ldots c_{n-1}[c+1]} = \inf \Delta^{-\widetilde{Q}}_{c_1c_2\ldots c_{n-1}c} \quad \textit{if } n \textit{ is even};$$

(5) *for any* $n \in \mathbb{N}$,

$$|\Delta^{-\widetilde{Q}}_{c_1c_2\ldots c_n}| = \prod_{j=1}^{n} \widetilde{q}_{c_j,j};$$

(6) *for an arbitrary* $x \in [0,1]$,

$$\bigcap_{n=1}^{\infty} \Delta^{-\widetilde{Q}}_{c_1c_2\ldots c_n} = x = \Delta^{-\widetilde{Q}}_{c_1c_2\ldots c_n\ldots}.$$

Lemma 2. *For the nega-*$\widetilde{Q}$*-representation, the following identities are true:*

$$\Delta^{-\widetilde{Q}}_{i_1i_2\ldots i_n\ldots} \equiv \Delta^{\widetilde{Q}}_{i_1[m_2-i_2]\ldots i_{2k-1}[m_{2k}-i_{2k}]\ldots},$$
$$\Delta^{\widetilde{Q}}_{i_1i_2\ldots i_n\ldots} \equiv \Delta^{-\widetilde{Q}}_{i_1[m_2-i_2]\ldots i_{2k-1}[m_{2k}-i_{2k}]\ldots}.$$

That is

$$x = \Delta^{-\widetilde{Q}}_{i_1i_2\ldots i_n\ldots} \equiv a_{i_1,1} + \sum_{n=2}^{\infty}\left[\widetilde{a}_{i_n,n}\prod_{j=1}^{n-1}\widetilde{q}_{i_j,j}\right],$$

where

$$\widetilde{a}_{i_n,n} = \begin{cases} a_{i_n,n} & \text{if } n \text{ is odd} \\ a_{m_n-i_n,n} & \text{if } n \text{ is even,} \end{cases} \qquad \widetilde{q}_{i_n,n} = \begin{cases} q_{i_n,n} & \text{if } n \text{ is odd} \\ q_{m_n-i_n,n} & \text{if } n \text{ is even.} \end{cases}$$

4. The main object of the research is an infinite system of functional equations

Let us have matrixes of the same dimension $\widetilde{Q} = \|q_{i,n}\|$ (properties of the last-mentioned matrix were considered earlier) and $P = \|p_{i,n}\|$, where $i = \overline{0, m_n}$, $m_n \in \mathbb{N} \cup \{0\}$, $n = 1, 2, \ldots$, and for elements $p_{i,n}$ of P the following system of conditions is true:

1°. $p_{i,n} \in (-1, 1)$;
2°. for all $n \in \mathbb{N} : \sum_{i=0}^{m_n} p_{i,n} = 1$;
3°. for any $(i_n), i_n \in \mathbb{N} \cup \{0\} : \prod_{n=1}^{\infty} |p_{i_n,n}| = 0$;
4°. for all $i_n \in \mathbb{N} : 0 < \sum_{i=0}^{i_n - 1} p_{i,n} < 1$.

Let us consider the infinite system of functional equations

$$f\left(\hat{\varphi}^k(x)\right) = \widetilde{\beta}_{i_{k+1},k+1} + \widetilde{p}_{i_{k+1},k+1} f\left(\hat{\varphi}^{k+1}(x)\right), \tag{4}$$

where $k = 0, 1, \ldots$, $\hat{\varphi}$ is the shift operator of the $\widetilde{Q}$-expansion,

$$x = \Delta^{-\widetilde{Q}}_{i_1(x)i_2(x)\ldots i_n(x)\ldots} \equiv \Delta^{\widetilde{Q}}_{i_1(x)[m_2 - i_2(x)]i_3\ldots i_{2k+1}(x)[m_{2k+2} - i_{2k+2}(x)]\ldots},$$

$$\widetilde{p}_{i_n,n} = \begin{cases} p_{i_n,n} & \text{if } n \text{ is odd} \\ p_{m_n - i_n,n} & \text{if } n \text{ is even,} \end{cases}$$

$$\beta_{i_n,n} = \begin{cases} \sum\limits_{i=0}^{i_n-1} p_{i,n} > 0 & \text{if } i_n \neq 0 \\ 0 & \text{if } i_n = 0, \end{cases} \qquad \widetilde{\beta}_{i_n,n} = \begin{cases} \beta_{i_n,n} & \text{if } n \text{ is odd} \\ \beta_{m_n - i_n,n} & \text{if } n \text{ is even.} \end{cases}$$

Since equality (2) is true, one can write system (4) as

$$f\Big(\widetilde{a}_{i_k,k} + \widetilde{q}_{i_k,k}\hat{\varphi}^k(x)\Big) = \widetilde{\beta}_{i_k,k} + \widetilde{p}_{i_k,k} f\left(\hat{\varphi}^k(x)\right), \tag{5}$$

where $k = 1, 2, \ldots$, $i \in N^0_{m_k}$.

Lemma 3. *The function*

$$F(x) = \beta_{i_1(x),1} + \sum_{k=2}^{\infty} \left[\widetilde{\beta}_{i_k(x),k} \prod_{j=1}^{k-1} \widetilde{p}_{i_j(x),j}\right] \tag{6}$$

is a well-defined function at an arbitrary point from $[0, 1]$.

P r o o f. Let x be a nega-$\widetilde{Q}$-irrational number. Series (6) is absolutely convergent. The last-mentioned statement follows from conditions

$$\widetilde{\beta}_{i_k(x),k} \in [0,1), \qquad |\widetilde{p}_{i_k(x),k}| < 1,$$

and from the convergence of the series

$$\beta_{i_1(x),1} + \sum_{k=2}^{\infty} \left[\widetilde{\beta}_{i_k(x),k} \prod_{j=1}^{k-1} |\widetilde{p}_{i_j(x),j}| \right].$$

The convergence is proved analogously to the proof of the convergence of series (1) when $\widetilde{p}_{i_k(x),k} \neq 0$ for all $k \in \mathbb{N}$.

