sciendo

TATRA MOUNTAINS
Mathematical Publications
DOI: 10.2478/tmmp-2021-0007
Tatra Mt. Math. Publ. **78** (2021), 85–118

SUPER AND HYPER PRODUCTS OF SUPER RELATIONS

Árpád Száz

Department of Mathematics, Faculty of Natural Sciences, University of Debrecen, Debrecen, HUNGARY

Dedicated to the memory of my faithful friend Roman Frič

ABSTRACT. If R is a relation on X to Y, U is a relation on $\mathcal{P}(X)$ to Y, and V is a relation on $\mathcal{P}(X)$ to $\mathcal{P}(Y)$, then we say that R is an ordinary relation, U is a super relation, and V is a hyper relation on X to Y.

Motivated by an ingenious idea of Emilia Przemska on a unified treatment of open- and closed-like sets, we shall introduce and investigate here four reasonable notions of product relations for super relations.

In particular, for any two super relations U and V on X, we define two super relations $U \star V$ and $U * V$, and two hyper relations $U \bigstar V$ and $U \divideontimes V$ on X such that:

$$(U \star V)(A) = \big(A \cup U(A)\big) \cap V(A),$$

$$(U * V)(A) = \big(A \cap V(A)\big) \cup U(A)$$

and

$$(U \bigstar V)(A) = \big\{ B \subseteq X : (U \star V)(A) \subseteq B \subseteq (U * V)(A) \big\},$$

$$(U \divideontimes V)(A) = \big\{ B \subseteq X : (U \cap V)(A) \subseteq B \subseteq (U \cup V)(A) \big\}$$

for all $A \subseteq X$.

By using the distributivity of the operation $\cap$ over $\cup$, we can at once see that $U \star V \subseteq U * V$. Moreover, if $U \subseteq V$, then we can also see that $U \star V = U * V$. The most simple case is when U is an interior relation on X and V is the associated closure relation defined such that $V(A) = U(A^c)^c$ for all $A \subseteq X$.

2010 Mathematics Subject Classification: 54C60, 06A15, 54E15, 54A05.
Keywords: relations on ordinary and power sets, unary and binary operations for relations, Galois connections, generalized uniformities, generalized open sets.

1. Introduction

A subset R of a product set $X \times Y$ is called a relation on X to Y. In particular, a relation on X to itself is called a relation on X. Thus, a relation on X to Y is also a relation on $X \cup Y$.

If R is a relation on X to Y, U is a relation on $\mathcal{P}(X)$ to Y, and V is a relation on $\mathcal{P}(X)$ to $\mathcal{P}(Y)$, then we say that R is an ordinary relation, U is a super relation, and V is a hyper relation on X to Y.

For any fixed $x \in X$ and $A \subseteq X$, the sets $R(x) = \{ y \in Y : \ (x, y) \in R \}$ and $R[A] = \bigcup_{x \in A} R(x)$ will be called the images or neighbourhoods of x and A under R, respectively.

The super relation U will be called quasi-increasing if $U\big(\{x\}\big) \subseteq U(A)$ for all $x \in A \subseteq X$. Moreover, the super relation U will be called union-preserving if $U\big(\bigcup \mathcal{A}\big) = \bigcup_{A \in \mathcal{A}} U(A)$ for all $\mathcal{A} \subseteq \mathcal{P}(X)$.

It will be shown that the super relation U is union-preserving if and only if $U(A) = \bigcup_{x \in A} U\big(\{x\}\big)$ for all $A \subseteq X$. Thus, the usual hull and closure relations, used in algebra and topology, are not union-preserving.

For the ordinary relation R, we define a super relation $R^{\triangleright}$ on X to Y such that $R^{\triangleright}(A) = R[A]$ for all $A \subseteq X$. While, for the super relation U, we define an ordinary relation $U^{\triangleleft}$ on X to Y such that $U^{\triangleleft}(x) = U\big(\{x\}\big)$ for all $x \in X$.

Thus, the mappings $\triangleright$ and $\triangleleft$ establish a partial Galois connection between ordinary and super relations in the sense that, for any ordinary relation R and quasi-increasing super relation U on X to Y, we have $R^{\triangleright} \subseteq U \iff R \subseteq U^{\triangleleft}$.

By using this Galois connection, for any super relation U on X to Y, we can at once define two closely related, union-preserving super relations $U^{\circ} = U^{\triangleleft \triangleright}$ and $U^{\boldsymbol{-1}} = U^{\triangleleft\, -1\, \triangleright}$. This greatly differs from the ordinary inverse U^{-1} of U.

Super relations on X to Y are more general objects than the ordinary relations on X to Y. Namely, by using the map $\triangleright$, the ordinary relations can only be identified with the union-preserving super relations.

While, hyper relations on X to Y are more general objects than the super relations on X to Y. Namely, by using the usual identification of relations with set-valued functions, super relations can only be identified with hyper functions.

For a hyper relation V on X to Y, we may naturally define a super relation $V^{\blacktriangleleft}$ on X to Y such that $V^{\blacktriangleleft}(A) = \{ y \in Y : \ \{y\} \in V(A) \}$. However, hyper relations can be derived from super relations in several natural ways [42, 43].

Motivated by an ingenious idea of Przemska [31, 32] on a unified treatment of open- and closed-like sets, we shall introduce and investigate here four reasonable notions of product relations for super relations.

For any two super relations U and V on X, we define two super relations $U \star V$ and $U * V$, and two hyper relations $U \bigstar V$ and $U \circledast V$ on X such that:

$$(U \star V) (A) = \big(A \cup U (A) \big) \cap V (A) ,$$

$$(U * V) (A) = \big(A \cap V (A) \big) \cup U (A)$$

and

$$(U \bigstar V) (A) = \big\{ B \subseteq X : \ (U \star V) (A) \subseteq B \subseteq (U * V) (A) \big\} ,$$

$$(U \circledast V) (A) = \big\{ B \subseteq X : \ (U \cap V) (A) \subseteq B \subseteq (U \cup V) (A) \big\}$$

for all $A \subseteq X$.

Thus, we shall show that the above operations are idempotent, and

$$U \star V \subseteq U * V \qquad \text{and} \qquad U \bigstar V \subseteq U \circledast V .$$

Moreover, the inclusions $U \star V \subseteq V$ and $U \subseteq U * V$ are also always true.

By using the dual super relation $U^\star$, defined such that $U^\star (A) = U (A^c)^c$ for all $A \subseteq X$, we shall also show that $U * V = \big(V^\star \star U^\star \big)^\star$. Therefore, the properties of the operation $*$ can actually be derived from those of the operations $\star$.

If in particular $U \subseteq V$, then we shall show that $U \bigstar V = U \star V = U * V$. Moreover, in this case, we can note that

$$(U \circledast V)(A) = \big\{ B \subseteq X : \ U (A) \subseteq B \subseteq V (A) \big\}$$

for all $A \subseteq X$.

Finally, we shall show that the super relations U and $U^\star$ and the hyper relations $U \circledast V$ and $(U \circledast V)^{-1}$ can be used to treat the various generalized open sets in a general unified framework.

Generalized open sets in topological and closure spaces, and their immediate generalizations, have been investigated by a great number of topologists. See, for instance, [1–3, 6, 8, 12, 19, 20, 24, 29, 38].

2. A few basic facts on relations and functions

A subset R of a product set $X \times Y$ is called a *relation on* X *to* Y. In particular, a relation on X to itself is called a *relation on* X. And, $\Delta_X = \{ (x, x) : \ x \in X \}$ is called the *identity relation of* X.

If R is a relation on X to Y, then by the above definitions we may also say that R is a relation on $X \cup Y$. However, the latter view of the relation R would be quite unnatural for several purposes.

If R is a relation on X to Y, then for any $x \in X$ and $A \subseteq X$ the sets $R(x) = \{ y \in Y : (x, y) \in R\}$ and $R[A] = \bigcup_{x \in A} R(x)$ are called the *images or neighbourhoods of x and A under R,* respectively.

If $(x, y) \in R$, then instead of $y \in R(x)$, we may also write $x\,R\,y$. However, instead of $R[A]$, we cannot write $R(A)$. Namely, it may occur that, in addition to $A \subseteq X$, we also have $A \in X$.

The sets $D_R = \{x \in X : R(x) \neq \emptyset\}$ and $R[X]$ are called the *domain* and *range* of R, respectively. If in particular $D_R = X$, then we say that R is a *relation of X to Y*, or that R is a *non-partial relation on X to Y.*

In particular, a relation f on X to Y is called a *function* if for each $x \in D_f$ there exists $y \in Y$ such that $f(x) = \{y\}$. In this case, by identifying singletons with their elements, we may simply write $f(x) = y$ instead of $f(x) = \{y\}$.

A function $\star$ of X to itself is called a *unary operation on X*. While, a function $*$ of X^2 to X is called a *binary operation on X*. For any $x, y \in X$, we usually write $x^\star$ and $x * y$ instead of $\star(x)$ and $*\big((x, y)\big)$, respectively.

For a relation R on X to Y, we may naturally define two *set-valued functions* φ_R of X to $\mathcal{P}(Y)$ and Φ_R of $\mathcal{P}(X)$ to $\mathcal{P}(Y)$ such that $\varphi_R(x) = R(x)$ for all $x \in X$ and $\Phi_R(A) = R[A]$ for all $A \subseteq X$.

Functions of X to $\mathcal{P}(Y)$ can be naturally identified with relations on X to Y. While, functions of $\mathcal{P}(X)$ to $\mathcal{P}(Y)$ are more general objects than relations on X to Y. In [50, 54, 55], they were briefly called *corelations on X to Y*.

However, if R is a relation on X to Y, U is a relation on $\mathcal{P}(X)$ to Y, and V is a relation on $\mathcal{P}(X)$ to $\mathcal{P}(Y)$, then it is better to say that R is an *ordinary relation*, U is a *super relation*, and V is a *hyper relation* on X to Y.

If R is a relation on X to Y, then $R = \bigcup_{x \in X} \{x\} \times R(x)$. Therefore, the images $R(x)$, where $x \in X$, uniquely determine R. Thus, a relation R on X to Y can also be naturally defined by specifying $R(x)$ for all $x \in X$.

For instance, the *complement* R^c and the *inverse* R^{-1} can be defined such that $R^c(x) = R(x)^c$ for all $x \in X$ and $R^{-1}(y) = \{x \in X : y \in R(x)\}$ for all $y \in Y$. Thus we also have $R^c = X \times Y \setminus R$ and $R^{-1} = \{(y, x) : (x, y) \in R\}$.

Moreover, if in addition S is a relation on Y to Z, then the *composition* $S \circ R$ can be defined such that $(S \circ R)(x) = S[R(x)]$ for all $x \in X$. Thus, it can be easily seen that $(S \circ R)[A] = S\big[R[A]\big]$ also holds for all $A \subseteq X$.

While, if S is a relation on Z to W, then the *box product* $R \boxtimes S$ can be defined such that $(R \boxtimes S)(x, z) = R(x) \times S(z)$ for all $x \in X$ and $z \in Z$. Thus, it can be shown that $(R \boxtimes S)[A] = S \circ A \circ R^{-1}$ for all $A \subseteq X \times Z$ [49].

Hence, by taking $A = \{(x, z)\}$, and $A = \Delta_Y$ if $Y = Z$, one can at once see that the box and composition products are actually equivalent tools. However, the box product can be immediately defined for any family of relations.

Now, a relation R on X may be briefly defined to be *reflexive* on X if $\Delta_X \subseteq R$, and *transitive* if $R \circ R \subseteq R$. Moreover, R may be briefly defined to be *symmetric* if $R^{-1} \subseteq R$, and *antisymmetric* if $R \cap R^{-1} \subseteq \Delta_X$.

