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ON STAR-K-$\mathcal{I}$-HUREWICZ PROPERTY

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ABSTRACT. A space X is said to have the star-K-$\mathcal{I}$-Hurewicz property (SK$\mathcal{I}$H) [Tyagi, B. K.—Singh, S.—Bhardwaj, M. *Ideal analogues of some variants of Hurewicz property*, Filomat **33** (2019), no. 9, 2725–2734] if for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of X there is a sequence $(K_n : n \in \mathbb{N})$ of compact subsets of X such that for each $x \in X$, $\{n \in \mathbb{N} : x \notin St(K_n, \mathcal{U}_n)\} \in \mathcal{I}$, where $\mathcal{I}$ is the proper admissible ideal of $\mathbb{N}$. In this paper, we continue to investigate the relationship between the SK$\mathcal{I}$H property and other related properties and study the topological properties of the SK$\mathcal{I}$H property.

1. Introduction

Scheepers [16] and Just, Miller and Szeptycki in [9] initiated the systematic study of selection principles in topology and their relations to game theory and Ramsey theory. Kočinac and Scheepers [12] studied the Hurewicz property in detail and found its relations with function spaces, game theory, and Ramsey theory. Di Maio and Kočinac [14] introduced the statistical analogues of certain types of open covers and selection principles using actually the ideal of asymptotic density zero sets of $\mathbb{N}$. Das, Kočinac and Chandra [2, 3] extended this study to the arbitrary ideal of $\mathbb{N}$. Using the notions of ideals, they started a more general approach to study certain results of open covers and selection principles. Further, Das et al. [4] studied the ideal analogues of the Hurewicz, the star-Hurewicz, and the strongly star-Hurewicz properties called them the

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$\mathcal{I}$-Hurewicz (see [1–3, 17, 19, 24]), the star-$\mathcal{I}$-Hurewicz and the strongly star-$\mathcal{I}$--Hurewicz properties, respectively, where $\mathcal{I}$ is the proper admissible ideal of $\mathbb{N}$. In [18, 25], Singh et al. introduced the ideal versions of the star-K-Hurewicz and the star-C-Hurewicz properties called the star-K-$\mathcal{I}$-Hurewicz (SK$\mathcal{I}$H) and the star-C-$\mathcal{I}$-Hurewicz properties, respectively.

The purpose of this paper is to investigate the relationships between the SK$\mathcal{I}$H property and other related properties by giving some suitable examples and to study the topological properties of the SK$\mathcal{I}$H property.

Throughout the paper, X and Y stand for topological spaces and $\mathbb{N}$ denotes the set of all positive integers. Let A be a subset of X and $\mathcal{P}$ be a collection of subsets of X, then $\mathrm{St}(A, \mathcal{P}) = \bigcup\{U \in \mathcal{P} : U \cap A \neq \emptyset\}$. We usually write

$$\mathrm{St}(x, \mathcal{P}) = \mathrm{St}(\{x\}, \mathcal{P}).$$

We first recall some basic definitions.

A space X is said to have the Hurewicz property [8] if for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of X, there is a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and for each $x \in X$, $x \in \bigcup \mathcal{V}_n$ for all but finitely many n.

A space X is said to have the star-Hurewicz property [10, 11] if for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of X, there is a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and for each $x \in X$, $x \in \mathrm{St}(\cup\mathcal{V}_n, \mathcal{U}_n)$ for all but finitely many n.

A space X is said to have the strongly star-Hurewicz property [10, 11] if for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of X, there is a sequence $(F_n : n \in \mathbb{N})$ of finite subsets of X such that for each $x \in X$, $x \in \mathrm{St}(F_n, \mathcal{U}_n)$ for all but finitely many n.

A space X is said to have the star-K-Hurewicz property [10, 11, 20] if for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of X, there is a sequence $(K_n : n \in \mathbb{N})$ of compact subsets of X such that for each $x \in X$, $x \in \mathrm{St}(K_n, \mathcal{U}_n)$ for all but finitely many n.