Let x be a nega-$\widetilde{Q}$-rational number. Consider the difference

$$\Delta = F\big(\Delta^{-\widetilde{Q}}_{i_1 i_2 \ldots i_{n-1} i_n m_{n+1} 0 m_{n+3} 0 m_{n+5} \ldots}\big) - F\big(\Delta^{-\widetilde{Q}}_{i_1 i_2 \ldots i_{n-1} [i_n - 1] 0 m_{n+2} 0 m_{n+4} \ldots}\big).$$

Let n be even. Then

$$\begin{aligned}\Delta &= \left(\prod_{j=1}^{n-1} \widetilde{p}_{i_j,j}\right) \\ &\times \left(\beta_{m_n - i_n, n} - \beta_{m_n - i_n + 1, n} + p_{m_n - i_n, n} \left(\beta_{m_{n+1}, n+1} + \sum_{l=2}^{\infty} \beta_{m_{n+l}, n+l} \left[\prod_{j=n+1}^{n+l-1} p_{m_j, j}\right]\right)\right) \\ &= \left(\prod_{j=1}^{n-1} \widetilde{p}_{i_j,j}\right) \times (-p_{m_n - i_n, n} + p_{m_n - i_n, n}) = 0\,.\end{aligned}$$

If n is odd, then

$$\begin{aligned}\Delta = &\left(\prod_{j=1}^{n-1} \widetilde{p}_{i_j,j}\right) \left(\beta_{i_n, n} - \beta_{i_n - 1, n} - p_{i_n - 1, n} \left(\beta_{m_{n+1}, n+1} + \sum_{l=2}^{\infty} \beta_{m_{n+l}, n+l} \left[\prod_{j=n+1}^{n+l-1} p_{m_j, j}\right]\right)\right) \\ &= 0\,.\end{aligned}$$

□

Theorem 2. *Infinite system* (4) *of functional equations has a unique solution*

$$f(x) = \beta_{i_1(x),1} + \sum_{k=2}^{\infty} \left[\widetilde{\beta}_{i_k(x),k} \prod_{j=1}^{k-1} \widetilde{p}_{i_j(x),j} \right]$$

in the class of determined and bounded on $[0,1]$ *functions.*

P r o o f. Really, for an arbitrary $x = \Delta^{-\widetilde{Q}}_{i_1(x)i_2(x)\ldots i_k(x)\ldots}$ from $[0,1]$, we have

$$\begin{aligned} f(x) &= \beta_{i_1(x),1} + p_{i_1(x),1} f\Big(\hat{\varphi}(x)\Big) \\ &= \beta_{i_1(x),1} + p_{i_1(x),1}\Big(\beta_{m_2-i_2(x),2} + p_{m_2-i_2(x),2} f\left(\hat{\varphi}^2(x)\right)\Big) \\ &= \beta_{i_1(x),1} + \beta_{m_2-i_2(x),2} p_{i_1(x),1} \\ &\quad + p_{i_1(x),1} p_{m_2-i_2(x),2}\Big(\beta_{i_3(x),3} + p_{i_3(x),3} f\left(\hat{\varphi}^3(x)\right)\Big) \\ &= \cdots = \beta_{i_1(x),1} + \widetilde{\beta}_{i_2(x),2}\widetilde{p}_{i_1(x),1} + \widetilde{\beta}_{i_3(x),3}\widetilde{p}_{i_1(x),1}\widetilde{p}_{i_2(x),2} \\ &\quad + \cdots + \widetilde{\beta}_{i_k(x),k}\prod_{j=1}^{k-1}\widetilde{p}_{i_j(x),j} + \left(\prod_{j=1}^{k}\widetilde{p}_{i_j(x),j}\right) f\Big(\hat{\varphi}^k(x)\Big). \end{aligned}$$

Since $\hat{\varphi}^k(x) \in [0,1]$, for arbitraries $x \in [0,1]$ and $k \in Z_0$, the function f is determined at all points from $[0,1]$ and the function is bounded on the segment $[0,1]$ (i.e., there exists $M > 0$ such that for any $x \in [0,1]$: $|f(x)| \le M$) and the condition

$$\prod_{j=1}^{k}\widetilde{p}_{i_j(x),j} \le \prod_{j=1}^{k}|\widetilde{p}_{i_j(x),j}| \to 0 \qquad (k \to \infty)$$

holds, it follows that

$$\begin{aligned} f\Big(\Delta^{-\widetilde{Q}}_{i_1 i_2 \ldots i_k \ldots}\Big) &= \lim_{k\to\infty}\left(\beta_{i_1(x),1} + \sum_{n=2}^{k}\left[\widetilde{\beta}_{i_n(x),n}\prod_{j=1}^{n-1}\widetilde{p}_{i_j(x),j}\right]\right. \\ &\qquad\left. + \left(\prod_{j=1}^{k}\widetilde{p}_{i_j(x),j}\right) f\Big(\hat{\varphi}^k(x)\Big)\right) \\ &= \lim_{k\to\infty}\left(\beta_{i_1(x),1} + \sum_{n=2}^{k}\left[\widetilde{\beta}_{i_n(x),n}\prod_{j=1}^{n-1}\widetilde{p}_{i_j(x),j}\right]\right) \\ &= \beta_{i_1(x),1} + \sum_{k=2}^{\infty}\left[\widetilde{\beta}_{i_k(x),k}\prod_{j=1}^{k-1}\widetilde{p}_{i_j(x),j}\right]. \end{aligned}$$

□

So, by system (4), one can model the class Λ_F of determined and bounded on [0, 1] functions, where the fixed matrix P determines the unique function

$$y = F(x)$$

such that

$$x=\sum_{i=0}^{i_1-1} q_{i,1}+\sum_{n=2}^{\infty}\left[(-1)^{n-1}\widetilde{\delta}_{i_n,n}\prod_{j=1}^{n-1}\widetilde{q}_{i_j,j}\right]+\sum_{n=1}^{\infty}\left(\prod_{j=1}^{2n-1}\widetilde{q}_{i_j,j}\right),$$

$$y=F(x)=\sum_{i=0}^{i_1-1} p_{i,1}+\sum_{n=2}^{\infty}\left[(-1)^{n-1}\widetilde{\zeta}_{i_n,n}\prod_{j=1}^{n-1}\widetilde{p}_{i_j,j}\right]+\sum_{n=1}^{\infty}\left(\prod_{j=1}^{2n-1}\widetilde{p}_{i_j,j}\right),$$

where

$$\widetilde{\zeta}_{i_n,n}=\begin{cases} 1 & \text{if } n \text{ is even and } i_n=m_n\\ \sum\limits_{i=m_n-i_n}^{m_n} p_{i,n} & \text{if } n \text{ is even and } i_n\ne m_n\\ 0 & \text{if } n \text{ is odd and } i_n=0\\ \sum\limits_{i=0}^{i_n-1} p_{i,n} & \text{if } n \text{ is odd and } i_n\ne 0.\end{cases}$$

Remark 2. The argument of the function $F\in\Lambda_F$ is determined by the nega-$\widetilde{Q}$-representation, but an expansion of a value of the function has only "formal" view of the nega-P-representation and is the last only if all elements $p_{i,n}$ of the matrix P are positive numbers.

5. Continuity and monotonicity conditions of the solution of system (4) of functional equations

Theorem 3. *The function $y=F(x)$ is a continuous function on $[0,1]$.*

P r o o f. Let $[0,1]\ni x_0$ be a number. Let us consider the difference

$$F(x)-F(x_0)=\left(\prod_{j=1}^{n_0}\widetilde{p}_{i_j,j}\right)\left(\widetilde{\beta}_{i_{n_0+1}(x),n_0+1}-\widetilde{\beta}_{i_{n_0+1}(x_0),n_0+1}\right)+$$
$$\left(\prod_{j=1}^{n_0}\widetilde{p}_{i_j,j}\right)\left(\sum_{k=n_0+2}^{\infty}\left(\widetilde{\beta}_{i_k(x),k}\prod_{l=n_0+1}^{k-1}\widetilde{p}_{i_l(x),l}\right)-\right.$$
$$\left.\sum_{k=n_0+2}^{\infty}\left(\widetilde{\beta}_{i_k(x_0),k}\prod_{l=n_0+1}^{k-1}\widetilde{p}_{i_l(x_0),l}\right)\right),$$

where

$$i_{n_0+1}(x)\ne i_{n_0+1}(x_0),\qquad i_j(x)=i_j(x_0),\qquad j=\overline{1,n_0}.$$

Let x_0 be a nega-$\widetilde{Q}$-irrational point. Since F is bounded and the conditions $x \to x_0$, $n_0 \to \infty$ are equivalent, it is easy to see that

$$\lim_{x\to x_0} |F(x) - F(x_0)| = \lim_{n_0\to\infty} \left(\prod_{j=1}^{n_0} \left| \widetilde{p}_{i_j,j} \right| \right) = 0.$$

So, $\lim_{x\to x_0} F(x) = F(x_0)$.