Thus, a reflexive and transitive (symmetric) relation may be called a *preorder (tolerance) relation.* And, a symmetric (antisymmetric) preorder relation may be called an *equivalence (partial order) relation.*

For any relation R on X, we may also define $R^0 = \Delta_X$ and $R^n = R \circ R^{n-1}$ if $n \in \mathbb{N}$. Moreover, we may also define $R^\infty = \bigcup_{n=0}^{\infty} R^n$. Thus, it can be shown that R^∞ is the smallest preorder relation on X containing R [17].

For $A \subseteq X$, the *Pervin relation* $R_A = A^2 \cup A^c \times X$ is a preorder on X [30]. While, for a *pseudometric* d on X, the *Weil surrounding* $B_r = \{(x, y) \in X^2 : d(x, y) < r\}$, with $r > 0$, is a tolerance on X [60].

Note that $S_A = R_A \cap R_A^{-1} = R_A \cap R_{A^c} = A^2 \cap \big(A^c\big)^2$ is already an equivalence relation on X. And, more generally, if $\mathcal{A}$ is a *cover (partition)* of X, then $S_{\mathcal{A}} = \bigcup_{A \in \mathcal{A}} A^2$ is a tolerance (equivalence) relation on X.

As an important generalization of the Pervin relation R_A, for any $A \subseteq X$ and $B \subseteq Y$, we may also naturally consider the *Hunsaker-Lindgren relation* $R_{(A,B)} = A \times B \,\cap\, A^c \times Y$ [18]. Namely, thus we evidently have $R_A = R_{(A,A)}$.

The Pervin relations R_A and the Hunsaker-Lindgren relations $R_{(A,B)}$ were actually first used by Davis [11] and Császár [7, pp. 42 and 351] in some less explicit and convenient forms, respectively.

3. Some basic properties of super relations

Notation 3.1. *In this section, we shall assume that U is a super relation on X to Y.*

Remark 3.2. Thus, by our former definitions, U is actually an ordinary relation on $\mathcal{P}(X)$ to Y, i.e., it is an arbitrary subset of $\mathcal{P}(X) \times Y$.

Moreover, U can be identified with the set-valued function φ_U, defined by $\varphi_U(A) = U(A)$ for all $A \subseteq X$, which is a particular subset of $\mathcal{P}(X) \times \mathcal{P}(Y)$.

Thus, several properties of the super relation U can be easily defined with the help of the set-valued function φ_U. For instance, we may naturally introduce

Definition 3.3. The super relation U will be called

1) *increasing* if $U(A) \subseteq U(B)$ for all $A \subseteq B \subseteq X$;

2) *quasi-increasing* if $U\big(\{x\}\big) \subseteq U(A)$ for all $x \in A \subseteq X$;

3) *union-preserving* if $U\big(\bigcup \mathcal{A}\big) = \bigcup_{A \in \mathcal{A}} U(A)$ for all $\mathcal{A} \subseteq \mathcal{P}(X)$.

Thus, we can at once state the following two theorems.

Theorem 3.4. *The following assertions are equivalent:*

1) U *is quasi-increasing;*

2) $\bigcup_{x \in A} U(\{x\}) \subseteq U(A)$ *for all* $A \subseteq X$.

Theorem 3.5. *The following assertions are equivalent:*

1) U *is increasing;*

2) $U\big(\bigcap \mathcal{A}\big) \subseteq \bigcap_{A \in \mathcal{A}} U(A)$ *for all* $\mathcal{A} \subseteq \mathcal{P}(X)$;

3) $\bigcup_{A \in \mathcal{A}} U(A) \subseteq U\big(\bigcup \mathcal{A}\big)$ *for all* $\mathcal{A} \subseteq \mathcal{P}(X)$.

Proof. If $A_1 \subseteq A_2 \subseteq X$, then by using a particular case of 3) we can see that

$$U(A_1) \subseteq U(A_1) \cup U(A_2) \subseteq U(A_1 \cup A_2) = U(A_2),$$

and thus 1) also holds. □

Moreover, by using Definition 3.3 and Theorem 3.5, we can also easily prove

Theorem 3.6. *The following assertions are equivalent:*

1) U *is union-preserving;*

2) $U(A) = \bigcup_{x \in A} U(\{x\})$ *for all* $A \subseteq X$.

Proof. Since $A = \bigcup_{x \in A} \{x\}$ for all $A \subseteq X$, it is clear that 1) implies 2).

While, if 2) holds, then we can at once see that U is increasing. Thus, by Theorem 3.5, we have $\bigcup_{A \in \mathcal{A}} U(A) \subseteq U\big(\bigcup \mathcal{A}\big)$ for all $\mathcal{A} \subseteq \mathcal{P}(X)$. Therefore, to obtain 1), we need only prove the converse inclusion.

For this, note that if $\mathcal{A} \subseteq \mathcal{P}(X)$, then by 2) we have

$$U\big(\bigcup \mathcal{A}\big) = \bigcup_{x \in \bigcup \mathcal{A}} U(\{x\}).$$

Therefore, if $y \in U\big(\bigcup \mathcal{A}\big)$, then there exists $x \in \bigcup \mathcal{A}$ such that $y \in U(\{x\})$. Thus, in particular, there exists $A_0 \in \mathcal{A}$ such that $x \in A_0$, and so, $\{x\} \subseteq A_0$. Hence, by using the increasingness of U, we can already see that

$$y \in U(\{x\}) \subseteq U(A_0) \subseteq \bigcup_{A \in \mathcal{A}} U(A).$$

Therefore, $U\big(\bigcup \mathcal{A}\big) \subseteq \bigcup_{A \in \mathcal{A}} U(A)$ also holds. □

Remark 3.7. In particular, a super relation U on X to itself may be simply called a super relation on X.

Thus, a super relation U on X may be called *extensive, intensive, involutive* and *idempotent* if $A \subseteq U(A)$, $U(A) \subseteq A$, $U\big(U(A)\big) = A$ and $U\big(U(A)\big) = U(A)$ for all $A \subseteq X$, respectively.

Moreover, an increasing involutive (idempotent) super relation may be called an *involution (projection) relation.* While, an extensive (intensive) projection relation may be called a *closure (interior) relation.*

4. Relationships between ordinary and super relations

Notation 4.1. *In this and the next two sections, we shall assume that R and S are ordinary relations, and U and V are super relations on X to Y.*

In [54], having in mind Galois connections [10, 48], we have introduced

Definition 4.2. For the ordinary relation R, we define a super relation $R^{\triangleright}$ on X to Y such that

$$R^{\triangleright}(A) = R[A] \quad \text{for all} \quad A \subseteq X.$$

While, for the super relation U, we define an ordinary relation $U^{\triangleleft}$ on X to Y such that

$$U^{\triangleleft}(x) = U\big(\{x\}\big) \quad \text{for all} \quad x \in X.$$

The appropriateness of the above definitions is apparent from the following two theorems whose proofs are included here only for the reader's convenience.

Theorem 4.3. $R^{\triangleright} \subseteq U$ *implies* $R \subseteq U^{\triangleleft}$.

Proof. If $R^{\triangleright} \subseteq U$, then, in particular, we have

$$R(x) = R\big[\{x\}\big] = R^{\triangleright}\big(\{x\}\big) \subseteq U\big(\{x\}\big) = U^{\triangleleft}(x)$$

for all $x \in X$. Therefore, $R \subseteq U^{\triangleleft}$ also holds. □

Remark 4.4. For the latter inclusion, we have only needed that $R^{\triangleright\triangleleft} \subseteq U^{\triangleleft}$. However, later we shall see that $R^{\triangleright\triangleleft} = R$, and thus $R \subseteq U^{\triangleleft}$ is actually equivalent to $R^{\triangleright\triangleleft} \subseteq U^{\triangleleft}$.

Theorem 4.5. *The following assertions are equivalent:*

1) *U is quasi-increasing;*

2) *$R \subseteq U^{\triangleleft}$ implies $R^{\triangleright} \subseteq U$ for any relation R on X to Y.*

P r o o f. If $R \subseteq U^{\triangleleft}$ and 1) holds, then

$$R^{\triangleright}(A) = R[A] = \bigcup_{x \in A} R(x) \subseteq \bigcup_{x \in A} U^{\triangleleft}(x) = \bigcup_{x \in A} U\big(\{x\}\big) \subseteq U(A)$$

for all $A \subseteq X$. Therefore, $R^{\triangleright} \subseteq U$, and thus 2) also holds.

While, if 2) holds, then because of $U^{\triangleleft} \subseteq U^{\triangleleft}$ we have $U^{\triangleleft\triangleright} = (U^{\triangleleft})^{\triangleright} \subseteq U$. Therefore, for any $A \subseteq X$, we have $U^{\triangleleft\triangleright}(A) \subseteq U(A)$. Moreover, by using the corresponding definitions, we can see that

$$U^{\triangleleft\triangleright}(A) = \big(U^{\triangleleft}\big)^{\triangleright}(A) = U^{\triangleleft}[A] = \bigcup_{x \in A} U^{\triangleleft}(x) = \bigcup_{x \in A} U\big(\{x\}\big).$$

Therefore, $\bigcup_{x \in A} U\big(\{x\}\big) \subseteq U(A)$, and thus assertion 1) also holds. $\square$

Now, as an immediate consequence of the above two theorems, we can also state

Corollary 4.6. *If U is quasi-increasing, then*

$$R^{\triangleright} \subseteq U \quad \iff \quad R \subseteq U^{\triangleleft}.$$

Remark 4.7. This shows that the operations $\triangleright$ and $\triangleleft$ establish a partial Galois connection between the power sets $\mathcal{P}(X \times Y)$ and $\mathcal{P}\big(\mathcal{P}(X) \times Y\big)$.

Therefore, we may also naturally introduce the following

Definition 4.8. The super relation

$$U^{\circ} = U^{\triangleleft\triangleright}$$

will be called the *Galois interior* of U.

Thus, by the proof of Theorem 4.5, we can at once state the following

Theorem 4.9. *We have*

$$U^{\circ}\big(A\big) = \bigcup_{x \in A} U\big(\{x\}\big).$$

for all $A \subseteq X$.

Hence, it is clear that, in particular, we also have

Corollary 4.10. *We have $U^{\circ}\big(\{x\}\big) = U\big(\{x\}\big)$ for all $x \in X$.*

Example 4.11. If in particular $U(A) = A^c$ for all $A \subseteq X$, then for any $A \subseteq X$ we have

$$U^{\circ}(A) = \begin{cases} \emptyset & \text{if} \quad \operatorname{card}(A) = 0, \\ A^c & \text{if} \quad \operatorname{card}(A) = 1, \\ X & \text{if} \quad \operatorname{card}(A) > 1. \end{cases}$$

Namely, by Theorem 4.9 and De Morgan's law, we have

$$U^{\circ}(A)=\bigcup_{x\in A}U\big(\{x\}\big)=\bigcup_{x\in A}\{x\}^{c}=\Big(\bigcap_{x\in A}\{x\}\Big)^{c},$$

whence the required equalities immediately follow.

5. Further theorems on the operations $\triangleright$, $\triangleleft$ and $\circ$

Several properties of the operations $\triangleright$, $\triangleleft$ and $\circ$ can be immediately derived from the general theory of Galois and Pataki connections [57].

However, because of the simplicity of Definition 4.2, it is now more convenient to use some direct proofs to establish the following four theorems.

