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OSCILLATION RESULTS FOR THIRD-ORDER QUASI-LINEAR EMDEN-FOWLER DIFFERENTIAL EQUATIONS WITH UNBOUNDED NEUTRAL COEFFICIENTS

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ABSTRACT. Some new oscillation criteria are obtained for a class of third-order quasi-linear Emden-Fowler differential equations with unbounded neutral coefficients of the form

$$\Big(a(t)\big(z''(t)\big)^{\alpha}\Big)' + f(t)y^{\lambda}\big(g(t)\big) = 0,$$

where $z(t) = y(t) + p(t)y\big(\sigma(t)\big)$ and α, λ are ratios of odd positive integers. The established results generalize, improve and complement to known results.

1. Introduction

In the present paper, we are dealing with the oscillatory behaviour of solutions of the third-order quasi-linear Emden-Fowler neutral differential equation

$$\Big(a(t)\big(z''(t)\big)^{\alpha}\Big)' + f(t)y^{\lambda}\big(g(t)\big) = 0,\ t \ge t_0 > 0, \tag{E}$$

where $z(t) = y(t) + p(t)y\big(\sigma(t)\big)$ and α, λ are ratios of odd positive integers, subject to the following conditions:

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(C_1) $p, f : [t_0, \infty) \to \mathbb{R}$ are continuous functions, $p(t) \geq 1$, $p(t) \neq 1$ for large t, $f(t) \geq 0$ and $f(t)$ is not identically zero for large t;

(C_2) $\sigma, g : [t_0, \infty) \to \mathbb{R}$ are continuous functions, $\sigma(t) \leq t$, σ is strictly increasing $\big(\sigma^{-1}\big(g(t)\big)\big)' > 0$ and $\lim_{t\to\infty} \sigma(t) = \lim_{t\to\infty} g(t) = \infty$;

(C_3) $a(t) \in C\big([t,\infty), (0, \infty)\big)$ and $\int_{t_0}^{\infty} a^{-1/\alpha}(t)\,\mathrm{d}t = \infty$.

By a solution of (E), we mean a real valued function $y(t)$ that is continuous on $[t_y, \infty)$ for some $t_y \geq t_0$, such that $z \in C^3([t_y, \infty), \mathbb{R})$ and $y(t)$ of (E) that satisfy $\sup\{|y(t)| : t \geq T\} > 0$ for all $T \geq t_y$. Moreover, we tacitly assume that (E) possesses such solutions. Such a solution $x(t)$ of (E) is called *oscillatory* if it has arbitrarily large zeros on $[t_y, \infty)$; otherwise it is called *nonoscillatory*. Equation (E) is called oscillatory if all its proper solutions oscillate.

Investigating the oscillatory behaviour of (E) is important due to its practical importance in the development of oscillation theory of functional differential equation. In particular Emden-Fowler type differential equations have many applications in physics, engineering and technology, see for example [1, 2, 20] and the references cited therein.

In recent years, there is a lot of research activity concerning the oscillation and asymptotic behaviour of solutions to various types of third-order neutral differential equations, see for example [1,5–12,16–18,21] and the references cited therein. Most of the papers concerned with the case where p is bounded, that is, the cases where $-1 < p_0 \leq p(t) \leq 0$, $0 \leq p(t) \leq p_0 < 1$ and $0 \leq p(t) \leq p_0 < \infty$ were considered. In a very recent paper [2], the authors studied the equation (E) where $\alpha = 1$ and $a(t) = 1$, and established sufficient conditions for the oscillation of all solutions of (E) for the cases $\lambda = 1$ and $0 < \lambda < 1$, while the case $\lambda > 1$ was left as an open problem.

In view of the above observations, in this paper our aim is to obtain explicit sufficient conditions for the oscillation of all solutions of (E) where α and λ satisfy different values. Thus, the results established in this paper are new and extend the results in [2, 5–12, 16–18, 21].