Let x_0 be a nega-$\widetilde{Q}$-rational number. Let us denote

$$\begin{aligned} x_0 = x_0^{(1)} &= \Delta^{-\widetilde{Q}}_{i_1 i_2 \ldots i_{n-1}[i_n-1]0m_{n+2}0m_{n+4}\ldots} \\ &= \Delta^{-\widetilde{Q}}_{i_1 i_2 \ldots i_{n-1} i_n m_{n+1}0m_{n+3}\ldots} = x_0^{(2)} \qquad \text{if } n \text{ is odd,} \\ x_0 = x_0^{(1)} &= \Delta^{-\widetilde{Q}}_{i_1 i_2 \ldots i_{n-1} i_n m_{n+1}0m_{n+3}\ldots} \\ &= \Delta^{-\widetilde{Q}}_{i_1 i_2 \ldots i_{n-1}[i_n-1]0m_{n+2}0m_{n+4}\ldots} = x_0^{(2)} \qquad \text{if } n \text{ is even,} \end{aligned}$$

and consider the limits

$$\lim_{x\to x_0+0} F(x) = \lim_{x\to x_0^{(2)}} F(x), \qquad \lim_{x\to x_0-0} F(x) = \lim_{x\to x_0^{(1)}} F(x)$$

by considerations for the case of a nega-$\widetilde{Q}$-irrational number x_0.

Therefore, F is a continuous function on $[0, 1]$. □

Lemma 4. *A value of the increment*

$$\mu_F\left(\Delta^{-\widetilde{Q}}_{c_1c_2\ldots c_n}\right) = F\left(\sup \Delta^{-\widetilde{Q}}_{c_1c_2\ldots c_n}\right) - F\left(\inf \Delta^{-\widetilde{Q}}_{c_1c_2\ldots c_n}\right)$$

of the function F on the cylinder $\Delta^{-\widetilde{Q}}_{c_1c_2\ldots c_n}$ is calculated by the formula

$$\mu_F\left(\Delta^{-\widetilde{Q}}_{c_1c_2\ldots c_n}\right) = \prod_{j=1}^{n} \widetilde{p}_{i_j,j}. \tag{7}$$

P r o o f. From the definition and properties of cylinders $\Delta^{-\widetilde{Q}}_{c_1c_2\ldots c_n}$, it follows that

$$\mu_F\left(\Delta^{-\widetilde{Q}}_{c_1c_2\ldots c_n}\right) = \begin{cases} \quad F\left(\Delta^{-\widetilde{Q}}_{c_1c_2\ldots c_n 0m_{n+2}0m_{n+4}\ldots}\right) & \\ -F\left(\Delta^{-\widetilde{Q}}_{c_1c_2\ldots c_n m_{n+1}0m_{n+3}\ldots}\right) & \text{if } n \text{ is odd} \\ \quad F\left(\Delta^{-\widetilde{Q}}_{c_1c_2\ldots c_n m_{n+1}0m_{n+3}\ldots}\right) & \\ -F\left(\Delta^{-\widetilde{Q}}_{c_1c_2\ldots c_n 0m_{n+2}0m_{n+4}\ldots}\right) & \text{if } n \text{ is even.} \end{cases}$$

Therefore,

$$\mu_F\left(\Delta^{-\widetilde{Q}}_{c_1c_2\ldots c_n}\right) = \Big(\beta_{m_{n+1},n+1} + \beta_{m_{n+2},n+2}p_{m_{n+1},n+1} + \beta_{m_{n+3},n+3}p_{m_{n+1},n+1}p_{m_{n+2},n+2} + \cdots\Big)\prod_{j=1}^{n}\widetilde{p}_{i_j,j}.$$

Equality (7) follows from the last-mentioned expression. □

Theorem 4. *The function $y = F(x)$ is:*

- *a monotonic non-decreasing function whenever elements $p_{i,n}$ of the matrix P are non-negative, and a strictly increasing function whenever all elemets of the matrix P are positive;*
- *a non-monotonic function that has at least one interval of monotonicity on $[0,1]$ whenever the matrix P does not have zeros and there exists only the finite tuple $\{p_{i,n}\}$ of the elements $p_{i,n} < 0$ of the matrix P;*
- *a function that does not have intervals of monotonicity on $[0,1]$ whenever the matrix P does not have zeros and there exists an infinite subsequence (n_k) of positive integers such that for an arbitrary number $k \in \mathbb{N}$ there exists at least one element $p_{i,n_k} < 0$ of P, where $i = \overline{0, m_{n_k}}$;*
- *a constant almost everywhere on $[0,1]$ function whenever there exists an infinite subsequence (n_k) of positive integers such that for an arbitrary number $k \in \mathbb{N}$ there exists at least one element $p_{i,n_k} = 0$ of the matrix P, where $i = \overline{0, m_{n_k}}$.*

P r o o f. *The first* and *the second statements* follow from (7).

The third statement. Let us choose a number n_0 such that the number $n_0 + 1$ belongs to a subsequence (n_k) of positive integers and the tuple $c_1, c_2, \ldots, c_{n_0}$ such that the condition $\mu_F(\Delta^{-\widetilde{Q}}_{c_1c_2\ldots c_{n_0}}) > 0$ holds. It is easy to see that there exist nega-$\widetilde{Q}$-symbols i_{n_0+1} and i'_{n_0+1} such that $\widetilde{p}_{i_{n_0+1},n_0} < 0$, $\widetilde{p}_{i'_{n_0+1},n_0} > 0$. In the case for

$$\Delta^{-\widetilde{Q}}_{c_1c_2\ldots c_{n_0}} \supset \left(\Delta^{-\widetilde{Q}}_{c_1c_2\ldots c_{n_0}i_{n_0+1}} \cup \Delta^{-\widetilde{Q}}_{c_1c_2\ldots c_{n_0}i'_{c_{n_0}+1}}\right),$$

since equation (7), it follows that

$$\mu_F\left(\Delta^{-\widetilde{Q}}_{c_1c_2\ldots c_{n_0}i_{n_0+1}}\right) < 0 < \mu_F\left(\Delta^{-\widetilde{Q}}_{c_1c_2\ldots c_{n_0}i'_{n_0+1}}\right).$$

Since the function F is a monotonically increasing function on a some segment, the function does not have intervals of increasing and decreasing on the segment simultaneously. Therefore, the last-mentioned double inequality is a contradiction. So, the function F does not have intervals of monotonicity on $[0,1]$.