Theorem 5.1. *The operations $\triangleright$, $\triangleleft$ and $\circ$ are increasing.*

Proof. For instance, if $U\subseteq V$, then $U(A)\subseteq V(A)$ for all $A\subseteq X$. Thus, in particular, we also have

$$U^{\triangleleft}(x)=U\big(\{x\}\big)\subseteq V\big(\{x\}\big)=V^{\triangleleft}(x)$$

for all $X\in X$. Therefore, $U^{\triangleleft}\subseteq V^{\triangleleft}$ also holds. □

Theorem 5.2. *$R^{\triangleright}$ is a union-preserving super relation on X to Y such that*

1) $R^{\triangleright\triangleleft}=R$; 2) $R^{\triangleright\circ}=R^{\triangleright}$.

Proof. By the corresponding definitions, we have

$$R^{\triangleright}(A)=R[A]=\bigcup_{x\in A}R(x)=\bigcup_{x\in A}R\big[\{x\}\big]=\bigcup_{x\in A}R^{\triangleright}\big(\{x\}\big)$$

for all $A\subseteq X$. Thus, by Theorem 3.6, the super relation $R^{\triangleright}$ is union-preserving.

Moreover, we can easily see that

$$R^{\triangleright\triangleleft}(x)=\big(R^{\triangleright}\big)^{\triangleleft}(x)=R^{\triangleright}\big(\{x\}\big)=R\big[\{x\}\big]=R(x)$$

for all $x\in X$. Thus, assertion (1) is also true.

Now, by using Definition 4.8 and assertion (1), we can also easily see that

$$R^{\triangleright\circ}=\big(R^{\triangleright}\big)^{\circ}=\big(R^{\triangleright}\big)^{\triangleleft\triangleright}=\big(R^{\triangleright\triangleleft}\big)^{\triangleright}=R^{\triangleright}.$$

□

Corollary 5.3. *We have $R\subseteq S$ if and only if $R^{\triangleright}\subseteq S^{\triangleright}$.*

Theorem 5.4. *U° is a union-preserving super relation on X to Y such that*

1) $U^{\circ\triangleleft}=U^{\triangleleft}$; 2) $U^{\circ\circ}=U^{\circ}$.

P r o o f. From Definition 4.8, by using Theorem 5.2, we can see that U° is union-preserving and

$$U^{\circ\triangleleft} = \big(U^\circ\big)^{\triangleleft} = \big(U^{\triangleleft\triangleright}\big)^{\triangleleft} = \big(U^{\triangleleft}\big)^{\triangleright\triangleleft} = U^{\triangleleft}.$$

Assertion 1) is also an immediate consequence of Definition 4.2 and Corollary 4.10.

Moreover, by using Theorem 4.9 and its corollary, we can easily see that

$$U^{\circ\circ}(A) = \big(U^\circ\big)^{\circ}(A) = \bigcup_{x\in A} U^\circ\big(\{x\}\big) = \bigcup_{x\in A} U\big(\{x\}\big) = U^\circ(A)$$

for all $A \subseteq X$. Therefore, assertion 2) is also true. □

Theorem 5.5. *The following assertions are equivalent:*

1) $U^\circ = U$; 2) U *is union-preserving;*

3) $U = R^{\triangleright}$ *for some relation* R *on* X *to* Y.

P r o o f. If 2) holds, then by Theorems 4.9 and 3.6 we can see that

$$U^\circ(A) = \bigcup_{x\in A} U\big(\{x\}\big) = U\,(A)$$

for all $A \subseteq X$. Therefore, 1) also holds.

Now, since 1) trivially implies 3), we need only note that if 3) holds, then by Theorem 5.2 assertion 2) also holds. □

Corollary 5.6. *If* U *and* V *are union-preserving, then* $U \subseteq V$ *if and only if* $U^{\triangleleft} \subseteq V^{\triangleleft}$.

Finally, we note that, by using our former results, the following four theorems can also be proved.

Theorem 5.7. *We have*

1) $U \subseteq U^\circ \iff U\,(A) \subseteq U^{\triangleleft}[\,A\,]$ *for all* $A \subseteq X$;

2) $U^\circ \subseteq U \iff U$ *is quasi-increasing* $\iff U^{\triangleleft}[\,A\,] \subseteq U\,(A)$ *for all* $A \subseteq X$;

3) $U^\circ = U \iff U$ *is union-preserving* $\iff U\,(A) = U^{\triangleleft}[\,A\,]$ *for all* $A \subseteq X$.

Theorem 5.8. *We have*

1) $U^\circ \subseteq V \implies U^\circ \subseteq V^\circ \iff U^{\triangleleft} \subseteq V^{\triangleleft}$;

2) $U^\circ \subseteq V^\circ \implies U^\circ \subseteq V$ *if* V *is quasi-increasing;*

3) $U \subseteq V \iff U^\circ \subseteq V^\circ$ *if* U *and* V *are union-preserving.*

Theorem 5.9. *If* $U = R^{\triangleright}$, *then*

1) U *is an union-preserving super relation on* X *to* Y *such that* $U^{\triangleleft} = R$;
2) U *is the smallest quasi-increasing super relation on* X *to* Y *such that* $R \subseteq U^{\triangleleft}$;
3) U *is the largest union-preserving super relation on* X *to* Y *such that* $U^{\triangleleft} \subseteq R$.

Theorem 5.10. *If* $R = U^{\triangleleft}$, *then*

1) $R^{\triangleright} \subseteq U$ *if and only if* U *is quasi-increasing;*
2) $R^{\triangleright} = U$ *if and only if* U *is union-preserving;*
3) *if* U *is quasi-increasing, then* R *is the largest relation on* X *to* Y *such that* $R^{\triangleright} \subseteq U$;
4) *if* U *is union-preserving, then* R *is the smallest relation on* X *to* Y *such that* $U \subseteq R^{\triangleright}$.

6. Relationally defined inverses of super relations

Because of Remark 4.7, we may also naturally introduce the following

Definition 6.1. The super relation

$$U^{\mathbf{-1}} = U^{\triangleleft\, -1\, \triangleright}$$

will be called the *relationally defined inverse* of U.

Remark 6.2. To feel the necessity of this bold inverse $U^{\mathbf{-1}}$, note that the ordinary inverse U^{-1} of U is not a super relation.

While, the ordinary inverse φ_U^{-1} of the associated set-valued function φ_U, which can be identified with U, is usually a hyper relation.

Now, using the corresponding definitions and Theorem 5.2, we can easily prove the following three theorems.

Theorem 6.3. *We have*

1) $R^{\triangleright\, \mathbf{-1}} = R^{-1\, \triangleright}$;   2) $R^{\triangleright\, \mathbf{-1}\, \triangleleft} = R^{-1}$.

P r o o f. By Definition 6.1 and Theorem 5.2, we have

$$R^{\triangleright\, \mathbf{-1}} = R^{\triangleright \triangleleft\, -1\, \triangleright} = R^{-1\, \triangleright}, \quad \text{and thus also} \quad R^{\triangleright\, \mathbf{-1}\, \triangleleft} = R^{-1\, \triangleright \triangleleft} = R^{-1}. \quad \square$$

Theorem 6.4. U^{-1} *is a union-preserving super relation on* Y *to* X *such that*

1) $U^{-1\,\triangleleft} = U^{\triangleleft\,-1}$;   2) $U^{\circ\,-1} = U^{-1}$.

P r o o f. By Definitions 6.1 and 4.8 and Theorem 5.2 we have

$$U^{-1\,\triangleleft} = U^{\triangleleft\,-1\,\triangleright\,\triangleleft} = U^{\triangleleft\,-1} \qquad \text{and} \qquad U^{\circ\,-1} = U^{\triangleleft\triangleright\,\triangleleft\,-1\,\triangleright} = U^{\triangleleft\,-1\,\triangleright} = U^{-1}. \ \square$$

Remark 6.5. Note that if $U^{\triangleleft}$ is symmetric, then $U^{-1} = U^{\triangleleft\,-1\,\triangleright} = U^{\triangleleft\triangleright} = U^{\circ}$. Thus, if in addition U is union-preserving, then $U^{-1} = U$.

In this respect, it is also worth noticing that if, in particular, U is as in Example 4.11, then $U^{\triangleleft}$ is symmetric. Thus, by the above observation,

$$U^{-1} = U^{\circ} .$$

Theorem 6.6. *We have* $\left(U^{-1}\right)^{-1} = U^{\circ}$.

P r o o f. By the corresponding definitions and Theorem 5.2, we have

$$\left(U^{-1}\right)^{-1} = U^{\,\triangleleft\,-1\,\triangleright\,\triangleleft\,-1\,\triangleright} = U^{\,\triangleleft\,-1\,-1\,\triangleright} = U^{\,\triangleleft\triangleright} = U^{\circ} . \qquad \square$$

Hence, by using Theorem 5.5, we can immediately derive

Corollary 6.7. *The following assertions are equivalent:*

1) $U = \left(U^{-1}\right)^{-1}$;   2) U *is union-preserving.*

Moreover, as a counterpart of Theorem 4.9, we can also prove the following

Theorem 6.8. *For any* $B \subseteq Y$, *we have*

$$U^{-1}(B) = \left\{ x \in X : \quad U(\{x\}) \cap B \neq \emptyset \right\} .$$

P r o o f. By the corresponding definitions, we have

$$U^{-1}(B) = \left(U^{\triangleleft\,-1\,\triangleright}\right)(A) = U^{\triangleleft\,-1}\,[\,B\,] .$$

Moreover, it is clear that, for any $x \in X$, we have

$$x \in U^{\triangleleft\,-1}\,[\,B\,] \iff U^{\triangleleft}(x) \cap B \neq \emptyset \iff U\left(\{x\}\right) \cap B \neq \emptyset .$$

Therefore, the required equality is true. $\square$

Remark 6.9. From the above proof, by Theorem 6.4, we can also see that

$$U^{-1}(B) = U^{\triangleleft\,-1}\,[\,B\,] = U^{-1\,\triangleleft}\,[\,B\,] .$$

7. Functionally and relationally defined compositions of super relations

Notation 7.1. *In this section, we shall assume that R is an ordinary relation and U is super relation on X to Y.*

Moreover, we shall also assume that S is an ordinary relation and V is super relation on Y to Z.

By the usual identification of U with φ_U, we may also naturally introduce

Definition 7.2. The super relation $V \circ U$, defined such that

$$(V \circ U)(A) = V\big(U(A)\big)$$

for all $A \subseteq X$, will be called the *functionally defined composition* of V and U.

Remark 7.3. Namely, thus we have

$$\varphi_{V \circ U}(A) = (V \circ U)(A) = V\big(U(A)\big) = \varphi_V\big(\varphi_U(A)\big) = (\varphi_V \circ \varphi_U)(A)$$

for all $A \subseteq X$, and thus $\varphi_{V \circ U} = \varphi_V \circ \varphi_U$.

The appropriateness of Definition 7.2 is also quite obvious from the following three simple theorems and their corollaries.

Theorem 7.4. *We have* $\big(S \circ R\big)^{\triangleright} = S^{\triangleright} \circ R^{\triangleright}$.

Corollary 7.5. *We have*

1) $(S \circ U^{\triangleleft})^{\triangleright} = S^{\triangleright} \circ U$ *if* U *is union-preserving;*

2) $(V^{\triangleleft} \circ R)^{\triangleright} = V \circ R^{\triangleright}$ *if* V *is union-preserving.*

Theorem 7.6. *If* V *is union-preserving, then* $\big(V \circ U\big)^{\triangleleft} = V^{\triangleleft} \circ U^{\triangleleft}$.