2. Main Results

We begin with the following lemmas which play an important role in proving our main results. For the sake of convenience, we define

$$A(t) = \int_{t_1}^{t} \frac{1}{a^{1/\alpha}(s)}\,\mathrm{d}s, \quad B(t) = \int_{t_1}^{t} A(s)\,\mathrm{d}s, \quad C(t) = \exp\left(\int_{t_1}^{t} \frac{A(s)}{B(s)}\,\mathrm{d}s\right),$$

$$\phi(t) = \frac{1}{p(\sigma^{-1}(t))}\left(1 - \frac{1}{p(\sigma^{-1}(\sigma^{-1}(t)))}\right) > 0,$$

$$\psi(t) = \frac{1}{p(\sigma^{-1}(t))}\left(1 - \frac{c(\sigma^{-1}(\sigma^{-1}(t)))}{p(\sigma^{-1}(\sigma^{-1}(t)))c(\sigma^{-1}(t))}\right) > 0$$

and

$$R(t) = \int_{\sigma^{-1}\left(g(t)\right)}^{\sigma^{-1}(\eta(t))} \left(\int_{s}^{\sigma^{-1}(\eta(t))} a^{-1/\alpha}(u)\,\mathrm{d}u \right) \mathrm{d}s$$

for any $t_1, t_1 \geq t_0$, and σ^{-1} is the inverse function of σ.

Lemma 2.1. *Let* $(\mathrm{C}_1) - (\mathrm{C}_3)$ *be satisfied and assume that* y *is an eventually positive solution of* (E). *Then the function* $z(t)$ *satisfies either*

(I) $z(t) > 0$, $z'(t) > 0$, $z''(t) > 0$ *and* $\left(a(t)\big(z''(t)\big)^{\alpha}\right)' \leq 0$; *or*

(II) $z(t) > 0$, $z'(t) < 0$, $z''(t) > 0$ *and* $\left(a(t)\big(z''(t)\big)^{\alpha}\right)' \leq 0$

for sufficiently large t.

Proof. The proof follows by the similar argument as in Lemma 1 of [7] and hence is omitted. □

Lemma 2.2 ([3]). *Let* $\gamma > 1$ *be a quotient of odd positive integers and* $\delta > 0$. *If*

$$\lim_{t\to\infty} \inf \gamma^{-t/\delta} \log b(t) > 0,$$

where $b \in C\big([t_0, \infty), (0, \infty)\big)$, *then equation*

$$x'(t) + b(t)x^{\gamma}(t - \delta) = 0$$

is oscillatory.

Lemma 2.3. *Assume that* $z(t)$ *satisfies Case (I) of Lemma 2.1 for all* $t \geq t_1$. *Then*

$$z'(t) \geq A(t)a^{1/\alpha}(t)z''(t), \tag{2.1}$$

$$z(t) \geq B(t)a^{1/\alpha}(t)z''(t), \tag{2.2}$$

$$z(t) \geq \frac{B(t)}{A(t)}z'(t) \tag{2.3}$$

and

$$\frac{z(t)}{C(t)} \text{ is nonincreasing for all } t \geq t_1. \tag{2.4}$$

Proof. Since $\big(a(t)\big(z''(t)\big)^{\alpha}\big)' leq 0$, $a(t)\big(z''(t)\big)^{\alpha}$ is nonincreasing and hence

$$z'(t) = z'(t_1) + \int_{t_1}^{t} \frac{(a(s)(z''(s))^{\alpha})^{1/\alpha}}{a^{1/\alpha}(s)} \, \mathrm{d}s \geq A(t)a^{1/\alpha}(t)z''(t).$$

Integrating again, we have

$$z(t) \geq a^{1/\alpha}(t)z''(t) \int_{t_1}^{t} A(s) \, \mathrm{d}s = B(t)a^{1/\alpha}(t)z''(t).$$

From (2.1), we see that

$$\frac{z'(t)}{A(t)} \text{ is nonincreasing}$$

and therefore

$$z(t) \;=\; z(t_1) + \int_{t_1}^{t} \frac{A(s)z'(s)}{A(s)} \;\geq\; \frac{B(t)}{A(t)} z'(t).$$