Since equality (7) holds and the value of the Lebesgue measure of the set

$$C\left[-\widetilde{Q},\overline{(V_{n_k})}\right]\equiv\left\{x : x=\Delta^{-\widetilde{Q}}_{i_1 i_2\ldots i_n\ldots}, i_{n_k}\notin V_{n_k}\right\},$$

where (n_k) is a fixed subsequence of positive integers and (V_{n_k}) is a sequence of the subsets V_{n_k} of the sets $N^0_{m_{n_k}}$ of nega-$\widetilde{Q}$-symbols such that

$$V_{n_k}\equiv\{i : i\in N^0_{m_{n_k}}, p_{i,n_k}=0\},$$

equals zero, *the fourth statement* is true.

Proposition 2. *Let us have the set*

$$C\left[\widetilde{Q},(N^0_{m_n}\setminus\{i^*_n\})\right]\equiv\left\{x : x=\Delta^{-\widetilde{Q}}_{i_1 i_2\ldots i_n\ldots}, i_n\in N^0_{m_n}\setminus\{i^*_n\}\right\},$$

where (i^*_n) *is a fixed sequence of* $\widetilde{Q}$*-symbols such that* $i^*_1\in N^0_{m_1}, i^*_2\in N^0_{m_2},\ldots$

The following equality

$$\lambda\left(C\left[\widetilde{Q},(N^0_{m_n}\setminus\{i^*_n\})\right]\right)=0$$

holds, where $\lambda(\cdot)$ *is the Lebesgue measure of a set.*

P r o o f. Let us denote by

$$V_{0,n}=N^0_{m_n}\setminus\{i^*_n\},$$

$$E_n=\bigcup_{i_1\neq i^*_1,\ldots,i_n\neq i^*_n}\Delta^{-\widetilde{Q}}_{i_1 i_2\ldots i_n},$$

and that

$$E_n=E_{n+1}\cup\overline{E_{n+1}},$$

$$C[\widetilde{Q},(V_{0,n})]=\cap_{n=1}^{\infty}E_n,$$

where

$$E_{n+1}\subset E_n \quad\text{and}\quad E_0=[0,1].$$

From the property of continuity of the Lebesgue measure, we obtain that

$$\lambda\left(C[\widetilde{Q},(V_{0,n})]\right)=\lim_{n\to\infty}\lambda(E_n)=\lim_{n\to\infty}\prod_{k=1}^{n}(1-q_{i^*_k,k})=0,$$

because

$$\lambda(E_{n+1})=\sum_{\substack{i_1\neq i^*_1,\\ \ldots,\\ i_{n+1}\neq i^*_{n+1}}}\left(q_{i_1,1}\ldots q_{i_{n+1},n+1}\right)=\lambda(E_n)-q_{i^*_{n+1},n+1}\sum_{\substack{i_1\neq i^*_1,\\ \ldots,\\ i_n\neq i^*_n}}(q_{i_1,1}\ldots q_{i_n,n})$$

and

$$1=\sum_{i_1\in N^0_{m_1},\dots,i_n\in N^0_{m_n}}(q_{i_1,1}q_{i_2,2}\dots q_{i_n,n})$$
$$=\sum_{\substack{i_1\neq i_1^*,\\ \dots,\\ i_n\neq i_n^*}}(q_{i_1,1}\dots q_{i_n,n})+q_{i_n^*,n}\sum_{\substack{i_1\neq i_1^*,\\ \dots,\\ i_{n-1}\neq i_{n-1}^*}}\big(q_{i_1,1}\dots q_{i_{n-1},n-1}\big)+\dots+q_{i_1^*,1}q_{i_2^*,2}\dots q_{i_n^*,n}.$$

So, Proposition 2 is true. □

Since the relation between the $\widetilde{Q}$- and nega-$\widetilde{Q}$-representation is given, the proofs of equalities

$$\lambda\left(C\left[-\widetilde{Q},\overline{(V_{n_k})}\right]\right)=0,\qquad \lambda\left(C\left[\widetilde{Q},(N^0_{m_n}\setminus\{i_n^*\})\right]\right)=0$$

are analogous.

□

Corollary 1. *The function F is a bijective mapping on $[0,1]$ whenever all elements of the matrix P are positive numbers.*

6. Integral properties

Theorem 5. *The Lebesgue integral of the function F can be calculated by the following formula*

$$\int_0^1 F(x)\mathrm{d}x=z_1+\sum_{n=2}^{\infty}\left(z_n\prod_{k=1}^{n-1}\sigma_k\right),$$

where

$$z_n=\widetilde{\beta}_{0,n}\widetilde{q}_{0,n}+\widetilde{\beta}_{1,n}\widetilde{q}_{1,n}+\dots+\widetilde{\beta}_{m_n,n}\widetilde{q}_{m_n,n}=\beta_{0,n}q_{0,n}+\beta_{1,n}q_{1,n}+\dots+\beta_{m_n,n}q_{m_n,n},$$

$$\sigma_n=\widetilde{p}_{0,n}\widetilde{q}_{0,n}+\widetilde{p}_{1,n}\widetilde{q}_{1,n}+\dots+\widetilde{p}_{m_n,n}\widetilde{q}_{m_n,n}=p_{0,n}q_{0,n}+p_{1,n}q_{1,n}+\dots+p_{m_n,n}q_{m_n,n}.$$

P r o o f. From equality (2), it follows that

$$\mathrm{d}\big(\hat{\varphi}^n(x)\big)=\widetilde{q}_{i_{n+1},n+1}d\big(\hat{\varphi}^{n+1}(x)\big).$$

By the definition of the function and the additive property of the Lebesgue integral, we obtain that