Proof. By the corresponding definitions and Theorem 5.5, we have

$$(V \circ U)^{\triangleleft}(x) = (V \circ U)\big(\{x\}\big) = V\Big(U\big(\{x\}\big)\Big) = V\big(U^{\triangleleft}(x)\big) = \\ V^{\circ}\big(U^{\triangleleft}(x)\big) = V^{\triangleleft\triangleright}\big(U^{\triangleleft}(x)\big) = V^{\triangleleft}\big[U^{\triangleleft}(x)\big] = \big(V^{\triangleleft} \circ U^{\triangleleft}\big)(x)$$

for all $x \in X$. Therefore, the required equality is also true. $\square$

Corollary 7.7. *We have*

1) $(S^{\triangleright} \circ U)^{\triangleleft} = S \circ U^{\triangleleft}$;

2) $(V \circ R^{\triangleright})^{\triangleleft} = V^{\triangleleft} \circ R$ *if* V *is union-preserving.*

Theorem 7.8. *If* V *is union-preserving, then* $(V \circ U)^{-1} = U^{-1} \circ V^{-1}$.

P r o o f. By Definition 6.1 and Theorems 7.6, and 7.4, we have

$$(V \circ U)^{-1} = (V \circ U)^{\lhd -1 \rhd} = (V^{\lhd} \circ U^{\lhd})^{-1 \rhd} = \big(U^{\lhd -1} \circ V^{\lhd -1} \big)^{\rhd} = U^{\lhd -1 \rhd} \circ V^{\lhd -1 \rhd} = U^{-1} \circ V^{-1} . \qquad \square$$

Corollary 7.9. *We have* $(S^{\rhd} \circ U)^{-1} = U^{-1} \circ S^{-1 \rhd}$.

Remark 7.10. By using Definition 7.2, it can also be easily seen that the functionally defined composition of super relations is associative.

Now, analogously to Definition 6.1, we may also naturally introduce

Definition 7.11. The super relation

$$V \bullet U = (V^{\lhd} \circ U^{\lhd})^{\rhd}$$

will be called the *relationally defined composition* of V and U.

The appropriateness of this definition is apparent from the following theorems.

Theorem 7.12. *We have*

1) $S^{\rhd} \bullet R^{\rhd} = \big(S \circ R \big)^{\rhd}$; 2) $(S^{\rhd} \bullet R^{\rhd})^{\lhd} = S \circ R$.

Theorem 7.13. $V \bullet U$ *is a union-preserving super relation such that*

$$V \bullet U = V^{\circ} \circ U^{\circ} .$$

P r o o f. From Definition 7.11, by Theorem 5.2, it is clear that $V \bullet U$ is a union-preserving. Moreover, by using Theorem 7.4 and Definition 4.8, we can see that

$$V \bullet U = (V^{\lhd} \circ U^{\lhd})^{\rhd} = V^{\lhd \rhd} \circ U^{\lhd \rhd} = V^{\circ} \circ U^{\circ} . \qquad \square$$

Thus, in particular, by Theorem 5.5, we can also state the following

Corollary 7.14. *If both* U *and* V *are union-preserving, then* $V \bullet U = V \circ U$.

Remark 7.15. From Theorem 7.13, by using Theorems 4.9 and 5.4, we can also infer that

$$\big(V \bullet U \big)(A) = \bigcup_{x \in A} \ \bigcup_{y \in U(\{x\})} V\big(\{ y \} \big)$$

for all $A \subseteq X$.

Now, by using our former results, we can also prove the following

Theorem 7.16. *We have*

$$\big(V \bullet U \big)^{-1} = U^{-1} \circ V^{-1} .$$

P r o o f. By Theorems 7.13, 5.4, 7.8 and 6.4, it is clear that

$$\big(V \bullet U \big)^{-1} = (V^{\circ} \circ U^{\circ})^{-1} = U^{\circ -1} \circ V^{\circ -1} = U^{-1} \circ V^{-1} . \qquad \square$$

Remark 7.17. Moreover, by using Theorem 7.13 and Remark 7.10, it can be easily seen that the relationally defined composition of super relations is also associative.

8. Lower and upper super products of super relations

Notation 8.1. *In this and the next six sections, we shall assume that U, V and W are super relations on X.*

Motivated by an ingenious idea of Przemska [31, 32], we may naturally introduce the following

Definition 8.2. We define two super relations $U \star V$ and $U * V$ on X such that

$$(U \star V)(A) = \big(A \cup U(A) \big) \cap V(A)$$

and

$$(U * V)(A) = \big(A \cap V(A) \big) \cup U(A)$$

for all $A \subseteq X$.

Thus, in particular, we can easily establish the following

Theorem 8.3. *We have*

1) $U \star U = U$; 2) $U * U = U$.

Proof. By the corresponding definitions, it is clear that

$$(U \star U)(A) = \big(A \cup U(A) \big) \cap U(A) = U(A)$$

and

$$(U * U)(A) = \big(A \cap U(A) \big) \cup U(A) = U(A)$$

for all $A \subseteq X$, and thus the required equalities are true. □

Remark 8.4. This theorem shows that the above binary operations $\star$ and $*$ are idempotent.

By the distributivity of the operations $\cap$ and $\cup$, we evidently have the following

Theorem 8.5. *For any $A \subseteq X$, we have*

1) $(U \star V)(A) = \big(A \cap V(A) \big) \cup \big(U(A) \cap V(A) \big)$;

2) $(U * V)(A) = \big(A \cup U(A) \big) \cap \big(U(A) \cup V(A) \big)$.

Hence, it is clear that, in particular, we can also state

Corollary 8.6. *For any* $A \subseteq X$, *we have*

1) $(U \star V)(A) = (V \star U)(A)$ *if* $A \cap U(A) = A \cap V(A)$;

2) $(U * V)(A) = (V * U)(A)$ *if* $A \cup U(A) = A \cup V(A)$.

Proof. For instance, if $A \cap U(A) = A \cap V(A)$, then by using Theorem 8.5 we can see that

$$(U \star V)(A) = \big(A \cap V(A)\big) \cup \big(U(A) \cap V(A)\big) = \big(A \cap U(A)\big) \cup \big(V(A) \cap U(A)\big) = (V \star U)(A). \quad \square$$

Remark 8.7. Note that if for instance $A \subseteq U(A)$, then by the corresponding definitions we have

$$(U \star V)(A) = \big(A \cup U(A)\big) \cap V(A) = U(A) \cap V(A).$$

Therefore, if $A \subseteq U(A) \cap V(A)$, then we also have

$$(U \star V)(A) = U(A) \cap V(A) = V(A) \cap U(A) = (V \star U)(A).$$

However, it is now more important to note that, by using Theorem 8.5, we can also easily prove the following

Theorem 8.8. *We have*

1) $U \star V \subseteq U * V$; 2) $U \star V \subseteq V$; 3) $U \subseteq U * V$.

Proof. By using Theorem 8.5, we can see that

$$(U \star V)(A) \subseteq \big(A \cap V(A)\big) \cup U(A) = (U * V)(A),$$

and quite similarly

$$(U \star V)(A) \subseteq \big(A \cap V(A)\big) \cup V(A) = V(A),$$
$$U(A) = \big(A \cup U(A)\big) \cap U(A) \subseteq (U * V)(A)$$

for all $A \subseteq X$. Thus, the required inclusions are true. $\square$

Remark 8.9. Because of inclusion 1), the super relations $U \star V$ and $U * V$ may be naturally called the *lower and upper super products* of the super relations U and V, respectively.

9. An illustrating example and two general theorems

The following example shows that the operations $\star$ and $*$ are not, in general, equal and commutative.

Example 9.1. If in particular

$$U(A) = A \qquad \text{and} \qquad V(A) = A^c$$

for all $A \subseteq X$, then

$$\begin{aligned}
(U \star V)(A) &= \big(A \cup U(A) \big) \cap V(A) = (A \cup A) \cap A^c = A \cap A^c = \emptyset\,,\\
(V \star U)(A) &= \big(A \cup V(A) \big) \cap U(A) = (A \cup A^c) \cap A = X \cap A = A\,,\\
(U * V)(A) &= \big(A \cap V(A) \big) \cup U(A) = (A \cap A^c) \cup A = \emptyset \cup A = A\,,\\
(V * U)(A) &= \big(A \cap U(A) \big) \cup V(A) = (A \cap A) \cup A^c = A \cup A^c = X
\end{aligned}$$

for all $A \subseteq X$.

Thus, if $\operatorname{card}(X) > 1$, then we have

$$U \star V \neq V \star U\,, \qquad U \star V \neq U * V\,, \qquad U * V \neq V * U\,,$$

despite that $U = \Delta$, $V \star U = \Delta$ and $U * V = \Delta$, and moreover $U \star V$ and $V * U$ are also very particular.

However, by using Theorem 8.5, we can also prove the following

Theorem 9.2. *If $U \subseteq V$, then*

$$U \star V = U * V\,.$$

Proof. Since $U \subseteq V$, we have $U(A) \subseteq V(A)$, and thus $U(A) \cap V(A) = U(A)$ for all $A \subseteq X$.

Hence, by Theorem 8.5, we can see that

$$(U \star V)(A) = \big(A \cap V(A) \big) \cup \big(U(A) \cap V(A) \big) = \big(A \cap V(A) \big) \cup U(A) = (U * V)(A)$$

for all $A \subseteq X$, and thus the required equality is true. □

Analogously to this theorem, we can also prove the following

Theorem 9.3. *If $V \subseteq U$, then*

1) $U \star V = V$; 2) $U * V = U$.

Proof. Since $V \subseteq U$, we have $V(A) \subseteq U(A)$, and thus

$$U(A) \cap V(A) = V(A) \qquad \text{and} \qquad U(A) \cup V(A) = U(A)$$

for all $A \subseteq X$.

Hence, by Theorem 8.5, we can see that

$$(U \star V)(A) = \big(A \cap V(A) \big) \cup \big(U(A) \cap V(A) \big) = \big(A \cap V(A) \big) \cup V(A) = V(A)$$

and

$$(U * V)(A) = \big(A \cup U(A) \big) \cap \big(U(A) \cup V(A) \big) = \big(A \cup U(A) \big) \cap U(A) = U(A)$$

for all $A \subseteq X$. Therefore, the required equalities are true. □

Remark 9.4. Note that $U \subseteq U$, and thus Theorem 8.3 can be derived from Theorem 9.3.

From Theorems 9.2 and 9.3, we can also derive the following two corollaries.

Corollary 9.5. *If U is intensive, then*

1) $U \star \Delta = U * \Delta$; 2) $\Delta \star U = U$; 3) $\Delta * U = \Delta$.

Corollary 9.6. *If U is extensive, then*

1) $\Delta \star U = \Delta * U$; 2) $U \star \Delta = \Delta$; 3) $U * \Delta = U$.

10. Some further theorems on intensive and extensive super relations

Remark 10.1. By using that $\Delta(A) = A$ and

$$(U \cup V)(A) = U(A) \cup V(A) \qquad \text{and} \qquad (U \cap V)(A) = U(A) \cap V(A)$$

for all $A \subseteq X$, Definition 8.2 and Theorem 8.5 can be reformulated in some more concise forms.

Moreover, concerning the operations $\star$ and $*$, we can also prove the following two theorems.

Theorem 10.2. *We have*

1) $U \star V = \Delta \cap V$ *if* U *is intensive;*

2) $U \star V = U \cap V$ *if* U *is extensive.*

Proof. If U is intensive, then $U(A) \subseteq A$, and thus $A \cup U(A) = A$ for all $A \subseteq X$. Hence, we can see that

$$(U \star V)(A) = \big(A \cup U(A)\big) \cap V(A) = A \cap V(A) = \Delta(A) \cap V(A) = (\Delta \cap V)A),$$

for all $A \subseteq X$, and thus (1) is true.