From the last inequality, we see that

$$\left(\frac{z(t)}{C(t)}\right)' = \frac{\left[\frac{B(t)}{A(t)} z'(t) - z(t)\right] \frac{A(t)}{B(t)}}{C(t)} \;\leq\; 0.$$

Hence $\frac{z(t)}{C(t)}$ is nonincreasing. This completes the proof. □

Theorem 2.4. *Let conditions* $(\mathrm{C}_1) - (\mathrm{C}_3)$ *hold and assume that there exists a function* $\eta \in C([t_0, \infty), \mathbb{R})$ *such that* $g(t) \leq \eta(t) < \sigma(t)$ *for all* $t \geq t_0$*. If both first order delay differential equations*

$$X'(t) + f(t)\psi^{\lambda}\big(g(t)\big)B^{\lambda}\Big(\sigma^{-1}\big(g(t)\big)\Big)X^{\lambda/\alpha}\Big(\sigma^{-1}\big(g(t)\big)\Big) = 0 \tag{2.5}$$

and

$$W'(t) + f(t)\phi^{\lambda}\big(g(t)\big)R^{\lambda}(t)W^{\lambda/\alpha}\Big(\sigma^{-1}\big(\eta(t)\big)\Big) = 0 \tag{2.6}$$

oscillate, then (E) *oscillates.*

Proof. Let y be a nonoscillatory solution of (E). Without loss of generality, we may assume that there is a $t_1 \geq t_0$ such that $y(t) > 0$, $y\big(\sigma(t)\big) > 0$ and $y\big(g(t)\big) > 0$ for all $t \geq t_1$. Then as in Lemma 2.1, the function z satisfies either Case (I) or Case (II).

Case (I): In view of definition of z, we have

$$\begin{aligned} y(t) &= \frac{1}{p\big(\sigma^{-1}(t)\big)}\Big(z\big(\sigma^{-1}(t)\big) - y\big(\sigma^{-1}(t)\big)\Big) \\ &\geq \frac{1}{p(\sigma^{-1}(t))}\left(z(\sigma^{-1}(t)) - \frac{z(\sigma^{-1}(\sigma^{-1}(t)))}{p(\sigma^{-1}(\sigma^{-1}(t)))}\right). \end{aligned} \tag{2.7}$$

From (2.4), we see that $\frac{z(t)}{C(t)}$ is nonincreasing and $\sigma^{-1}\big(\sigma^{-1}(t)\big) \geq \sigma^{-1}(t)$. Thus, from (2.7) we have

$$y(t) \geq \frac{1}{p(\sigma^{-1}(t))}\left(1 - \frac{C(\sigma^{-1}(\sigma^{-1}(t)))}{p(\sigma^{-1}(\sigma^{-1}(t)))C(\sigma^{-1}(t))}\right) z\big(\sigma^{-1}(t)\big),$$

that is,

$$y(t) \geq \psi(t) z\big(\sigma^{-1}(t)\big)$$

and thus

$$y\big(g(t)\big) \geq \psi\big(g(t)\big) z\Big(\sigma^{-1}\big(g(t)\big)\Big). \tag{2.8}$$

Combining (2.8) with (E), we obtain

$$\Big(a(t)\big(z''(t)\big)^{\alpha}\Big)' + f(t)\psi^{\lambda}\big(g(t)\big) z^{\lambda}\Big(\sigma^{-1}\big(g(t)\big)\Big) \leq 0, \quad t \geq t_1. \tag{2.9}$$

From (2.2), we have

$$z\big(\sigma^{-1}(t)\big) \geq B\big(\sigma^{-1}(t)\big) a^{1/\alpha}\big(\sigma^{-1}(t)\big) z''\Big(\sigma^{-1}\big((t)\big)\Big), \quad t \geq t_1$$

and hence

$$z\Big(\sigma^{-1}\big(g(t)\big)\Big) \geq B\Big(\sigma^{-1}\big(g(t)\big)\Big) a^{1/\alpha}\Big(\sigma^{-1}\big(g(t)\big)\Big) z''\Big(\sigma^{-1}\big(g(t)\big)\Big), \quad t \geq t_1. \tag{2.10}$$