$$
\begin{aligned}
\int_0^1 F(x)\mathrm{d}x &= \int_0^{a_{1,1}} F(x)\mathrm{d}x + \int_{a_{1,1}}^{a_{2,1}} F(x)\mathrm{d}x + \cdots + \int_{a_{m_1,1}}^{1} F(x)\mathrm{d}x \\
&= \int_0^{a_{1,1}} p_{0,1}F\big(\hat{\varphi}(x)\big)\mathrm{d}x + \int_{a_{1,1}}^{a_{2,1}} \big[\beta_{1,1} + p_{1,1}F\big(\hat{\varphi}(x)\big)\big]\,\mathrm{d}x \\
&\qquad + \cdots + \int_{a_{m_1,1}}^{1} \big[\beta_{m_1,1} + p_{m_1,1}F\big(\hat{\varphi}(x)\big)\big]\,\mathrm{d}x \\
&= p_{0,1}q_{0,1}\int_0^1 F\big(\hat{\varphi}(x)\big)d\big(\hat{\varphi}(x)\big) + \beta_{1,1}q_{1,1} \\
&\quad + p_{1,1}q_{1,1}\int_0^1 F\big(\hat{\varphi}(x)\big)\mathrm{d}\big(\hat{\varphi}(x)\big) + \cdots + \beta_{m_1,1}q_{m_1,1} \\
&\quad + p_{m_1,1}q_{m_1,1}\int_0^1 F\big(\hat{\varphi}(x)\big)\mathrm{d}\big(\hat{\varphi}(x)\big) \\
&= (\beta_{0,1}q_{0,1} + \beta_{1,1}q_{1,1} + \cdots + \beta_{m_1,1}q_{m_1,1}) \\
&\quad + (p_{0,1}q_{0,1} + p_{1,1}q_{1,1} + \cdots + p_{m_1,1}q_{m_1,1})\int_0^1 F\big(\hat{\varphi}(x)\big)\mathrm{d}\big(\hat{\varphi}(x)\big) \\
&= z_1 + \sigma_1\int_0^1 F\big(\hat{\varphi}(x)\big)\mathrm{d}\big(\hat{\varphi}(x)\big) \\
&= z_1 + \sigma_1\Bigg(\int_0^{a_{1,2}} \Big(\widetilde{\beta}_{m_2,2} + \widetilde{p}_{m_2,2}F\big(\hat{\varphi}^2(x)\big)\Big)\mathrm{d}\big(\hat{\varphi}(x)\big) \\
&\qquad + \cdots + \int_{a_{m_2,2}}^{1} \Big(\widetilde{\beta}_{0,2} + \widetilde{p}_{0,2}F\big(\hat{\varphi}^2(x)\big)\Big)\mathrm{d}\big(\hat{\varphi}(x)\big)\Bigg) \\
&= z_1 + \sigma_1\Big(\beta_{0,2}q_{0,2} + \beta_{1,2}q_{1,2} + \cdots + \beta_{m_2,2}q_{m_2,2} \\
&\qquad + (p_{0,2}q_{0,2} + \cdots + p_{m_2,2}q_{m_2,2})\int_0^1 F\big(\hat{\varphi^2}(x)\big)\mathrm{d}\big(\hat{\varphi^2}(x)\big)\Big) \\
&= z_1 + z_2\sigma_1 + \sigma_1\sigma_2\int_0^1 F\big(\hat{\varphi^2}(x)\big)\mathrm{d}\big(\hat{\varphi^2}(x)\big) \\
&= \cdots = z_1 + z_2\sigma_1 + z_3\sigma_1\sigma_2 + \cdots + z_n\sigma_1\sigma_2\ldots\sigma_{n-1} \\
&\quad + \sigma_1\sigma_2\ldots\sigma_n\int_0^1 F\big(\hat{\varphi^n}(x)\big)\mathrm{d}\big(\hat{\varphi^n}(x)\big) = \cdots
\end{aligned}
$$

Continuing the process indefinitely, we obtain that

$$\int_0^1 F(x)\mathrm{d}x = z_1 + \sum_{n=2}^{\infty}\left(z_n \prod_{k=1}^{n-1}\sigma_k\right). \qquad \square$$

7. Modeling singular distribution functions

Lemma 5. *If the function F has a derivative $F'(x_0)$ at a nega-$\widetilde{Q}$-irrational point x_0, then*

$$F'(x_0) = \lim_{n\to\infty}\left(\prod_{j=1}^{n}\frac{\widetilde{p}_{i_j(x_0),j}}{\widetilde{q}_{i_j(x_0),j}}\right) = \prod_{n=1}^{\infty}\frac{\widetilde{p}_{i_n(x_0),n}}{\widetilde{q}_{i_n(x_0),n}}.$$

P r o o f. The statement is true, because the Property 6 of cylinders $\Delta^{-\widetilde{Q}}_{c_1c_2\ldots c_n}$ is true and the conditions $x \to x_0, n \to \infty$ are equivalent. $\square$

Theorem 6. *If, for all $n \in \mathbb{N}$ and $i = \overline{0, m_n}$, it is true that $p_{i,n} \geq 0$, then the unique solution of functional equations system* (4) *is a continuous function of probabilities distribution on* $[0, 1]$.

P r o o f. It is easy to see that

$$F(0) = F\left(\Delta^{-\widetilde{Q}}_{0m_20m_4\ldots 0m_{2k}\ldots}\right) = \beta_{0,1} + \sum_{n=2}^{\infty}\left[\beta_{0,n}\prod_{j=1}^{n-1}p_{0,j}\right] = 0,$$

$$F(1) = F\left(\Delta^{-\widetilde{Q}}_{m_10m_3\ldots 0m_{2k-1}\ldots}\right) = \beta_{m_1,1} + \sum_{n=2}^{\infty}\left[\beta_{m_n,n}\prod_{j=1}^{n-1}p_{m_j,j}\right] = 1.$$

Let

$$x_1 = \Delta^{-\widetilde{Q}}_{i_1(x_1)i_2(x_1)\ldots i_n(x_1)\ldots} \quad \text{and} \quad x_2 = \Delta^{-\widetilde{Q}}_{i_1(x_2)i_2(x_2)\ldots i_n(x_2)\ldots}$$

be some numbers from $[0, 1]$ such that $x_1 < x_2$. Therefore, there exists a number n_0 such that $i_j(x_1) = i_j(x_2)$ for all $j = \overline{1, n_0 - 1}$ and $i_{n_0}(x_1) < i_{n_0}(x_2)$ whenever n_0 is odd, or $i_{n_0}(x_1) > i_{n_0}(x_2)$ whenever n_0 is even.

Whence,

$$\begin{aligned}
F(x_2) - F(x_1) &= \left(\prod_{j=1}^{n_0-1} \widetilde{p}_{i_j(x_2),j}\right) \cdot \Bigg(\widetilde{\beta}_{i_{n_0}(x_2),n_0} - \widetilde{\beta}_{i_{n_0}(x_1),n_0} \\
&\quad + \sum_{k=1}^{\infty} \left(\widetilde{\beta}_{i_{n_0+k}(x_2),n_0+k} \prod_{j=0}^{k-1} \widetilde{p}_{i_{n_0+j}(x_2),n_0+j}\right) \\
&\quad - \sum_{k=1}^{\infty} \left(\widetilde{\beta}_{i_{n_0+k}(x_1),n_0+k} \prod_{j=0}^{k-1} \widetilde{p}_{i_{n_0+j}(x_1),n_0+j}\right)\Bigg) \\
&\geq \left(\prod_{j=1}^{n_0-1} \widetilde{p}_{i_j(x_2),j}\right) \cdot \Bigg(\widetilde{\beta}_{i_{n_0}(x_2),n_0} - \widetilde{\beta}_{i_{n_0}(x_1),n_0} \\
&\quad - \sum_{k=1}^{\infty} \left(\widetilde{\beta}_{i_{n_0+k}(x_1),n_0+k} \prod_{j=0}^{k-1} \widetilde{p}_{i_{n_0+j}(x_1),n_0+j}\right)\Bigg) = \kappa,
\end{aligned}$$

where for odd n_0

$$\begin{aligned}
\kappa &\geq \left(\prod_{j=1}^{n_0-1} \widetilde{p}_{i_j(x_2),j}\right)\Big(p_{i_{n_0}(x_1),n_0} + p_{i_{n_0}(x_1)+1,n_0} + \cdots \\
&\quad \cdots + p_{i_{n_0}(x_2)-1,n_0} - p_{i_{n_0}(x_1),n_0} \max_{x\in[0,1]} F\big(\hat{\varphi}^{n_0}(x_1)\big)\Big) \\
&= \left(\prod_{j=1}^{n_0-1} \widetilde{p}_{i_j(x_2),j}\right)\big(p_{i_{n_0}(x_1)+1,n_0} + \cdots + p_{i_{n_0}(x_2)-1,n_0}\big) \geq 0.
\end{aligned}$$