While, if U is extensive, then $A \subseteq U(A)$, and thus $A \cup U(A) = U(A)$ for all $A \subseteq X$. Hence, we can see that

$$(U \star V)(A) = \big(A \cup U(A)\big) \cap V(A) = U(A) \cap V(A) = (U \cap V)A),$$

for all $A \subseteq X$, and thus 2) is also true. $\square$

Theorem 10.3. *We have*

1) $U * V = U \cup V$ *if* V *is intensive;*

2) $U * V = U \cup \Delta$ *if* V *is extensive.*

P r o o f. If V is intensive, then $V(A) \subseteq A$, and thus $A \cap V(A) = V(A)$ for all $A \subseteq X$. Hence, we can see that

$$(U * V)(A) = \big(A \cap V(A)\big) \cup U(A) = V(A) \cup U(A) = (U \cup V) A),$$

for all $A \subseteq X$, and thus 1) is true.

While, if V is extensive, then $A \subseteq V(A)$, and thus $A \cap V(A) = A$ for all $A \subseteq X$. Hence, we can see that

$$(U * V)(A) = \big(A \cap V(A)\big) \cup U(A) = A \cup U(A) = U(A) \cup \Delta(A) = (U \cup \Delta)(A)$$

for all $A \subseteq X$, and thus 2) is also true. $\square$

Remark 10.4. From the above two theorems, we can see that

1) $U \star V = V \star U$ if both U and V are extensive;

2) $U * V = V * U$ if both U and V are intensive.

From Theorems 10.2 and 10.3, by using that $\Delta \subseteq \Delta$, we can also derive

Corollary 10.5. *We have*

1) $\Delta \star U = \Delta \cap U$; 2) $U * \Delta = U \cup \Delta$.

Moreover, by using Theorems 10.2 and 9.2, we can easily prove the following

Theorem 10.6. *If U is intensive and V is extensive, then*

1) $U \star V = \Delta$; 2) $U * V = \Delta$.

P r o o f. By Theorem 10.2, for any $A \subseteq X$, we have

$$(U \star V)(A) = \big(\Delta \cap V\big)(A) = \Delta(A) \cap V(A) = A \cap V(A) = A = \Delta(A).$$

Therefore, 1) is true. Thus, by Theorem 9.2, assertion 2) is also true. $\square$

11. Associativity properties of the operations $\star$ and $*$

By using Theorem 8.5 and the distributivity properties of $\cap$ and $\cup$, we can prove the following two theorems

Theorem 11.1. *We have*

1) $(U \star V) \star W = U \star (V \star W)$;

2) $\big((U \star V) \star W\big)(A) = \big(A \cap W(A)\big) \cup \big(U(A) \cap V(A) \cap W(A)\big)$ *for all* $A \subseteq X$.

P r o o f. For any $A \subseteq X$, we have

$$\big((U \star V) \star W\big)(A) = \big(A \cap W(A)\big) \cup \big((U \star V)(A) \cap W(A)\big)$$

and

$$(U \star V)(A) \cap W(A) = \Big(\big(A \cap V(A)\big) \cup \big(U(A) \cap V(A)\big)\Big) \cap W(A)\big) = \big(A \cap V(A) \cap W(A)\big) \cup \big(U(A) \cap V(A) \cap W(A)\big).$$

Therefore,

$$\big((U \star V) \star W\big)(A) = \big(A \cap W(A)\big) \cup \big(A \cap V(A) \cap W(A)\big) \cup \big(U(A) \cap V(A) \cap W(A)\big) = \big(A \cap W(A)\big) \cup \big(U(A) \cap V(A) \cap W(A)\big)$$

For any $A \subseteq X$, we also have

$$\big(U \star (V \star W)\big)(A) = \big(A \cap (V \star W)(A)\big) \cup \big(U(A) \cap (V \star W)(A)\big),$$

and moreover,

$$A \cap (V \star W)(A) = A \cap \Big(\big(A \cap W(A)\big) \cup \big(V(A) \cap W(A)\big)\Big) = \big(A \cap W(A)\big) \cup \big(A \cap V(A) \cap W(A)\big) = A \cap W(A)$$

and

$$\big(U(A) \cap (V \star W)(A)\big) = U(A) \cap \Big(\big(A \cap W(A)\big) \cup \big(V(A) \cap W(A)\big)\Big) = \big(U(A) \cap A \cap W(A)\big) \cup \big(U(A) \cap V(A) \cap W(A)\big).$$

Therefore,

$$\big(U \star (V \star W)\big)(A) = \big(A \cap W(A)\big) \cup \big(U(A) \cap A \cap W(A)\big) \cup \big(U(A) \cap V(A) \cap W(A)\big) = \big(A \cap W(A)\big) \cup \big(U(A) \cap V(A) \cap W(A)\big).$$

Thus, the required assertions are true. $\square$

Theorem 11.2. *We have*

1) $(U * V) * W = U * (V * W)$*;*
2) $\big((U * V) * W\big)(A) = \big(A \cup U(A)\big) \cap \big(U(A) \cup V(A) \cup W(A)\big)$ *for all* $A \subseteq X$*.*

P r o o f. For any $A \subseteq X$, we have

$$\big((U * V) * W\big)(A) = \big(A \cup (U * V)(A)\big) \cap \big((U * V)(A) \cup W(A)\big),$$

and moreover,

$$A \cup (U * V)(A) = A \cup \Big(\big(A \cup U(A)\big) \cap \big(U(A) \cup V(A)\big)\Big) = \big(A \cup U(A)\big) \cap \big(A \cup U(A) \cup V(A)\big) = A \cup U(A)$$

and

$$(U * V)(A) \cup W(A) = \Big(\big(A \cup U(A)\big) \cap \big(U(A) \cup V(A)\big)\Big) \cup W(A) = \big(A \cup U(A) \cup W(A)\big) \cap \big(U(A) \cup V(A) \cup W(A)\big).$$

Therefore,

$$\big((U * V) * W\big)(A) = \big(A \cup U(A)\big) \cap \big(A \cup U(A) \cup W(A)\big) \cap \big(U(A) \cup V(A) \cup W(A)\big) = \big(A \cup U(A)\big) \cap \big(U(A) \cup V(A) \cup W(A)\big).$$

For any $A \subseteq X$, we also have

$$\big(U * (V * W)\big)(A) = \big(A \cup U(A)\big) \cap \big(U(A) \cup (V * W)(A)\big)$$

and

$$U(A) \cup (V * W)(A) = \Big(U(A) \cup \big(A \cup V(A)\big) \cap \big(V(A) \cup W(A)\big)\Big) = \big(U(A) \cup A \cup V(A)\big) \cap \big(U(A) \cup V(A) \cup W(A)\big).$$

Therefore,

$$\big(U * (V * W)\big)(A) = \big(A \cup U(A)\big) \cap \big(U(A) \cup A \cup V(A)\big) \cap \big(U(A) \cup V(A) \cup W(A)\big) = \big(A \cup U(A)\big) \cap \big(U(A) \cup V(A) \cup W(A)\big).$$

Thus, the required assertions are true. $\square$

Remark 11.3. In addition to the above associativity properties, we can easily establish some increasingness properties of the operations $\star$ and $*$.

For instance, if $U \subseteq V$, then by using Definition 8.2 we can at once see that

$$(U \star W)(A) = \big(A \cup U(A)\big) \cap W(A) \subseteq \big(A \cup V(A)\big) \cap W(A) = (V \star W)(A)$$

for all $A \subseteq X$, and thus $U \star W \subseteq V \star W$.

Remark 11.4. However, we could neither prove nor disprove the distributivity properties of the operations $\star$ and $*$ over each other.

12. Subsets and supersets of the sets $(U \star V)(A)$ and $(U * V)(A)$

Now, by extending an argument of Przemska [31,32], we can also prove the following two theorems and their corollaries.

Theorem 12.1. *For any* $A,\, B \subseteq X$, *we have*

1) $B \subseteq (U \star V)(A)$ *if and only if* $B \subseteq A \cup U(A)$ *and* $B \subseteq V(A)$;

2) $(U \star V)(A) \subseteq B$ *if and only if* $A \cap V(A) \subseteq B$ *and* $U(A) \cap V(A) \subseteq B$.

P r o o f. By Definition 8.2 and Theorem 8.5, we have

$$B \subseteq (U \star V)(A) \iff B \subseteq \big(A \cup U(A)\big) \cap V(A) \iff B \subseteq A \cup U(A),\ \ B \subseteq V(A)$$

and

$$\begin{aligned}(U \star V)(A) \subseteq B \quad &\iff \quad \big(A \cap V(A)\big) \cup \big(U(A) \cap V(A)\big) \subseteq B \\ &\iff \quad A \cap V(A) \subseteq B, \quad U(A) \cap V(A) \subseteq B.\end{aligned}$$

Therefore, assertions 1) and 2) are true. □

Corollary 12.2. *For any* $A \subseteq X$, *we have*

1) $A \subseteq (U \star V)(A)$ *if and only if* $A \subseteq V(A)$;

2) $(U \star V)(A) \subseteq A$ *if and only if* $U(A) \cap V(A) \subseteq A$.

Corollary 12.3. *For any* $A \subseteq X$, *we have*

$$A = (U \star V)(A) \quad \iff \quad U(A) \cap V(A) \subseteq A \subseteq V(A).$$

Theorem 12.4. *For any* $A,\, B \subseteq X$, *we have*

1) $(U * V)(A) \subseteq B$ *if and only if* $A \cap V(A) \subseteq B$ *and* $U(A) \subseteq B$;

2) $B \subseteq (U * V)(A)$ *if and only if* $B \subseteq A \cup U(A)$ *and* $B \subseteq U(A) \cup V(A)$.

P r o o f. By Definition 8.2 and Theorem 8.5, we have

$$(U * V)(A) \subseteq B \iff \big(A \cap V(A)\big) \cup U(A) \subseteq B \iff A \cap V(A) \subseteq B,\ \ U(A) \subseteq B$$

and

$$\begin{aligned}B \subseteq (U * V)(A) \quad &\iff \quad B \subseteq \big(A \cup U(A)\big) \cap \big(U(A) \cup V(A)\big) \iff \\ &\qquad B \subseteq A \cup U(A), \quad B \subseteq U(A) \cup V(A).\end{aligned}$$

Therefore, assertions 1) and 2) are true. □

Corollary 12.5. *For any* $A \subseteq X$, *we have*

1) $(U * V)(A) \subseteq A$ *if and only if* $U(A) \subseteq A$ *;*

2) $A \subseteq (U * V)(A)$ *if and only if* $A \subseteq U(A) \cup V(A)$ *.*

Corollary 12.6. *For any* $A \subseteq X$, *we have*

$$A = (U * V)(A) \quad \Longleftrightarrow \quad U(A) \subseteq A \subseteq U(A) \cup V(A) .$$

Now, by Corollaries 12.3 and 12.6, we can also state the following two theorems.

Theorem 12.7. *If* $V \subseteq U$, *then for any* $A \subseteq X$ *we have*

1) $A = (U \star V)(A)$ *if and only if* $A = V(A)$*;*

2) $A = (U * V)(A)$ *if and only if* $A = U(A)$ *.*

Theorem 12.8. *If* $U \subseteq V$, *then for any* $A \subseteq X$ *the following assertions are equivalent:*

1) $A = (U \star V)(A)$ *;* 2) $A = (U * V)(A)$ *;* 3) $U(A) \subseteq A \subseteq V(A)$ *.*

Remark 12.9. By using Corollaries 12.3 and 12.6 and Theorems 12.7 and 12.8, the fixed points of the super relations $U \star V$ and $U * V$ can be easily determined.