A family $\mathcal{I} \subset 2^Y$ of subsets of a non-empty set Y is said to be an ideal in Y if
i) $A, B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$,
ii) $A \in \mathcal{I}$, $B \subset A$ implies $B \in \mathcal{I}$,
while an ideal is said to be admissible ideal or free ideal $\mathcal{I}$ of Y if $\{y\} \in \mathcal{I}$ for each $y \in Y$. If $\mathcal{I}$ is a proper ideal in Y (that is, $Y \notin \mathcal{I}, \mathcal{I} \neq \{\emptyset\}$), then the family of sets $\mathcal{F}(\mathcal{I}) = \{M \subset Y :$ there exists $A \in \mathcal{I} : M = Y \setminus A\}$ is a filter in Y, that is, the family of all subsets of Y outside $\mathcal{I}$. Throughout the paper $\mathcal{I}$, will stand for proper admissible ideal of $\mathbb{N}$. We denote the ideals of all finite subsets of $\mathbb{N}$ by $\mathcal{I}_{fin}$.

A space X is said to have the $\mathcal{I}$-Hurewicz [2, 4] property (in short, $\mathcal{I}$H) if for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of X, there is a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and for each $x \in X$, $\{n \in \mathbb{N} : x \notin \bigcup \mathcal{V}_n\} \in \mathcal{I}$.

A space X is said to have the star-$\mathcal{I}$-Hurewicz [4] property (in short, S$\mathcal{I}$H) if for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of X, there is a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and for each $x \in X$, $\{n \in \mathbb{N} : x \notin \mathrm{St}(\cup\mathcal{V}_n, \mathcal{U}_n)\} \in \mathcal{I}$.

A space X is said to have the strongly star-$\mathcal{I}$-Hurewicz [4] property (in short, SS$\mathcal{I}$H) if for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of X, there is a sequence $(F_n : n \in \mathbb{N})$ of finite subsets of X such that for each $x \in X$, $\{n \in \mathbb{N} : x \notin \mathrm{St}(F_n, \mathcal{U}_n)\} \in \mathcal{I}$.

A space X is said to have the star-K-$\mathcal{I}$-Hurewicz [25] property (in short, SK$\mathcal{I}$H) if for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of X, there is a sequence $(K_n : n \in \mathbb{N})$ of compact subsets of X such that for each $x \in X$, $\{n \in \mathbb{N} : x \notin \mathrm{St}(K_n, \mathcal{U}_n)\} \in \mathcal{I}$.

A space X is said to be starcompact [5, 13] (resp., $\mathcal{K}$-starcompact [23]) if for each open cover $\mathcal{U}$ of X, there exists a finite subset $\mathcal{V}$ of $\mathcal{U}$ (resp., a compact subset K of X) such that $\mathrm{St}(\cup\mathcal{V}, \mathcal{U}) = X$ (resp., $\mathrm{St}(K, \mathcal{U}) = X$).

Recall that a collection $\mathcal{A} \subset P(\omega)$ is said to be almost disjoint if each set $A \in \mathcal{A}$ is infinite and the sets $A \bigcap B$ are finite for all distinct elements $A, B \in \mathcal{A}$. For an almost disjoint family $\mathcal{A}$, put $\psi(\mathcal{A}) = \mathcal{A} \bigcup \omega$ and topologize $\psi(\mathcal{A})$ as follows: for each element $A \in \mathcal{A}$ and each finite set $F \subset \omega$, $\{A\} \bigcup (A \setminus F)$ is a basic open neighbourhood of A and the natural numbers are isolated. The spaces of this type are called Isbell-Mrówka ψ-spaces [15] or $\psi(\mathcal{A})$ spaces.

Throughout the paper, let ω denote the first infinite cardinal, ω_1 the first uncountable cardinal, $\mathfrak{c}$ the cardinality of the set of all real numbers. For a cardinal κ, let κ^+ be the smallest cardinal greater than κ. For each pair of ordinals α, β with $\alpha < \beta$, we write

$$[\alpha, \beta) = \{\gamma : \alpha \leq \gamma < \beta\},$$

$$(\alpha, \beta] = \{\gamma : \alpha < \gamma \leq \beta\},$$

$$(\alpha, \beta) = \{\gamma : \alpha < \gamma < \beta\},$$

$$[\alpha, \beta] = \{\gamma : \alpha \leq \gamma \leq \beta\}.$$

As usual, a cardinal is an initial ordinal and an ordinal is the set of smaller ordinals. A cardinal is often viewed as a space with the usual order topology. Other terms and symbols follow [6]. The extent e(X) denotes the minimal cardinal number κ such that $|A| \leq \kappa$, where A is any discrete closed subset of a space X.