Using (2.10) in (2.9) yields

$$\Big(a(t)\big(z''(t)\big)^{\alpha}\Big)' + f(t)\psi^{\lambda}\big(g(t)\big) B^{\lambda}\Big(\sigma^{-1}\big(g(t)\big)\Big)\left(a\Big(\sigma^{-1}\big(g(t)\big)\Big)\Big(z''\Big(\sigma^{-1}\big(g(t)\big)\Big)\Big)^{\alpha}\right)^{\lambda/\alpha} \leq 0.$$

Letting $X(t) = a(t)\big(z''(t)\big)^{\alpha}$, we have that X is a positive solution of the first--order delay differential inequality

$$X'(t) + f(t)\psi^{\lambda}\big(g(t)\big) B^{\lambda}\Big(\sigma^{-1}\big(g(t)\big)\Big) X^{\lambda/\alpha}\Big(\sigma^{-1}\big(g(t)\big)\Big) \leq 0.$$

Therefore, by Corollary 1 of [19], we conclude that equation (2.5) also has a positive solution, which is a contradiction.

Case (II): Since z is strictly decreasing and $\sigma(t) \le t$, we have

$$z\big(\sigma^{-1}(t)\big) \ge z\Big(\sigma^{-1}\big(\sigma^{-1}(t)\big)\Big)$$

and using this in (2.7), we obtain

$$y(t) \ge \phi(t) z\big(\sigma^{-1}(t)\big).$$

Thus

$$y\big(g(t)\big) \ge \phi\big(g(t)\big) z\Big(\sigma^{-1}\big(g(t)\big)\Big). \tag{2.11}$$

Substituting (2.11) into (E), we get

$$\Big(a(t)\big(z''(t)\big)^{\alpha}\Big)' + f(t)\phi^{\lambda}\big(g(t)\big) z^{\lambda}\Big(\sigma^{-1}\big(g(t)\big)\Big) \le 0. \tag{2.12}$$

For $t \ge s \ge t_1$, we have

$$z'(t) - z'(s) = \int_s^t \frac{a^{1/\alpha}(u) z''(u)}{a^{1/\alpha}(u)} \,\mathrm{d}u,$$

or

$$-z'(s) \ge \left(\int_s^t \frac{1}{a^{1/\alpha}(u)} \,\mathrm{d}u\right) a^{1/\alpha}(t) z''(t).$$

Again integrating, we have

$$-z(t) + z(s) \ge \left(\int_s^t \left(\int_u^t a^{1/\alpha}(v) \,\mathrm{d}v\right) \mathrm{d}u\right) a^{1/\alpha}(t) z''(t),$$

or

$$z(s) \ge \left(\int_s^t \left(\int_u^t a^{-1/\alpha}(v) \,\mathrm{d}v\right) \mathrm{d}u\right) a^{1/\alpha}(t) z''(t). \tag{2.13}$$

Since $g(t) \le \eta(t)$ and the fact that σ is strictly increasing, we see that $\sigma^{-1}\big(g(t)\big) \le \sigma^{-1}\big(\eta(t)\big)$. Setting $s = \sigma^{-1}\big(g(t)\big)$ and $t = \sigma^{-1}\big(\eta(t)\big)$ into (2.13), we get

$$z\Big(\sigma^{-1}\big(g(t)\big)\Big) \ge \left(\int_{\sigma^{-1}(g(t))}^{\sigma^{-1}(\eta(t))} \left(\int_s^{\sigma^{-1}(\eta(t))} a^{-1/\alpha}(u) \,\mathrm{d}u\right) \mathrm{d}s\right) a^{1/\alpha}\Big(\sigma^{-1}\big(\eta(t)\big)\Big) z''\Big(\sigma^{-1}\big(\eta(t)\big)\Big).$$

Using the last inequality in (2.12), we obtain

$$\Big(a(t)\big(z''(t)\big)^{\alpha}\Big)' + f(t)\phi^{\lambda}\big(g(t)\big) R^{\lambda}(t) \left[a\Big(\sigma^{-1}\big(\eta(t)\big)\Big)\Big(z''\Big(\sigma^{-1}\big(\eta(t)\big)\Big)\Big)^{\alpha}\right]^{\lambda/\alpha} \le 0.$$