13. Lower and upper hyper products of super relations

Because of our former results and some closely related definitions of Levine [24] , Corson and Michael [6] and Andrijević [2] , we may also naturally introduce

Definition 13.1. We define two hyper relations $U \bigstar V$ and $U \circledast V$ on X such that

$$(U \bigstar V)(A) = \big\{ B \subseteq X : \ (U \star V)(A) \subseteq B \subseteq (U * V)(A) \big\}$$

and

$$(U \circledast V)(A) = \big\{ B \subseteq X : \ (U \cap V)(A) \subseteq B \subseteq (U \cup V)(A) \big\}$$

for all $A \subseteq X$.

Thus, by the above definition and Theorem 8.3, we can at once state

Theorem 13.2. *We have*

1) $U \bigstar U = U$ *;* 2) $U \circledast U = U$ *.*

P r o o f. By Theorem 8.3, for any $A \subseteq X$, we have

$$(U \star U)(A) = U(A) = (U * U)(A).$$

Hence, by using Definition 13.1, we can infer that

$$(U \bigstar U)(A) = \big\{ U(A) \big\}.$$

Thus, $U \bigstar U$ is actually a function of $\mathcal{P}(X)$ such that $(U \bigstar U)(A) = U(A)$ for all $A \subseteq X$.

Therefore, by the usual identification of relations with set-valued functions, equality 1) can be stated without any danger of confusions. $\square$

Remark 13.3. This theorem shows that the above binary operations $\bigstar$ and $\divideontimes$ are, in a certain sense, also idempotent.

Moreover, from Definition 13.1, we can see that, in contrast to the operations $\star$, $*$ and $\bigstar$, the operation $\divideontimes$ is commutative.

By using Theorem 8.5, we can easily prove the following

Theorem 13.4. *We have*

$$U \bigstar V \subseteq U \divideontimes V.$$

P r o o f. If $A \subseteq X$ and $B \in (U \bigstar V)(A)$, then by Definition 13.1 we have

$$(U \star V)(A) \subseteq B \subseteq (U * V)(A).$$

Hence, by using Theorem 8.5, we can infer that

$$\big(A \cap V(A) \big) \cup (U \cap V)(A) \subseteq B \subseteq \big(A \cup U(A) \big) \cap (U \cup V)(A).$$

This implies that

$$(U \cap V)(A) \subseteq B \subseteq (U \cup V)(A)$$

and thus by Definition 13.1 we also have $B \in (U \divideontimes V)(A)$.

Therefore, $(U \bigstar V)(A) \subseteq (U \divideontimes V)(A)$ for all $A \subseteq X$, and thus the required inclusion is also true. $\square$

Remark 13.5. Because of this theorem, the hyper relations $U \bigstar V$ and $U \divideontimes V$ may be naturally called the *lower and upper hyper products* of the super relations U and V, respectively.

Now, by using Example 9.1 and Definition 13.1, we can also easily establish

Example 13.6. If U and V are as in Example 9.1, then for any $A \subseteq X$ we have

$$(U \bigstar V)(A) = \mathcal{P}(A), \quad (V \bigstar U)(A) = \mathcal{P}^{-1}(A) \quad \text{and} \quad (U \divideontimes V)(A) = \mathcal{P}(X).$$

To prove the second equality, note that by Definition 13.1 and Example 9.1, for any $B \subseteq X$, we have

$$B \in (V \bigstar U)(A) \iff (V \star U)(A) \subseteq B \subseteq (V \ast U)(A) \iff$$
$$A \subseteq B \subseteq X \iff A \in \mathcal{P}(B) \iff B \in \mathcal{P}^{-1}(A)\,.$$

Moreover, by using Definition 13.1 and Theorems 9.2 and 9.3, we can easily establish the following two theorems.

Theorem 13.7. *If* $U \subseteq V$, *then we have*

1) $U \bigstar V = U \star V = U \ast V$;

2) $(U \divideontimes V)(A) = \big\{ B \subseteq X : \quad U(A) \subseteq B \subseteq V(A) \big\}$ *for all* $A \subseteq X$.

Theorem 13.8. *If* $V \subseteq U$, *then we have*

1) $U \bigstar V = V \divideontimes U$;

2) $(V \divideontimes U)(A) = \big\{ B \subseteq X : \quad V(A) \subseteq B \subseteq U(A) \big\}$ *for all* $A \subseteq X$.

Remark 13.9. Note that $U \subseteq U$, and thus Theorem 13.2 can be derived from Theorem 13.8.

14. Some further theorems on the operations $\bigstar$ and $\divideontimes$

By using Definition 13.1 and Corollaries 9.5 and 9.6, we can easily establish the following two theorems.

Theorem 14.1. *If* U *is intensive, then*

1) $U \bigstar \Delta = U \star \Delta = U \ast \Delta$; 2) $\Delta \bigstar U = U \divideontimes \Delta$;

3) $(U \divideontimes \Delta)(A) = \big\{ B \subseteq X : \quad U(A) \subseteq B \subseteq A \big\}$ *for all* $A \subseteq X$.

Theorem 14.2. *If* U *is extensive, then*

1) $\Delta \bigstar U = \Delta \star U = \Delta \ast U$; 2) $U \bigstar \Delta = \Delta \divideontimes U$;

3) $(\Delta \divideontimes U)(A) = \big\{ B \subseteq X : \quad A \subseteq B \subseteq U(A) \big\}$ *for all* $A \subseteq X$.

Moreover, by Definition 13.1 and Theorem 10.6, we can also state

Theorem 14.3. *If* U *is intensive and* V *is extensive, then*

1) $U \bigstar V = \Delta$;

2) $(U \divideontimes V)(A) = \big\{ B \subseteq X : \quad U(A) \subseteq B \subseteq V(A) \big\}$ *for all* $A \subseteq X$.

Now, as an immediate consequence of Definition 13.1 and Theorems 12.8 and 13.7, we can also state

Theorem 14.4. *If $U \subseteq V$, then for any $A \subseteq X$ the following assertions are equivalent:*

1) $A = (U \star V)(A)$; 2) $A = (U * V)(A)$;

3) $A = (U \bigstar V)(A)$; 4) $A \in (U \circledast V)(A)$;

5) $U(A) \subseteq A \subseteq V(A)$.

Thus, in particular, we can also state

Corollary 14.5. *If $U \subseteq V$, then the following assertions are equivalent:*

1) $U \star V = \Delta$; 2) $U * V = \Delta$;

3) $U \bigstar V = \Delta$; 4) $U \circledast V$ *is reflexive;*

5) U *is intensive and* V *is extensive.*

In this respect, it is also worth proving the following

Theorem 14.6. *If $U \subseteq V$ and*

1) U *is extensive or increasing and lower semi-idempotent;*

2) V *is intensive or increasing and upper semi-idempotent;*

then $U \circledast V$ is transitive.

P r o o f. If $A, B, C \subseteq X$ such that

$$B \in (U \circledast V)(A) \qquad \text{and} \qquad C \in (U \circledast V)(B),$$

then by Theorem 13.7 we have

$$U(A) \subseteq B \subseteq V(A) \qquad \text{and} \qquad U(B) \subseteq C \subseteq V(B).$$

Hence, if, for instance, U is extensive, we can infer that

$$U(A) \subseteq B \subseteq U(B) \subseteq C.$$

Moreover, if, for instance, V is increasing and upper semi-idempotent, then we can also note that

$$C \subseteq V(B) \subseteq V\big(V(A)\big) \subseteq V(A).$$

Therefore,

$$U(A) \subseteq C \subseteq V(A), \qquad \text{and thus} \qquad C \in (U \circledast V)(A).$$

Consequently, in this particular case, $U \circledast V$ is transitive. □

15. The duals of super and hyper relations

Having in mind the relationship between the usual closure and interior operations, we may naturally introduce the following

Definition 15.1. For a super relation U on X to Y, we define a dual super relation $U^\star$ on X to Y such that

$$U^\star(A) = U\,(\,A^c\,)^{\,c}$$

for all $A \subseteq X$.

Thus, we can easily prove the following four theorems.

Theorem 15.2. *If U and V are super relations on X to Y, then*

1) $U = U^{\star\star}$; 2) $U \subseteq V$ *implies* $V^\star \subseteq U^\star$.

P r o o f. To prove 2), note that if $U \subseteq V$, then $U\,(A^c) \subseteq V\,(A^c)$, and thus

$$V^\star(A) = V\,(A^c)^c \subseteq U\,(A^c)^c = U^\star(A)$$

for all $A \subseteq X$. Therefore, $V^\star \subseteq U^\star$ also holds. □

Theorem 15.3. *If U is a super relation on X to Y, then*

1) $U^\star$ *is increasing if and only if* U *is increasing;*

2) $U^\star$ *is union-preserving if and only if* U *is intersection-preserving;*

3) $U^\star$ *is intersection-preserving if and only if* U *is union-preserving.*

P r o o f. If, for instance, U is union-preserving, then by the corresponding definitions and De Morgan's law we have

$$U^\star\big(\textstyle\bigcap_{A\in\mathcal{A}} A\,\big) = U\Big(\big(\textstyle\bigcap_{A\in\mathcal{A}} A\,\big)^c\Big)^c = U\,\big(\textstyle\bigcup_{A\in\mathcal{A}} A^c\big)^c =$$

$$\big(\textstyle\bigcup_{A\in\mathcal{A}} U\,(A^c)\big)^c = \textstyle\bigcap_{A\in\mathcal{A}} U\,(A^c)^c = \textstyle\bigcap_{A\in\mathcal{A}} U^\star(A)$$

for all $\mathcal{A} \subseteq \mathcal{P}(X)$. Therefore, $U^\star$ is intersection-preserving.

Thus, the "if part" of assertion 3) is true. Hence, since $U^{\star\star} = U$, it is clear that the "only if part" of assertion 2) is also true. □

Theorem 15.4. *If U is a super relation on X to Y, then*

1) $U^\star$ *is intensive if and only if* U *is extensive;*

2) $U^\star$ *is extensive if and only if* U *is intensive;*

3) $U^\star$ *is involutive if and only if* U *is involutive;*

4) $U^\star$ *is idempotent if and only if* U *is idempotent.*

Proof. For instance, if U is idempotent, then by the corresponding definitions

$$U^{\star}\big(U^{\star}(A)\big)=U^{\star}\big(U\,(A^{c})^{c}\big)=U\,\big(U\,(A^{c})\big)^{c}=U\,(A^{c})^{c}=U^{\star}(A)$$

for all $A\subseteq X$. Therefore, $U^{\star}$ is also idempotent.

Thus, the "if part" of assertion (4) is true. Hence, since $U^{\star\star}=U$, it is clear that the "only if part" of assertion (4) is also true. $\square$

Theorem 15.5. *If U and V are super relations on X to Y, then*

1) $(U\star V)^{\star}=V^{\star}*U^{\star}$; 2) $(U*V)^{\star}=V^{\star}\star U^{\star}$.

Proof. By Definitions 15.1 and 8.2 and De Morgan's laws, we have

$$(U\star V)^{\star}(A)=(U\star V)(A^{c})^{c}=\Big(\big(A^{c}\cup U\,(A^{c})\big)\cap V\,(A^{c})\Big)^{c}=$$
$$\big(A^{c}\cup U\,(A^{c})\big)^{c}\cup V\,(A)^{c}=\big(A\cap U\,(A^{c})^{c}\big)\cup V\,(A^{c})^{c}=$$
$$\big(A\cap U^{\star}(A)\big)\cup V^{\star}(A)=(V^{\star}*U^{\star})(A)$$

for all $A\subseteq X$. Therefore, assertion (1) is true.