Let n_0 be even. Then

$$\begin{aligned}
\kappa &\geq \left(\prod_{j=1}^{n_0-1} \widetilde{p}_{i_j(x_2),j}\right)\Big(p_{m_{n_0}-i_{n_0}(x_1),n_0} + p_{m_{n_0}-i_{n_0}(x_1)+1,n_0} + \cdots \\
&\quad \cdots + p_{m_{n_0}-i_{n_0}(x_2)-1,n_0} - p_{m_{n_0}-i_{n_0}(x_1),n_0} \max_{x\in[0,1]} F\big(\hat{\varphi}^{n_0}(x_1)\big)\Big) \\
&= \left(\prod_{j=1}^{n_0-1} \widetilde{p}_{i_j(x_2),j}\right)\big(p_{m_{n_0}-i_{n_0}(x_1)+1,n_0} + \cdots + p_{m_{n_0}-i_{n_0}(x_2)-1,n_0}\big) \geq 0.
\end{aligned}$$

If all elements $p_{i,n}$ of the matrix P are positive, then the inequality

$$F(x_2) - F(x_1) > 0$$

holds.

Since the function F is a continuous function at all points from $[0,1]$, it follows that the function is a continuous function of probabilities distribution on $[0,1]$. □

Let η be a random variable defined by the following form

$$\eta = \Delta^{\widetilde{Q}}_{\xi_1\xi_2\ldots\xi_n\ldots},$$

where

$$\xi_n = \begin{cases} i_n & \text{if } n \text{ is odd} \\ m_n - i_n & \text{if } n \text{ is even,} \end{cases}$$

$n = 1, 2, 3, \ldots$, the digits ξ_n are random and taking the values $0, 1, \ldots, m_n$ with probabilities $p_{0,n}, p_{1,n}, \ldots, p_{m_n,n}$. That is, ξ_n are independent and $P\{\xi_n = i_n\} = p_{i_n,n}$, $i_n \in N^0_{m_n}$.

Theorem 7. *The distribution function $\widetilde{F}_\eta$ of the random variable η can be represented by*

$$\widetilde{F}_\eta(x) = \begin{cases} 0, & x < 0; \\ \beta_{i_1(x),1} + \sum\limits_{n=2}^{\infty} \left[\widetilde{\beta}_{i_n(x),n} \prod\limits_{j=1}^{n-1} \widetilde{p}_{i_j(x),j}\right], & 0 \le x < 1; \\ 1, & x \ge 1. \end{cases}$$

Proof. Let $k \in \mathbb{N}$. The statement follows from the equalities

$$\{\eta < x\} = \{\xi_1 < i_1(x)\} \cup \{\xi_1 = i_1(x), \xi_2 < m_2 - i_2(x)\} \cup \ldots$$
$$\cdots \cup \{\xi_1 = i_1(x), \xi_2 = m_2 - i_2(x), \ldots, \xi_{2k-1} < i_{2k-1}(x)\}$$
$$\cup \{\xi_1 = i_1(x), \xi_2 = m_2 - i_2(x), \ldots, \xi_{2k-1} = i_{2k-1}(x),$$
$$\xi_{2k} < m_{2k} - i_{2k}(x)\} \cup \ldots,$$

$$P\{\xi_1 = i_1(x), \xi_2 = m_2 - i_2(x), \ldots, \xi_{2k-1} < i_{2k-1}(x)\}$$
$$= \beta_{i_{2k-1}(x),2k-1} \prod_{j=1}^{2k-2} \widetilde{p}_{i_j(x),j},$$
$$P\{\xi_1 = i_1(x), \xi_2 = m_2 - i_2(x), \ldots, \xi_{2k} < m_{2k} - i_{2k}(x)\}$$
$$= \beta_{m_{2k}-i_{2k}(x),2k} \prod_{j=1}^{2k-1} \widetilde{p}_{i_j(x),j}$$

and the definition of a distribution function. □

One can formulate the following conclusions by the statements in [11, p. 170].

Lemma 6. *Let the inequality* $p_{i,n} \geq 0$ *holds for any* $n \in \mathbb{N}$ *and* $i = \overline{0, m_n}$.

The function F *is a singular function of Cantor type if and only if*

(1)
$$\prod_{n=1}^{\infty} \left(\sum_{i:\widetilde{p}_{i,n}>0} \widetilde{q}_{i,n} \right) = 0$$

or

(2)
$$\sum_{n=1}^{\infty} \left(\sum_{i:\widetilde{p}_{i,n}=0} \widetilde{q}_{i,n} \right) = \infty.$$

8. Modeling functions that does not have the derivative at any nega-$\widetilde{Q}$-rational point

Theorem 8. *If the following properties of the matrix* P *hold, then the unique solution to system* (4) *of functional equations does not have a finite or an infinite derivative at any nega-*$\widetilde{Q}$*-rational point from the segment* $[0,1]$:

- *for all* $n \in \mathbb{N}$, $i_n \in N^1_{m_n} \equiv \{1, 2, \ldots, m_n\}$
$$p_{i_n,n} \cdot p_{i_n-1,n} < 0;$$
- *the conditions*
$$\lim_{n\to\infty} \prod_{k=1}^{n} \frac{p_{0,k}}{q_{0,k}} \neq 0, \qquad \lim_{n\to\infty} \prod_{k=1}^{n} \frac{p_{m_k,k}}{q_{m_k,k}} \neq 0$$

hold simultaneously.

P r o o f. Let x_0 be a nega-$\widetilde{Q}$-rational point. That is

$$x_0 = \Delta^{-\widetilde{Q}}_{i_1 i_2 \ldots i_{n-1} i_n m_{n+1} 0 m_{n+3} 0 m_{n+5} \ldots} = \Delta^{-\widetilde{Q}}_{i_1 i_2 \ldots i_{n-1} [i_n - 1] 0 m_{n+2} 0 m_{n+4} \ldots}, \quad i_n \neq 0.$$

In the case of odd n, let us denote

$$x_0 = x_0^{(1)} = \Delta^{-\widetilde{Q}}_{i_1 i_2 \ldots i_{n-1} i_n m_{n+1} 0 m_{n+3} 0 m_{n+5} \ldots}$$
$$= \Delta^{-\widetilde{Q}}_{i_1 i_2 \ldots i_{n-1} [i_n - 1] 0 m_{n+2} 0 m_{n+4} - \ldots} = x_0^{(2)}.$$