2. The star-K-$\mathcal{I}$-Hurewicz property

In this section, we study the topological properties of spaces having the SK$\mathcal{I}$H property. We divide this section into four subsections, that is, subspaces, Alexandorff duplicate, images-preimages, and the products.

2.1. Subspaces

In the following example, we show that SK$\mathcal{I}$H property is not preserved under closed subspaces.

Example. Assume $\omega_1 < \mathfrak{b}(\mathcal{I}) = \mathfrak{c}$. Let $X = \psi(\mathcal{A}) = \omega \cup \mathcal{A}$ be the Isbell--Mrówka ψ-space, where $\mathcal{A}$ is the almost disjoint family with $|\mathcal{A}| = \omega_1$. Then by [4, Theorem 4.4], X has the SS$\mathcal{I}$H property, hence SK$\mathcal{I}$H property. Since $\mathcal{A}$ is a closed discrete space with $|\mathcal{A}| = \omega_1$, $\mathcal{A}$ does not have the SK$\mathcal{I}$H property.

Now, we will see that the regular-closed subspace of a Tychonoff space with the SK$\mathcal{I}$H property need not have the SK$\mathcal{I}$H property. By a minor modification in the proof of [21, Example 3.1], we obtain the following example.

Example. There exists a Tychonoff space with the SK$\mathcal{I}$H property having a regular-closed subspace which does not have the SK$\mathcal{I}$H property.

Now, in the following example, we show that the regular-closed G_δ-subspace of a Tychonoff space with the SK$\mathcal{I}$H property does not have the SK$\mathcal{I}$H property.

By a minor modification in the proof of [22, Example 2.6], we obtain the following example.

Example. Assume $\omega_1 < \mathfrak{b}(\mathcal{I}) = \mathfrak{c}$, there exists a Tychonoff space with the SK$\mathcal{I}$H property having regular closed G_δ-subspace which does not have the SK$\mathcal{I}$H property.

In [25], Tyagi et al. give positive result on SK$\mathcal{I}$H property:

Theorem 2.1 ([25]). *The SK$\mathcal{I}$H property is preserved under clopen subspaces.*

The following result generalized Theorem 2.1.

Theorem 2.2. *The SK$\mathcal{I}$H property is preserved under open F_σ-subsets.*

Proof. Let X be a space having the SK$\mathcal{I}$H property, let $Y = \cup\{H_n : n \in \mathbb{N}\}$ be an open F_σ-subset of X, where H_n is closed in X for each $n \in \mathbb{N}$. Without loss of generality, we can assume that $H_n \subseteq H_{n+1}$ for each $n \in \mathbb{N}$. To show that Y has SK$\mathcal{I}$H property, let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of open covers of Y. We have to find a sequence $(K_n : n \in \mathbb{N})$ of compact subsets of Y such that for each $y \in Y$, $\{n \in \mathbb{N} : y \notin \mathrm{St}(K_n, \mathcal{U}_n)\} \in \mathcal{I}$. For each $n \in \mathbb{N}$, let

$$\mathcal{V}_n = \mathcal{U}_n \bigcup \{X \setminus H_n\}.$$

Then, $(\mathcal{V}_n : n \in \mathbb{N})$ is a sequence of open covers of X and by the SK$\mathcal{I}$H property of X, there exists a sequence $(K'_n : n \in \mathbb{N})$ of compact subsets of X such that for each $x \in X$, $\{n \in \mathbb{N} : x \notin \mathrm{St}(K'_n, \mathcal{V}_n)\} \in \mathcal{I}$. Consequently,

$$\{n \in \mathbb{N} : x \in \mathrm{St}(K'_n, \mathcal{V}_n)\} \in \mathcal{F}(\mathcal{I}).$$