From assertion (1), by using Theorem 15.2 we can infer that

$$(V^{\star}*U^{\star})^{\star}=(U\star V)^{\star\star}=U\star V.$$

Hence, by writing $U^{\star}$ in place of V and $V^{\star}$ in place of U, we can see that assertion (2) is also true. $\square$

Remark 15.6. From assertion 2), by using Theorem 15.2, we can infer that $U*V=\big(V^{\star}\star U^{\star}\big)^{\star}$.

Therefore, the properties of the binary operation $*$ can be derived from those of the binary operation $\star$ and the unary operation $\star$.

Analogously to Definition 15.1, we may also naturally introduce the following

Definition 15.7. For a hyper relation V on X to Y, we define two *dual hyper relations* $V^{\star}$ and $V^{\bigstar}$ on X to Y such that

$$V^{\star}(A)=V\,(A^{c})^{c}=\mathcal{P}\,(Y)\setminus V\,(A^{c})$$

and

$$V^{\bigstar}(A)=\big[\,V\,(A^{c})\big]^{c}=\big\{\,B^{c}:\ B\in V\,(A^{c})\big\}$$

for all $A\subseteq X$.

Remark 15.8. Thus, some properties of the hyper relations $V^{\star}$ and $V^{\bigstar}$ can also be easily derived from those of the hyper relation V.

Moreover, having in mind the derivations of small closures and interiors from the big ones [41, 42], we may also naturally introduce the following

Definition 15.9. For a hyper relation V on X to Y, we define a super relation $V^{\blacktriangleleft}$ on X to Y such that

$$V^{\blacktriangleleft}(A) = \{ y \in Y : \{y\} \in V(A) \}$$

for all $A \subseteq X$.

Remark 15.10. Thus, in some cases, we may also naturally consider the ordinary relation $V^{\blacktriangleleft \triangleleft}$.

However, for instance, for two super relations U and V it may be very hard to determine the super relation $(U \bigstar V)^{\blacktriangleleft}$.

Example 15.11. Of course, if U and V are as in Example 13.6, then for any $A \subseteq X$ and $y \in X$ we have

$$y \in (U \bigstar V)^{\blacktriangleleft}(A) \iff \{y\} \in (U \bigstar V)(A) \iff \{y\} \in \mathcal{P}(A) \iff$$
$$\{y\} \subseteq A \iff y \in A \iff y \in U(A).$$

Therefore, $(U \bigstar V)^{\blacktriangleleft}(A) = U(A)$ for all $A \subseteq X$, and thus $(U \bigstar V)^{\blacktriangleleft} = U$.

Hence, by Corollary 10.5, we can see that $(U \bigstar V)^{\blacktriangleleft} \star W = U \star W = U \cap W$ for any super relation W on X. Thus, in particular, $(U \bigstar V)^{\blacktriangleleft} \star W = U$ if W is extensive.

16. Applications to generalized topologically open sets

Now, following the ideas of [33, 34, 59], we shall show that the ordinary, super and hyper relations can be used to treat, in a general, unified framework, the various generalized open sets studied by a great number of topologists.

For this, we shall assume that $\mathcal{R}$, $\mathcal{U}$ and $\mathcal{V}$ are *ordinary, super and hyper relators* on X in the sense that they are arbitrary families of ordinary, super and hyper relations on X, respectively.

And, following the ideas of [42, 54], in the *non-conventional three relator space* $X(\mathcal{R}, \mathcal{U}, \mathcal{V})$, for any $A, B \subseteq X$ and $x \in X$ we define

1) $B \in \operatorname{Int}_{\mathcal{U}}(A)$ if $U(B) \subseteq A$ for some $U \in \mathcal{U}$;

2) $B \in \operatorname{Cl}_{\mathcal{U}}(A)$ if $U(B) \cap A \neq \emptyset$ for all $U \in \mathcal{U}$;

3) $x \in \operatorname{int}_{\mathcal{U}}(A)$ if $\{x\} \in \operatorname{Int}_{\mathcal{U}}(A)$;

4) $x \in \operatorname{cl}_{\mathcal{U}}(B)$ if $\{x\} \in \operatorname{Cl}_{\mathcal{U}}(B)$;

5) $A \in \tau_{\mathcal{U}}$ if $A \in \operatorname{Int}_{\mathcal{U}}(A)$;

6) $A \in \tau\!\!\!-_{\mathcal{U}}$ if $A^c \notin \operatorname{Cl}_{\mathcal{U}}(A)$;

7) $A \in \mathcal{T}_{\mathcal{U}}$ if $A \subseteq \operatorname{int}_{\mathcal{U}}(A)$;

8) $A \in \mathcal{F}_{\mathcal{U}}$ if $\operatorname{cl}_{\mathcal{U}}(A) \subseteq A$;

9) $A \in \mathcal{E}_{\mathcal{U}}$ if $\operatorname{int}_{\mathcal{U}}(A) \neq \emptyset$;

10) $A \in \mathcal{D}_{\mathcal{U}}$ if $\operatorname{cl}_{\mathcal{U}}(A) = X$;

11) $A \in \mathcal{N}_{\mathcal{U}}$ if $\operatorname{cl}_{\mathcal{U}}(A) \notin \mathcal{E}_{\mathcal{U}}$;

12) $A \in \mathcal{M}_{\mathcal{U}}$ if $\operatorname{int}_{\mathcal{U}}(A) \in \mathcal{D}_{\mathcal{U}}$.

Thus, by using the plausible definitions $\operatorname{Int}_{\mathcal{R}} = \operatorname{Int}_{\mathcal{R}^{\triangleright}}$ and $\operatorname{int}_{\mathcal{R}} = \operatorname{int}_{\mathcal{R}^{\triangleright}}$, with $\mathcal{R}^{\triangleright} = \{R^{\triangleright} : \ R \in \mathcal{R}\}$, it can be easily shown that $\operatorname{Int}_{\mathcal{U}^{\circ}} = \operatorname{Int}_{\mathcal{U}^{\triangleleft}}$ and $\operatorname{int}_{\mathcal{U}} = \operatorname{int}_{\mathcal{U}^{\circ}} = \operatorname{int}_{\mathcal{U}^{\triangleleft}}$. However, $\operatorname{Int}_{\mathcal{U}}$ is usually a more general tool than $\operatorname{Int}_{\mathcal{R}}$.

Moreover, it can also be easily shown that

$$\mathcal{T}_{\mathcal{U}} = \big\{ A \subseteq X : \quad \exists\ U \in \mathcal{U} : \quad A \subseteq U^{\star}(A) \big\}.$$

Thus, in particular

$$\mathcal{T}_{U} = \big\{ A \subseteq X : \ A \subseteq U^{\star}(A) \big\} \qquad \text{and} \qquad \mathcal{T}_{U^{\star}} = \big\{ A \subseteq X : \ A \subseteq U(A) \big\}$$

for any super relation U on X.

Hence, by taking

$$U(A) = \operatorname{cl}_{\mathcal{R}}\big(\operatorname{int}_{\mathcal{R}}(A)\big) \quad \text{for all} \quad A \subseteq X,$$

and using that $U^{\star}(A) = \operatorname{int}_{\mathcal{R}}\big(\operatorname{cl}_{\mathcal{R}}(A)\big)$ for all $A \subseteq X$, we can at once see that $\mathcal{T}_{U}$ and $\mathcal{T}_{U^{\star}}$ are just the families of all *topologically preopen and semi-open subsets* of the relator space $X(\mathcal{R})$ considered in [35].

Thus, by taking

$$U_1(A) = \operatorname{cl}_{\mathcal{R}}\Big(\operatorname{int}_{\mathcal{R}}\big(\operatorname{cl}_{\mathcal{R}}(A)\big)\Big) \quad \text{and} \quad U_2(A) = \operatorname{cl}_{\mathcal{R}}\big(\operatorname{int}_{\mathcal{R}}(A)\big) \cup \operatorname{int}_{\mathcal{R}}\big(\operatorname{cl}_{\mathcal{R}}(A)\big)$$

for all $A \subseteq X$, we can quite similarly obtain some further important classes of generalized topologically open subsets of the relator space $X(\mathcal{R})$ [35,36].

However, it is now more important to note that, for any $\mathcal{A} \subseteq \mathcal{P}(X)$, we may also naturally define

$$\mathcal{A}^{k} = \mathcal{A}^{k_{\mathcal{V}}} = \operatorname{cl}_{\mathcal{V}}(\mathcal{A}) \qquad \text{and} \qquad \mathcal{A}^{\ell} = \mathcal{A}^{\ell_{\mathcal{V}}} = \operatorname{cl}_{\mathcal{V}^{-1}}(\mathcal{A}).$$

Namely, thus it can be easily seen that, for any $\mathcal{A} \subseteq \mathcal{P}(X)$ and $B \subseteq X$, we have

a) $B \in \mathcal{A}^{k}$ if and only if for each $V \in \mathcal{V}$ there exists $A \in \mathcal{A}$ such that $A \in V(B)$;

b) $B \in \mathcal{A}^{\ell}$ if and only if for each $V \in \mathcal{V}$ there exists $A \in \mathcal{A}$ such that $B \in V(A)$.

Therefore, if, in particular, V is a hyper relation on X such that

$$V(A) = \big\{ B \subseteq X : \quad A \subseteq B \subseteq \operatorname{cl}_{\mathcal{R}}(A) \big\} \qquad \text{for all} \qquad A \subseteq X,$$

then we can at once see that $\mathcal{T}_{\mathcal{R}}^{\ell_V}$ and $\mathcal{T}_{\mathcal{R}}^{k_V}$ are just the families of all *topologically quasi-open and pseudo-open subsets* of the relator space $X(\mathcal{R})$ considered also in [35].

Concerning the above particular relations

$$U = \mathrm{cl}_{\mathcal{R}} \circ \mathrm{int}_{\mathcal{R}} \qquad \text{and} \qquad V = \Delta \circledast \mathrm{cl}_{\mathcal{R}}\,,$$

in [33, Section 32] we have proved the following assertions and their counterparts.

A) $\mathcal{T}_{\mathcal{R}}^{k_V} \subseteq \mathcal{F}_U$ is always true;

B) $\mathcal{T}_{\mathcal{R}} \subseteq \mathcal{T}_{\mathcal{R}}^{k_V} \subseteq \mathcal{F}_U^{k_V}$ if $\mathcal{R}$ is *reflexive* in the sense that $x \in R(x)$ for all $x \in X$ and $R \in \mathcal{R}$;

C) $\mathcal{F}_U^{k_V} \subseteq \mathcal{F}_U$ if $\mathcal{R}$ is *quasi-topological* in the sense that $x \in \mathrm{int}_{\mathcal{R}}\big(\mathrm{int}_{\mathcal{R}}(R(x))\big)$ for all $x \in X$ and $R \in \mathcal{R}$;

D) $\mathcal{T}_{\mathcal{R}}^{k_V} = \mathcal{F}_U^{k_V} = \mathcal{F}_U$ if $\mathcal{R}$ is *topological* in the sense that for each $x \in X$ and $R \in \mathcal{R}$ there exists $V \in \mathcal{T}_{\mathcal{R}}$ such that $x \in V \subseteq R(x)$.