Letting $W(t) = a(t)\big(z''(t)\big)^{\alpha}$, we see that W is a positive solution of the first order delay differential inequality

$$W'(t) + f(t)\phi^{\lambda}\big(g(t)\big)R^{\lambda}(t)W^{\lambda/\alpha}\Big(\sigma^{-1}\big(\eta(t)\big)\Big) \le 0.$$

The rest of the proof is similar to that of Case(I) and hence the details are not repeated. The proof of the theorem is complete. □

Corollary 2.5. *Let conditions* $(\mathrm{C}_1)-(\mathrm{C}_3)$ *hold and* $\alpha = \lambda$. *Assume that there exists a function* $\eta \in C([t_0,\infty),\mathbb{R})$ *such that* $g(t) \le \eta(t) < \sigma(t)$ *for* $t \ge t_0$. *If*

$$\liminf_{t\to\infty} \int\limits_{\sigma^{-1}(g(t))}^{t} f(s)\psi^{\lambda}\big(g(s)\big)B^{\lambda}\Big(\sigma^{-1}\big(g(s)\big)\Big)\,\mathrm{d}s > \frac{1}{e} \tag{2.14}$$

and

$$\liminf_{t\to\infty} \int\limits_{\sigma^{-1}(\eta(t))}^{t} f(s)\phi^{\lambda}\big(g(s)\big)R^{\lambda}(s)\,\mathrm{d}s > \frac{1}{e}, \tag{2.15}$$

then (E) *oscillates.*

P r o o f. The proof follows from a well-known result in [14] and Theorem 2.4, and hence the details are omitted. □

Corollary 2.6. *Let conditions* $(\mathrm{C}_1)-(\mathrm{C}_3)$ *hold and* $\alpha > \lambda$. *Assume that there exists a function* $\eta \in C([t_0,\infty),\mathbb{R})$ *such that* $g(t) \le \eta(t) < \sigma(t)$ *for* $t \ge t_0$. *If*

$$\int\limits_{T}^{\infty} f(t)\psi^{\lambda}\big(g(t)\big)B^{\lambda}\Big(\sigma^{-1}\big(g(t)\big)\Big)\,\mathrm{d}t = \infty \tag{2.16}$$

and

$$\int\limits_{T}^{\infty} f(t)\phi^{\lambda}\big(g(t)\big)R^{\lambda}(t)\,\mathrm{d}t = \infty \tag{2.17}$$

for all $t \ge T \ge t_0$, *then* (E) *oscillates.*

P r o o f. Applications of (2.16), (2.17) and [13, Theorem 2], show that (2.5) and (2.6) oscillate. So, by Theorem 2.4, equation (E) oscillates. □

In our next result, assume that $g(t) = t - \delta_1$, $\sigma(t) = t - \delta_3$ and $\eta(t) = t - \delta_2$, where δ_1, δ_2 and δ_3 are positive real numbers.

Corollary 2.7. *Let conditions* $(\mathrm{C}_1)-(\mathrm{C}_3)$ *hold and* $\alpha<\lambda$. *If* $\delta_1\geq\delta_2>\delta_3$,

$$\liminf_{t\to\infty}\left(\frac{\lambda}{\alpha}\right)^{-t/(\delta_1-\delta_3)}\log\bigl(f(t)\psi^{\lambda}(t-\delta_1)B^{\lambda}(t+\delta_3-\delta_1)\bigr)>0 \tag{2.18}$$

and

$$\liminf_{t\to\infty}\left(\frac{\lambda}{\alpha}\right)^{-t/(\delta_2-\delta_3)}\log\bigl(f(t)\phi^{\lambda}(t-\delta_1)R^{\lambda}(t)\bigr)>0, \tag{2.19}$$

then (E) *oscillates.*

Proof. Applications of (2.18), (2.19) and Lemma 2.2 imply that (2.5) and (2.6) oscillate. So, by Theorem 2.4, (E) oscillates. □

Next, we present a result when $g(t)=\theta t, \sigma(t)=\mu t$ and $\eta(t)=\nu t$, where $\theta,\mu,\nu\in(0,1)$.