For each $n \in \mathbb{N}$, let $K_n = K'_n \cap Y$. Thus, $(K_n : n \in \mathbb{N})$ is a sequence of compact subsets of Y. Let $y \in Y$. If $y \in \mathrm{St}(K'_n, \mathcal{V}_n)$ for some n. By the construction of $\mathcal{V}_n$, choose $U \in \mathcal{U}_n$ such that $y \in U$ and $U \cap K'_n \cap Y \neq \emptyset$, which implies $U \cap K_n \neq \emptyset$. Therefore, for each $y \in Y$,

$$\{n \in \mathbb{N} : y \in \mathrm{St}(K'_n, \mathcal{V}_n)\} \subset \{n \in \mathbb{N} : y \in \mathrm{St}(K_n, \mathcal{U}_n)\}.$$

This completes the proof. □

A cozero-set in a space X is a set of form $f^{-1}(\mathbb{R} \setminus \{0\})$ for some real-valued continuous function f on X. Since a cozero-set is an open F_σ-subset, the following corollary follows.

Corollary 2.3. *The SK$\mathcal{I}$H property is preserved under cozero-subsets.*

2.2. Alexandorff Duplicate

Now, we consider Alexandorff duplicate $\mathsf{AD(X)} = X \times \{0, 1\}$ of a space X. The basic neighbourhood of a point $\langle x, 0\rangle \in X \times \{0\}$ is of the form

$$(U \times \{0\}) \bigcup (U \times \{1\} \setminus \{\langle x, 1\rangle\}),$$

where U is a neighbourhood of x in X and all points $\langle x, 1\rangle \in X \times \{1\}$ are isolated points.

Lemma 2.4. *Let $D = \{d_\alpha : \alpha < \mathfrak{c}\}$ be the discrete space of cardinality $\mathfrak{c}$. Then, the subspace*

$$X = (\beta D \times \mathfrak{c}^+) \cup (D \times \{\mathfrak{c}^+\})$$

of the product space $\beta D \times (\mathfrak{c}^+ + 1)$ is $\mathcal{K}$-starcompact (hence, it has the SK$\mathcal{I}$H property).

P r o o f. The proof follows form [22, Lemma 2.9]. □

Example. There exists a Tychonoff space X having the SK$\mathcal{I}$H property such that $\mathsf{AD(X)}$ does not have the SK$\mathcal{I}$H property.

P r o o f. Let X be the space in Lemma 2.4. Then, X is a Tychonoff space X having the SK$\mathcal{I}$H property. Now, we have to show that $\mathsf{AD(X)}$ does not have the SK$\mathcal{I}$H property. Since $D \times \{\mathfrak{c}^+\}$ is a discrete closed subset of X with $|D \times \{\mathfrak{c}^+\}| = \mathfrak{c}$ and each point $\langle\langle d_\alpha, \mathfrak{c}^+\rangle, 1\rangle$ is isolated for each $\alpha < \mathfrak{c}$, $(D \times \{\mathfrak{c}^+\}) \times \{1\}$ does not have the SK$\mathcal{I}$H property. Since the SK$\mathcal{I}$H property is preserved under clopen subsets, $\mathsf{AD(X)}$ does not have the SK$\mathcal{I}$H property. □

Lemma 2.5 ([22]). *For T_1-space, $e(X) = e(\mathsf{AD(X)})$.*

Recall a space X is said to have star-K-Menger property [21, 22] if for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of X there is a sequence $(K_n : n \in \mathbb{N})$ compact subsets of X such that

$$X = \bigcup_{n\in\mathbb{N}} \mathrm{St}(K_n, \mathcal{U}_n).$$

Theorem 2.6. *If X has the SK$\mathcal{I}$H property with $e(X) < \omega_1$, then $\mathsf{AD(X)}$ has the star-K-Menger property.*