If n is even, then let us denote

$$x_0 = x_0^{(1)} = \Delta^{-\widetilde{Q}}_{i_1 i_2 \ldots i_{n-1} [i_n - 1] 0 m_{n+2} 0 m_{n+4} \ldots}$$
$$= \Delta^{-\widetilde{Q}}_{i_1 i_2 \ldots i_{n-1} i_n m_{n+1} 0 m_{n+3} 0 m_{n+5} \ldots} = x_0^{(2)}.$$

Let us consider the sequences (x'_k), (x''_k) such that $x'_k \to x_0$, $x''_k \to x_0$ as $k \to \infty$ and for an odd number n

$$x'_k = \begin{cases} \Delta^{-\widetilde{Q}}_{i_1\ldots i_{n-1}i_n m_{n+1}0 m_{n+3}0\ldots m_{n+k-1}1 m_{n+k+1}0 m_{n+k+3}\ldots} & \text{if } k \text{ is even} \\ \Delta^{-\widetilde{Q}}_{i_1\ldots i_{n-1}i_n m_{n+1}0\ldots m_{n+k-2}0[m_{n+k}-1]0 m_{n+k+2}0 m_{n+k+4}\ldots} & \text{if } k \text{ is odd,} \end{cases}$$

$$x''_k = \begin{cases} \Delta^{-\widetilde{Q}}_{i_1\ldots i_{n-1}[i_n-1]0 m_{n+2}0\ldots m_{n+k-1}00 m_{n+k+2}0 m_{n+k+4}\ldots} & \text{if } k \text{ is odd} \\ \Delta^{-\widetilde{Q}}_{i_1\ldots i_{n-1}[i_n-1]0 m_{n+2}0\ldots m_{n+k} m_{n+k+1}0 m_{n+k+3}0 m_{n+k+5}\ldots} & \text{if } k \text{ is even,} \end{cases}$$

and for the case of even n

$$x'_k = \begin{cases} \Delta^{-\widetilde{Q}}_{i_1\ldots i_{n-1}[i_n-1]0 m_{n+2}0 m_{n+4}0\ldots m_{n+k-1}1 m_{n+k+1}0 m_{n+k+3}\ldots} & \text{if } k \text{ is an odd} \\ \Delta^{-\widetilde{Q}}_{i_1\ldots i_{n-1}[i_n-1]0 m_{n+2}0\ldots m_{n+k-2}0[m_{n+k}-1]0 m_{n+k+2}0 m_{n+k+4}\ldots} & \text{if } k \text{ is even,} \end{cases}$$

$$x''_k = \begin{cases} \Delta^{-\widetilde{Q}}_{i_1\ldots i_{n-1}i_n m_{n+1}0 m_{n+3}\ldots 0 m_{n+k} m_{n+k+1}0 m_{n+k+3}\ldots} & \text{if } k \text{ is odd} \\ \Delta^{-\widetilde{Q}}_{i_1\ldots i_{n-1}i_n m_{n+1}0\ldots m_{n+k-1}00 m_{n+k+2}0 m_{n+k+4}0 m_{n+k+6}\ldots} & \text{if } k \text{ is even.} \end{cases}$$

Therefore, if n is odd, then

$$x'_k - x_0^{(1)} = \Delta^{\widetilde{Q}}_{i_1[m_2-i_2]i_3[m_4-i_4]\ldots i_n \underbrace{0\ldots0}_{k-1}1(0)} - \Delta^{\widetilde{Q}}_{i_1[m_2-i_2]\ldots i_n(0)}$$

$$\equiv a_{1,n+k}\left(\prod_{j=1}^{n}\widetilde{q}_{i_j,j}\right)\left(\prod_{t=n+1}^{n+k-1} q_{0,t}\right),$$

$$F\left(x'_k\right) - F\left(x_0^{(1)}\right) = \beta_{1,n+k}\left(\prod_{j=1}^{n}\widetilde{p}_{i_j,j}\right)\left(\prod_{t=n+1}^{n+k-1} p_{0,t}\right) = \left(\prod_{j=1}^{n}\widetilde{p}_{i_j,j}\right)\left(\prod_{t=n+1}^{n+k} p_{0,t}\right),$$

$$x_0^{(2)} - x''_k = \Delta^{\widetilde{Q}}_{i_1[m_2-i_2]\ldots[m_{n-1}-i_{n-1}][i_n-1]m_{n+1}m_{n+2}\ldots}$$

$$- \Delta^{\widetilde{Q}}_{i_1[m_2-i_2]\ldots[m_{n-1}-i_{n-1}][i_n-1]m_{n+1}m_{n+2}\ldots m_{n+k}(0)}$$

$$\equiv q_{i_n-1,n}\left(\prod_{j=1}^{n-1}\widetilde{q}_{i_j,j}\right)\left(\prod_{t=n+1}^{n+k} q_{m_t,t}\right),$$

$$F\left(x_0^{(2)}\right) - F\left(x''_k\right) = p_{i_n-1,n}\left(\prod_{j=1}^{n-1}\widetilde{p}_{i_j,j}\right)\left(\prod_{t=n+1}^{n+k} p_{m_t,t}\right).$$

If n is even, then

$$
\begin{aligned}
x'_k - x_0^{(1)} &= \Delta^{\widetilde{Q}}_{i_1[m_2-i_2]\ldots i_{n-1}[m_n-i_n+1]\underbrace{0\ldots0}_{k-1}1(0)} \\
&\quad - \Delta^{\widetilde{Q}}_{i_1[m_2-i_2]\ldots i_{n-1}[m_n-i_n+1](0)} \\
&\equiv a_{1,n+k}\left(\prod_{j=1}^{n-1}\widetilde{q}_{i_j,j}\right)\left(\prod_{t=n+1}^{n+k-1} q_{0,t}\right) q_{m_n-i_n+1,n} \\
&= \left(\prod_{j=1}^{n-1}\widetilde{q}_{i_j,j}\right)\left(\prod_{t=n+1}^{n+k} q_{0,t}\right) q_{m_n-i_n+1,n},
\end{aligned}
$$

$$
\begin{aligned}
F\left(x'_k\right) - F\left(x_0^{(1)}\right) &= \beta_{1,n+k}\left(\prod_{j=1}^{n-1}\widetilde{p}_{i_j,j}\right)\left(\prod_{t=n+1}^{n+k-1} p_{0,t}\right) p_{m_n-i_n+1,n} \\
&= \left(\prod_{j=1}^{n-1}\widetilde{p}_{i_j,j}\right)\left(\prod_{t=n+1}^{n+k} p_{0,t}\right) p_{m_n-i_n+1,n},
\end{aligned}
$$

$$
\begin{aligned}
x_0^{(2)} - x''_k &= \Delta^{\widetilde{Q}}_{i_1[m_2-i_2]\ldots i_{n-1}[m_n-i_n]m_{n+1}m_{n+2}\ldots} \\
&\quad - \Delta^{\widetilde{Q}}_{i_1[m_2-i_2]\ldots i_{n-1}[m_n-i_n]m_{n+1}m_{n+2}\ldots m_{n+k}(0)} \\
&\equiv \left(\prod_{j=1}^{n}\widetilde{q}_{i_j,j}\right)\left(\prod_{t=n+1}^{n+k} q_{m_t,t}\right),
\end{aligned}
$$

$$
F\left(x_0^{(2)}\right) - F\left(x''_k\right) = \left(\prod_{j=1}^{n}\widetilde{p}_{i_j,j}\right)\left(\prod_{t=n+1}^{n+k} p_{m_t,t}\right).
$$