Corollary 2.8. *Assume that* $(\mathrm{C}_1)-(\mathrm{C}_3)$ *hold and* $\alpha<\lambda$. *If* $\theta\leq\nu<\mu$ *and there exists a* $\delta>-\ln\left(\frac{\lambda}{\alpha}\right)/\ln\left(\frac{\theta}{\mu}\right)$ *such that*

$$\liminf_{t\to\infty}\left[f(t)\psi^{\lambda}(\theta t)B^{\lambda}\left(\frac{\theta}{\mu}t\right)\exp(-t^{\delta})\right]>0 \tag{2.20}$$

and there exists a $\epsilon>-\ln\left(\frac{\lambda}{\alpha}\right)/\ln\left(\frac{\nu}{\mu}\right)$ *such that*

$$\liminf_{t\to\infty}\left[f(t)\phi^{\lambda}(\theta t)R^{\lambda}(t)\exp(-t^{\epsilon})\right]>0, \tag{2.21}$$

then (E) *oscillates.*

Proof. Applications of (2.20), (2.21) and Theorem 4 of [4] imply that (2.5) and (2.6) oscillate. Hence by Theorem 2.4, equation (E) oscillates. □

Theorem 2.9. *Let conditions* $(\mathrm{C}_1)-(\mathrm{C}_3)$ *hold and* $\alpha=\lambda$. *Assume that there exists a function* $\eta\in C([t_0,\infty),\mathbb{R})$ *such that* $g(t)\leq\eta(t)<\sigma(t)$ *for* $t\geq t_0$. *If* (2.15) *holds and there exists a positive nondecreasing differentiable function* $\rho(t)$ *such that*

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left[\rho(s)f(s)\psi^{\alpha}\bigl(g(s)\bigr)-\frac{(\rho'(s))^{\alpha+1}}{(\alpha+1)^{\alpha+1}(\rho(s))^{\alpha}A^{\alpha}(h(s))(h'(s))^{\alpha}}\right]\mathrm{d}s=\infty, \tag{2.22}$$

where $h(t)=\sigma^{-1}\bigl(g(t)\bigr)$, *then* (E) *oscillates.*

Proof. Let y be a nonoscillatory solution of (E). Without loss of generality we may assume that there exists a $t_1 > t_0$ such that $y(t) > 0$, $y\big(g(t)\big) > 0$ and $y\big(\sigma(t)\big) > 0$ for $t \geq t_1$. Then as in Lemma 2.1, the function z satisfies either Case (I) or Case (II).

Case (I): Proceeding as in Case (I) of Theorem 2.4, for $\alpha = \lambda$, we arrive at

$$\Big(a(t)\big(z''(t)\big)^{\alpha}\Big)' + f(t)\psi^{\alpha}\big(g(t)\big)z^{\alpha}\big(h(t)\big) \leq 0, t \geq t_1. \tag{2.23}$$

Define

$$F(t) = \frac{\rho(t)a(t)(z''(t))^{\alpha}}{z^{\alpha}(h(t))}, t \geq t_1. \tag{2.24}$$

Then $F(t) > 0$ for all $t \geq t_1$. Differentiating (2.24) and using (2.23), we obtain

$$\begin{aligned} F'(t) &= \frac{\rho(t)(a(t)(z''(t))^{\alpha})'}{z^{\alpha}(h(t))} + \frac{\rho'(t)}{\rho(t)}F(t) - \frac{\alpha F(t)z'(h(t))(h'(t))}{z(h(t))} \\ &\leq -\rho(t)f(t)\psi^{\alpha}\big(g(t)\big) + \frac{\rho'(t)}{\rho(t)}F(t) - \frac{\alpha F^{1+1/\alpha}(t)}{\rho^{1/\alpha}(t)}A\big(h(t)\big)\big(h'(t)\big), \end{aligned} \tag{2.25}$$