Proof. Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of open covers of $\mathsf{AD(X)}$. For each $n \in \mathbb{N}$ and each $x \in X$, choose an open neighbourhood $W_{n_x} = (V_{n_x} \times \{0, 1\}) \setminus \{\langle x, 1\rangle\}$ of $\langle x, 0\rangle$ satisfying that there exists some $U \in \mathcal{U}_n$ such that $W_{n_x} \subseteq U$, where V_{n_x} is an open subset of X containing x. For each $n \in \mathbb{N}$, let $\mathcal{V}_n = \{V_{n_x} : x \in X\}$, then $\mathcal{V}_n$ is an open cover of X. Applying the SK$\mathcal{I}$H property to the sequence $(\mathcal{V}_n : n \in \mathbb{N})$, thus there exists a sequence $(K_n : n \in \mathbb{N})$ of compact subsets of X such that for each $x \in X$,

$$\{n \in \mathbb{N} : x \notin \mathrm{St}(K_n, \mathcal{V}_n)\} \in \mathcal{I}.$$

For each $n \in \mathbb{N}$, let $K'_n = \mathsf{AD}(K_n)$. Then, K'_n is a compact subset of $\mathsf{AD(X)}$. Thus, we get a sequence $(K'_n : n \in \mathbb{N})$ of compact subsets of $\mathsf{AD(X)}$ and for each $x \in X$, $\{n \in \mathbb{N} : \langle x, 0\rangle \notin \mathrm{St}(K'_n, \mathcal{U}_n)\} \in \mathcal{I}$. Hence,

$$X \times \{0\} \subset \bigcup_{n\in\mathbb{N}} \mathrm{St}(K'_n, \mathcal{U}_n).$$

Let

$$A = \mathsf{AD(X)} \setminus \bigcup_{n\in\mathbb{N}} \mathrm{St}(K'_n, \mathcal{U}_n).$$

Then, A is a discrete closed subset of $\mathsf{AD(X)}$. By Lemma 2.5, the set A is countable, we can enumerate A as $\{a_n : n \in \mathbb{N}\}$. For each $n \in \mathbb{N}$, let $K''_n = K'_n \cup \{a_1, \ldots, a_n\}$. Then, K''_n is a compact subset of $\mathsf{AD(X)}$ and hence, for each $y \in \mathsf{AD(X)}$, $y \in \mathrm{St}(K''_n, \mathcal{U}_n)$ for some n. This shows that $\mathsf{AD(X)}$ has star-K-Menger property. $\square$

Theorem 2.7. *If X is a T_1-space and $\mathsf{AD(X)}$ has the SK$\mathcal{I}$H property, then $e(X) < \omega_1$.*

Proof. Suppose that $e(X) \geq \omega_1$. Then, there exists a discrete closed subset B of X such that $|B| \geq \omega_1$. Hence, $B \times \{1\}$ is an open and closed subset of $\mathsf{AD(X)}$ and every point of $B \times \{1\}$ is an isolated point. By Theorem 2.1, $\mathsf{AD(X)}$ does not have the SK$\mathcal{I}$H property, since $B \times \{1\}$ does not have the SK$\mathcal{I}$H property. $\square$

2.3. Images and preimages

Tyagi et al. [25, Theorem 5.3], proved the following theorem.

Theorem 2.8 ([25]). *The SK$\mathcal{I}$H property is preserved under continuous mappings.*

Now, we show that preimage of the space having the SK$\mathcal{I}$H property under closed 2-to-1 continuous map does not have the SK$\mathcal{I}$H property.

Example. There exists a closed 2-to-1 continuous map $f : \mathsf{AD(X)} \to X$ such that X has the SK$\mathcal{I}$H property, but $\mathsf{AD(X)}$ does not have the SK$\mathcal{I}$H property.

Proof. Let X be the space in Lemma 2.4. Then by Example 2.2, X has the SK$\mathcal{I}$H property, but $\mathsf{AD(X)}$ does not have the SK$\mathcal{I}$H property. Define $f : \mathsf{AD(X)} \to X$ by $f(\langle x, 0\rangle) = f(\langle x, 1\rangle) = x$ for each $x \in X$. Then, f is a closed 2-to-1 continuous map. □

In [25], Tyagi et al. give positive result on the preimages of the SK$\mathcal{I}$H property.

Theorem 2.9 ([25]). *The SK$\mathcal{I}$H property is inverse invariant under perfect open mappings.*

2.4. Products

By Theorem 2.9, we have the following corollary.