If $\mathcal{R}$ is topological, then $\mathcal{A} = \mathcal{F}_U$ is actually the smallest subset of $\mathcal{P}(X)$ such that $\mathcal{T}_{\mathcal{R}} \subseteq \mathcal{A}$ and $\mathcal{A}^{k_V} \subseteq \mathcal{A}$.

Moreover, if $\mathcal{R}$ is topological and *topologically filtered* in the sense that for any $X \in X$ and $R, S \in \mathcal{R}$ there exists $T \in \mathcal{R}$ such that $T(x) \subseteq R(x) \cap S(x)$, then for any $B \subseteq X$ we have $B \in \mathcal{F}_U$ if and only if there exist $A \in \mathcal{T}_{\mathcal{R}}$ and $D \in \mathcal{D}_{\mathcal{R}}$ such that $B = A \cap D$.

In this respect, it is curious that if $\mathcal{R}$ is topological and topologically filtered and $B \in \mathcal{F}_{U^\star}$, then there exist $A \in \mathcal{T}_{\mathcal{R}}$ and $N \in \mathcal{N}_{\mathcal{R}}$ such that $B = A \cup N$ and $A \cap N = \emptyset$.

However, the converse statement need not be true. Moreover, the genuine characterizations of $\mathcal{F}_{U^\star}$, established in [35, Section 24], do not require $\mathcal{R}$ to be topologically filtered.

REFERENCES

[1] ABD EL-MONSEF, M. E.—EL-DEEB, S. N.—MAHMOUD, R. A.: *β-open sets and β--continuous mappings*, Bull. Fac. Sci. Assiut Univ. **12** (1983), 77–90.

[2] ANDRIJEVIĆ, D.: *Semi-preopen sets*, Mat. Vesnik **38** (1986), 24–32.

[3] ANDRIJEVIĆ, D.: *On b-open sets*, Mat. Vesnik **48** (1996), 59–64.

[4] ARIANPOOR, H.: *Preorder relators and generalized topologies*, J. Linear Topol. Algebra **5** (2016), 271–277.

[5] ČECH, E.: *Topological Spaces*. Academia, Prague, 1966.

[6] CORSON, H. H.—MICHAEL, E.: *Metrizability of certain countable unions*, Illinois J. Math. **8** (1964), 351–360.

[7] CSÁSZÁR, Á.: *Foundations of General Topology*. Pergamon Press, London, 1963.

[8] CSÁSZÁR, Á.: *γ–quasi-open sets*, Studia Sci. Math. Hungar. **38** (2001), 171–176.

[9] CSÁSZÁR, Á.: *Weak structures*, Acta Math. Hungar. **131** (2011), 193–195.

[10] DAVEY, B. A.—PRIESTLEY, H. A.: *Introduction to Lattices and Order*. Cambridge University Press, Cambridge, 2002.

[11] DAVIS, A. S.: *Indexed systems of neighbordoods for general topological spaces*, Amer. Math. Monthly **68** (1961), 886–893.

[12] DONTCHEV, J.: *Survey on preopen sets*, in: Meetings on Topological Spaces, Theory and Applications, Yatsushiro College of Technology, Kumamoto, Japan, 1998, 18 pp.

[13] EFREMOVIČ, V. A.: *The geometry of proximity*, Mat. Sb. **31** (1952), 189–200. (In Russian.)

[14] FLETCHER, P.—LINDGREN, W. F.: *Quasi-Uniform Spaces*. Marcel Dekker, New York, 1982.

[15] GARDNER, B. J.—JACKSON, M. G.: *The Kuratowski closure-complement theorem*, New Zealand J. Math. **38** (2008), 9–44.

[16] GARGOURI, R.—REZGUI, A.: *A unification of weakening of open and closed subsets in a topological space*, Bull. Malays. Math. Sci. Soc. **40** (2017), 1219–1230.

[17] GLAVOSITS, T.: *Generated preorders and equivalences*, Acta Acad. Paed. Agrienses, Sect. Math. **29** (2002), 95–103.

[18] HUNSAKER, W.—LINDGREN, W.: *Construction of quasi-uniformities*, Math. Ann. **188** (1970), 39–42.

[19] ISOMICHI, Y.: *New concept in the theory of topological spaces–Supercondensed set, subcondensed set, and condensed set*, Pacific J. Math. **38** (1971), 657–668.

[20] JAMUNARANI, R.—JEYANTHI, P.—NOIRI, T.: *On generalized weak structures*, Journal of Algorithms and Computation (JAC) **47** (2016), 21–26.

[21] KURATOWSKI, K.: *Sur l'opération $\bar{A}$ del'Analysis Situs*, Fund. Math. **3** (1922), 182–199.

[22] KURATOWSKI, K.: *Topology I*. Academic Press, New York, 1966.

[23] LEVINE, N.: *On the commutivity of the closure and interior operators in topological spaces*, Amer. Math. Montly **68** (1961), 474–477.

[24] LEVINE, N.: *Semi-open sets and semi-continuity in topological spaces*, Amer. Math. Monthly **70** (1963), 36–41.

[25] LEVINE, N.: *On uniformities generated by equivalence relations*, Rend. Circ. Mat. Palermo **18** (1969), 62–70.

[26] LEVINE, N.: *On Pervin's quasi uniformity*, Math. J. Okayama Univ. **14** (1970), 97–102.

[27] NAIMPALLY, S. A.—WARRACK, B. D.: *Proximity Spaces*. Cambridge University Press, Cambridge, 1970.

[28] NAKANO, H.—NAKANO, K.: *Connector theory*, Pacific J. Math. **56** (1975), 195–213.

[29] NJÅSTAD, O.: *On some classes of nearly open sets*, Pacific J. Math. **15** (1965), no. 3, 965–970.

[30] PERVIN, W. J.: *Quasi-uniformization of topological spaces*, Math. Ann. **147** (1962), 316–317.

[31] PRZEMSKA, E.: *The lattices of families of regular sets in topological spaces*, Math. Slovaca **70** (2020), 477–488.

[32] PRZEMSKA, E.: *Regular sets in topological spaces*, in: The 34th International Summer Conference on Real Functions Theory, Mathematical Institute, Slovak Academy of Sciences, Bratislava and Košice, 2020, 5 pp.

[33] RASSIAS, TH. M.—SZÁZ, Á.: *A general framework for sudying certain generalized topologically open sets in relator spaces*, in: *Nonlinear Analysis, Differential Equations, and Applications*, (Th. M. Rassias, eds.), Springer-Verlag, Berlin (to appear).

[34] RASSIAS, TH. M.—SZÁZ, Á.: *A non-conventional three relator space whose very particular cases can be used to treat the various generalized open sets in a unified way*, in: The 34th International Summer Conference on Real Functions Theory, Mathematical Institute, Slovak Academy of Sciences, Bratislava and Košice, 2020, 6 pp.

[35] RASSIAS, TH. M.—SALIH, M. M.—SZÁZ, Á.: *Characterizations of generalized topologically open sets in relator spaces*, Montes Taurus J. Pure Appl. Math. **3** (2021), 39–94.

[36] RASSIAS, TH. M.—SALIH, M. M.—SZÁZ, Á.: *Set-theoretic properties of generalized topologically open sets in relator spaces*, in: *Mathematical Analysis in Interdisciplinary Research*, (I. N. Parasidis, E. Providas and Th. M. Rassias, eds.), Springer-Verlag, Berlin (to appear).

[37] SALIH, M. M.—SZÁZ, Á.: *Generalizations of some ordinary and extreme connectedness properties of topological spaces to relator spaces*, Elec. Res. Arch. **28** (2020), 471–548.

[38] SARSAK, M. S.: *On some properties of generalized open sets in generalized topological spaces*, Demonstr. Math. **46** (2013), 415–427.

[39] SHERMAN, D.: *Variations on Kuratowski's 14-set theorem*, Amer. Math. Monthly **117** (2010), 113–123.

[40] SMIRNOV, YU. M.: *On proximity spaces*, Math. Sb. **31** (1952), 543–574. (In Russian.)

[41] SZÁZ, Á.: *Basic tools and mild continuities in relator spaces*, Acta Math. Hungar. **50** (1987), 177–201.

[42] SZÁZ, Á.: *Structures derivable from relators*, Singularité **3** (1992), 14–30.

[43] SZÁZ, Á.: *Upper and lower bounds in relator spaces*, Serdica Math. J. **29** (2003), 239–270.

[44] SZÁZ, Á.: *Rare and meager sets in relator spaces*, Tatra Mt. Math. Publ. **28** (2004), 75–95.

[45] SZÁZ, Á.: *Galois-type connections on power sets and their applications to relators*, Technical Report, Inst. Math., Univ. Debrecen **2005/2**, 38 pp.

[46] SZÁZ, Á.: *Supremum properties of Galois-type connections*, Comment. Math. Univ. Carolin. **47** (2006), 569–583.

[47] SZÁZ, Á.: *Minimal structures, generalized topologies, and ascending systems should not be studied without generalized uniformities*, Filomat **21** (2007), 87–97.

[48] SZÁZ, Á.: *Galois type connections and closure operations on preordered sets*, Acta Math. Univ. Comenian. (N. S.) **78** (2009), 1–21.

[49] SZÁZ, Á.: *Inclusions for compositions and box products of relations*, J. Int. Math. Virt. Inst. **3** (2013), 97–125.

[50] SZÁZ, Á.: *A particular Galois connection between relations and set functions*, Acta Univ. Sapientiae Math. **6** (2014), 73–91.

[51] SZÁZ, Á.: *Generalizations of Galois and Pataki connections to relator spaces*, J. Int. Math. Virtual Inst. **4** (2014), 43–75.

[52] SZÁZ, Á.: *Four general continuity properties, for pairs of functions, relations and relators, whose particular cases could be investigated by hundreds of mathematicians*, Technical Report, Inst. Math., Univ. Debrecen **2017/1**, 17 pp.

[53] SZÁZ, Á.: *The closure-interior Galois connection and its applications to relational equations and inclusions*, J. Int. Math. Virt. Inst. **8** (2018), 181–224.

[54] SZÁZ, Á.: *Corelations are more powerful tools than relations*, in:*Applications of Nonlinear Analysis, Springer Optim. Appl. Vol. 134*, (Th. M. Rassias, ed.), Springer, Cham, 2018, pp. 711–779.

[55] SZÁZ, Á.: *Relationships between inclusions for relations and inequalities for corelations*, Math. Pannon. **26** (2018), 15–31.

[56] SZÁZ, Á.: *A unifying framework for studying continuity, increasingness, and Galois connections*, MathLab J. **1** (2018), 154–173.

[57] SZÁZ, Á.: *Galois and Pataki connections on generalized ordered sets*, Earthline J. Math. Sci. **2** (2019), 283–323.

[58] SZÁZ, Á.: *Birelator spaces are natural generalizations of not only bitopological spaces, but also ideal topological spaces*, in: *Mathematical Analysis and Applications, Springer Optimization and Its Applications Vol. 154*, (Th. M. Rassias and P. M. Pardalos, eds.), Springer Nature Switzerland AG, 2019, pp. 543–586.

[59] SZÁZ, Á.: *Super and hyper products of super relations*, in: The 34th International Summer Conference on Real Functions Theory, Mathematical Institute, Slovak Academy of Sciences, Bratislava and Košice, 2020, 12 pp.

[60] WEIL, A.: *Sur les espaces á structure uniforme et sur la topologie générale*, Actual. Sci. Ind. **551** (1937), 39 pp.

Received December 22, 2020

Department of Mathematics
Faculty of Natural Sciences
University of Debrecen
Egyetem tér 1
H-4002–Debrecen
HUNGARY
E-mail: szaz@science.unideb.hu