where we have used (2.1) and the nonincreasing behaviour of $a(t)\big(z''(t)\big)^{\alpha}$ for $t \geq t_1$. Using the inequality (see [15])

$$Au - Bu^{1+1/\alpha} \leq \frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}}\frac{A^{\alpha+1}}{B^{\alpha}}$$

for $B > 0$ and $u > 0$ in (2.25), we obtain

$$F'(t) \leq -\rho(t)f(t)\psi^{\alpha}\big(g(t)\big) + \frac{(\rho'(t))^{\alpha+1}}{(\alpha+1)^{\alpha+1}\rho^{\alpha}(t)A^{\alpha}(h(t))(h'(t))^{\alpha}}.$$

Integrating the last inequality from t_1 to t, we get

$$\int_{t_1}^{t}\left[\rho(s)f(s)\psi^{\alpha}\big(g(s)\big) - \frac{(\rho'(s))^{\alpha+1}}{(\alpha+1)^{\alpha+1}(\rho(s))^{\alpha}A^{\alpha}(h(s))(h'(s))^{\alpha}}\right]\mathrm{d}s < \infty.$$

Letting $t \to \infty$ and taking supremum in the last inequality, we obtain a contradiction to (2.22).

Case (II): In this case, by using condition (2.15) and by a known result in [14] we conclude that equation (2.6) oscillates. Hence by Theorem 2.4, equation (E) is oscillatory. The proof is now complete. □

Theorem 2.10. *In addition to conditions* $(C_1)-(C_3)$, *assume that* $\lambda \le \alpha$ *and the function* g *with* $g(t) < \sigma(t)$ *is nondecreasing for all* $t \ge t_0$. *If*

$$\lim_{t\to\infty} \sup A^\lambda\big(h(t)\big)\int_t^\infty f(s)\psi^\lambda\big(g(s)\big)\frac{B^\lambda(h(s))}{A^\lambda(h(s))}\,\mathrm{d}s \begin{cases} =\infty & \text{if} \quad \lambda<\alpha, \\ >1 & \text{if} \quad \lambda=\alpha, \end{cases} \tag{2.26}$$

and

$$\lim_{t\to\infty} \sup \int_{h(t)}^\infty f(s)\phi^\lambda\big(g(s)\big)R_1^\lambda(t,s)\,\mathrm{d}s \begin{cases} =\infty & \text{if} \quad \lambda<\alpha, \\ >1 & \text{if} \quad \lambda=\alpha, \end{cases} \tag{2.27}$$

where $R_1(t,s) = \int\limits_{h(s)}^{h(t)} \left(\int\limits_u^{h(t)} a^{-1/\alpha}(v)\,\mathrm{d}v \right) \mathrm{d}u$, *then* (E) *oscillates.*

P r o o f. Let y be a nonoscillatory solution of (E). With no loss of generality, we may assume that there exists a $t_1 \ge t_0$ such that $y(t) > 0$, $y\big(\sigma(t)\big) > 0$, and $y\big(g(t)\big) > 0$ for such all $t \ge t_1$. Then as in Lemma 2.1, the function z satisfies either Case (I) or Case (II).

Case (I): Proceeding as in the proof of Theorem 2.4, we get (2.9). Using (2.3) in (2.9), we obtain

$$\Big(a(t)\big(z''(t)\big)^\alpha\Big)' + f(t)\psi^\lambda\big(g(t)\big)\frac{B^\lambda(h(t))}{A^\lambda\big(h(t)\big)}\Big(z'\big(h(t)\big)\Big)^\lambda \le 0, t \ge t_1. \tag{2.28}$$