So,

$$
B'_k = \frac{F(x'_k) - F(x_0)}{x'_k - x_0} = \begin{cases} \frac{p_{i_n,n}}{q_{i_n,n}}\left(\prod_{j=1}^{n-1}\frac{\widetilde{p}_{i_j,j}}{\widetilde{q}_{i_j,j}}\right)\left(\prod_{t=n+1}^{n+k}\frac{p_{0,t}}{q_{0,t}}\right) & \text{if } n \text{ is odd} \\ \frac{p_{m_n-i_n+1,n}}{q_{m_n-i_n+1,n}}\left(\prod_{j=1}^{n-1}\frac{\widetilde{p}_{i_j,j}}{\widetilde{q}_{i_j,j}}\right)\left(\prod_{t=n+1}^{n+k}\frac{p_{0,t}}{q_{0,t}}\right) & \text{if } n \text{ is even,} \end{cases}
$$

$$
B''_k = \frac{F(x_0) - F(x''_k)}{x_0 - x''_k} = \begin{cases} \frac{p_{i_n-1,n}}{q_{i_n-1,n}}\left(\prod_{j=1}^{n-1}\frac{\widetilde{p}_{i_j,j}}{\widetilde{q}_{i_j,j}}\right)\left(\prod_{t=n+1}^{n+k}\frac{p_{m_t,t}}{q_{m_t,t}}\right) & \text{if } n \text{ is odd} \\ \frac{p_{m_n-i_n,n}}{q_{m_n-i_n,n}}\left(\prod_{j=1}^{n-1}\frac{\widetilde{p}_{i_j,j}}{\widetilde{q}_{i_j,j}}\right)\left(\prod_{t=n+1}^{n+k}\frac{p_{m_t,t}}{q_{m_t,t}}\right) & \text{if } n \text{ is even.} \end{cases}
$$

Let us denote

$$b_{0,k} = \prod_{t=n+1}^{n+k} \frac{p_{0,t}}{q_{0,t}}, \qquad b_{m_k,k} = \prod_{m=n+1}^{n+k} \frac{p_{m_t,t}}{q_{m_t,t}}.$$

Since the conditions

$$\prod_{j=1}^{n-1} (\widetilde{p}_{i_j,j}/\widetilde{q}_{i_j,j}) = \text{const}, \quad p_{i_n,n}p_{i_n-1,n} < 0, \quad p_{m_n-i_n+1,n}p_{m_n-i_n,n} < 0$$

hold and the sequences $(b_{0,k})$, $(b_{m_k,k})$ do not converge to 0 simultaneously, the inequality

$$\lim_{k\to\infty} B'_k \neq \lim_{k\to\infty} B''_k$$

holds for all cases. Therefore, F does not have a finite or an infinite derivative at an arbitrary nega-$\widetilde{Q}$-rational point from the segment $[0, 1]$. □

9. A graph of the continuous unique solution to system (4)

Theorem 9. *Let the elements $p_{i,n}$ of the matrix P do not equal 0.*

Let $x = \Delta^{-\widetilde{Q}}_{i_1(x)i_2(x)\dots i_n(x)\dots}$ be a fixed number and the sequence $\left(\psi_{i_n(x),n}\right)$ be a corresponding to its sequence of affine transformations of space $\mathbb{R}^2$:

$$\psi_{i_n(x),n} : \begin{cases} x' = \widetilde{a}_{i_n(x),n} + \widetilde{q}_{i_n(x),n}x \\ y' = \widetilde{\beta}_{i_n(x),n} + \widetilde{p}_{i_n(x),n}y, \end{cases}$$

where $i_n \in N^0_{m_n}$.

Then the graph Γ_F of the function F is the following set in space $\mathbb{R}^2$:

$$\Gamma_F = \bigcup_{x\in[0,1]} \left(\dots \circ \psi_{i_n(x),n} \circ \dots \circ \psi_{i_2(x),2} \circ \psi_{i_1(x),1}(\Gamma_F)\right).$$

P r o o f. Since the function F is the continuous unique solution to system (5), it is clear that

$$\psi_{i_1,1}\colon \begin{cases} x' = q_{i_1,1}x + a_{i_1,1} \\ y' = \beta_{i_1,1} + p_{i_1,1}y, \end{cases}$$

$$\psi_{i_2,2}\colon \begin{cases} x' = q_{m_2-i_2,2}x + a_{m_2-i_2,2} \\ y' = \beta_{m_2-i_2,2} + p_{m_2-i_2,2}y, \end{cases}$$

etc.,

whence,

$$\psi_{i_n,n} : \begin{cases} x' = \widetilde{q}_{i_n,n} x + \widetilde{a}_{i_n,n} \\ y' = \widetilde{\beta}_{i_n,n} + \widetilde{p}_{i_n,n} y. \end{cases}$$

Therefore,

$$\bigcup_{x\in[0,1]} \big(\cdots \circ \psi_{i_n(x),n} \circ \cdots \circ \psi_{i_2(x),2} \circ \psi_{i_1(x),1}(\Gamma_F)\big) \equiv G \subset \Gamma_{\widetilde{F}},$$

because

$$F(x') = F(\widetilde{a}_{i_n,n} + \widetilde{q}_{i_n,n} x) = \widetilde{\beta}_{i_n,n} + \widetilde{p}_{i_n,n} y = y'.$$

Let

$$T\big(x_0, F(x_0)\big) \in \Gamma_{\widetilde{F}},$$

$$x_0 = \Delta^{-\widetilde{Q}}_{i_1 i_2 \ldots i_n \ldots}$$

be a fixed point from $[0,1]$. Let x_n be a point from $[0,1]$ such that $x_n = \hat{\varphi}^n(x_0)$.

Since $i_1 \in N^0_{m_1}$, $i_2 \in N^0_{m_2}$, $\ldots$, $i_n \in N^0_{m_n}$ and system (4) is true and the condition

$$\overline{T}\Big(\hat{\varphi}^n(x_0), F\big(\hat{\varphi}^n(x_0)\big)\Big) \in \ \Gamma_{\widetilde{F}}$$

holds, it follows that

$$\psi_{i_n,n} \circ \cdots \circ \psi_{i_2,2} \circ \psi_{i_1,1}\big(\overline{T}\big) = T_0\big(x_0, F(x_0)\big) \in \Gamma_F, \qquad i_n \in N^0_{m_n}, \ \ n \to \infty.$$

Whence, $\Gamma_F \subset G$. So,

$$\Gamma_F = \bigcup_{x\in[0,1]} \big(\cdots \circ \psi_{i_n(x),n} \circ \cdots \circ \psi_{i_2(x),2} \circ \psi_{i_1(x),1}(\Gamma_F)\big).$$

□

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