Corollary 2.10 ([25]). *Let X have the SK$\mathcal{I}$H property and let Y be a compact space, then $X \times Y$ has the SK$\mathcal{I}$H property.*

Theorem 2.11. *The SK$\mathcal{I}$H property closed under countable unions.*

Proof. Let $X = \bigcup\{X_k : k \in \mathbb{N}\}$, where $X_k, k \in \mathbb{N}$ has the SK$\mathcal{I}$H property. Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of open covers of X. For each $k \in \mathbb{N}$, consider the sequence $(\mathcal{U}_n : n \geq k, n \in \mathbb{N})$ of open covers of X_k. Since X_k has the SK$\mathcal{I}$H property, there is a sequence $(H_{n,k} : n > k, n \in \mathbb{N})$ of compact subsets of X_k and for each $x \in X_k$, $\{n \in \mathbb{N} : x \notin \mathrm{St}(H_{n,k}, \mathcal{U}_n)\} \in \mathcal{I}$. For each $n \in \mathbb{N}$, $H_n = \bigcup\{H_{n,k} : k \leq n\}$. Then for each $n \in \mathbb{N}$, H_n is a compact subset of X. Since for each $x \in X = \bigcup\{X_k : k \in \mathbb{N}\}$, $x \in X_k$ for some $k \in \mathbb{N}$, thus $\{n \in \mathbb{N} : x \notin \mathrm{St}(H_{n,k}, \mathcal{U}_n)\} \in \mathcal{I}$. Since $\mathrm{St}(H_{n,k}, \mathcal{U}_n) \subset \mathrm{St}(H_n, \mathcal{U}_n)$ for all $n > k$ and hence, $\{n \in \mathbb{N} : x \notin \mathrm{St}(H_n, \mathcal{U}_n)\} \in \mathcal{I}$. □

We have the following corollary from Corollary 2.10 and Theorem 2.11.

Corollary 2.12. *Let X has the SK$\mathcal{I}$H property and Y is a σ-compact space, then $X \times Y$ has the SK$\mathcal{I}$H property.*

The following examples show that the σ-compact space in Corollary 2.12, cannot be replaced by countably compact or by Lindelöf space.

EXAMPLE. There exist two countably compact spaces X and Y such that $X \times Y$ does not have the SK$\mathcal{I}$H property.

P r o o f. Let $D(\mathfrak{c})$ be the discrete space of cardinality $\mathfrak{c}$. We can define $X = \bigcup_{\alpha<\omega_1} E_\alpha$, $Y = \bigcup_{\alpha<\omega_1} F_\alpha$, where E_α and F_α are subsets of $\beta D(\mathfrak{c})$ defined inductively so that the following three conditions are satisfied:

1) $E_\alpha \cap F_\beta = D(\mathfrak{c})$ *if* $\alpha \neq \beta$;
2) $|E_\alpha| \leq \mathfrak{c}$ and $|F_\alpha| \leq \mathfrak{c}$;
3) every infinite subset of E_α (resp., F_α) has an accumulation point in $E_{\alpha+1}$ (resp., $F_{\alpha+1}$).

Those sets E_α and F_α are well-defined since every infinite closed set in $\beta D(\mathfrak{c})$ has the cardinality $2^{\mathfrak{c}}$ (see [26]). The diagonal $\{\langle d, d\rangle : d \in D(\mathfrak{c})\}$ is a discrete open and closed subset of $X \times Y$ of cardinality $\mathfrak{c}$ so that it does not have the SK$\mathcal{I}$H property. Thus, $X \times Y$ does not have the SK$\mathcal{I}$H property, since by Theorem 2.1, the SK$\mathcal{I}$H property is preserved under clopen subsets. □

EXAMPLE. There exist a countably compact (hence, it has the SK$\mathcal{I}$H property) space X and a Lindelöf space Y such that $X \times Y$ does not have the SK$\mathcal{I}$H property.

P r o o f. Let $X = [0, \omega_1)$ be equipped with the order topology and $Y = [0, \omega_1]$ with the following topology. Each point $\alpha < \omega_1$ is isolated, and a set U containing ω_1 is open if and only if $Y \setminus U$ is countable. Then, X is countably compact and Y is Lindelöf. By a simple modification in [21, Example 3.7], we can show that $X \times Y$ does not have the SK$\mathcal{I}$H property. □

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