Let $x(t) = z'(t)$. Then $x(t) > 0$, $x'(t) > 0$ and $\big(a(t)\big(x'(t)\big)^\alpha\big)' \le 0$ for $t \ge t_1$. Since $a^{1/\alpha}(t)x'(t)$ is positive and decreasing, we have for $t \ge s \ge t_1$,

$$x'(s) \ge \frac{a^{1/\alpha}(t)x'(t)}{a^{1/\alpha}(s)}$$

and integrating from t_1 to t, we obtain

$$x(t) \ge A(t)a^{1/\alpha}(t)x'(t), t \ge t_1. \tag{2.29}$$

Integrating (2.28) from t to ∞, we obtain

$$a(t)\big(z''(t)\big)^\alpha \ge \left(\int_t^\infty f(s)\psi^\lambda\big(g(s)\big)\frac{B^\lambda\big(h(s)\big)}{A^\lambda\big(h(s)\big)}\,\mathrm{d}s\right) x^\lambda\Big(\sigma^{-1}\big(g(t)\big)\Big).$$

Using (2.29) in the last inequality, we obtain

$$a(t)\big(z''(t)\big)^\alpha \ge$$
$$\left(\int_t^\infty f(s)\psi^\lambda\big(g(s)\big)\frac{B^\lambda(h(s))}{A^\lambda(h(s))}\,\mathrm{d}s\right) A^\lambda\big(h(t)\big)\Big(a\big(h(t)\big)\Big)\Big(x\big(h(t)\big)\Big)^{\alpha}\Big)^{\lambda/\alpha}.$$

Since $h(t) \le t$, $a^{1/\alpha}(t)x'(t)$ is decreasing and using this in the last inequality, we get

$$\left(a(t_0)\big(x'(t_0)\big)^{\alpha}\right)^{1-\frac{\lambda}{\alpha}} \ge A^{\lambda}\big(h(t)\big)\int_t^{\infty} f(s)\psi^{\lambda}\big(g(s)\big)\frac{B^{\lambda}(h(s))}{A^{\lambda}(h(s))}\,\mathrm{d}s.$$

Taking lim sup as $t \to \infty$ in the last inequality, we obtain a contradiction to (2.26).

Case (II): From Case (II) of Theorem 2.4, we arrive at (2.12) and (2.13). Since $h(t) \ge h(s)$ for $t \ge s$ and putting $s = h(s)$ and $t = h(t)$ into (2.13), we get

$$z(h(s)) \ge \left(\int_{h(s)}^{h(t)}\left(\int_u^{h(t)} a^{-1/\alpha}(v)\,\mathrm{d}v\right)\mathrm{d}u\right) a^{1/\alpha}\big(h(t)\big)z''\big(h(t)\big). \tag{2.30}$$

Integrating (2.12) from $h(t)$ to t and using (2.30), we obtain

$$a\big(h(t)\big)\Big(z''\big(h(t)\big)\Big)^{\alpha} \ge \left(\int_{h(t)}^{t} f(s)\phi^{\lambda}\big(g(s)\big)R_1^{\lambda}(t,s)\,\mathrm{d}s\right)\left(a\big(h(t)\big)\Big(z''\big(h(t)\big)\Big)^{\alpha}\right)^{\lambda/\alpha},$$

which can be written as

$$\left[a\big(h(t_0)\big)\Big(z''\big(h(t_0)\big)\Big)^{\alpha}\right]^{1-\frac{\lambda}{\alpha}} \ge \int_{h(t)}^{t} f(s)\phi^{\lambda}\big(g(s)\big)R_1^{\lambda}(t,s)\,\mathrm{d}s.$$

Taking lim sup as $t \to \infty$ in the above inequality, we get a contradiction to (2.27). The proof of the theorem is complete. □

3. Examples

In this section, we present some examples to illustrate the significance of the main results.

Example 3.1. Consider the sublinear Emden-Fowler neutral differential equation

$$\left(\sqrt{t}\left(y(t) + ty\left(\frac{t}{2}\right)\right)''\right)' + \frac{d}{t^{\beta}}y^{1/3}\left(\frac{t}{4}\right) = 0, t \ge 4, \tag{3.1}$$

where $d > 0$ is a constant and $0 < \beta \le 7/6$.

Here $a(t) = \sqrt{t}$, $p(t) = t$, $f(t) = \frac{d}{t^{\beta}}$, $\sigma(t) = \frac{t}{2}$, $g(t) = \frac{t}{4}$, $\alpha = 1$ and $\lambda = 1/3$. Then $A(t) \approx 2\sqrt{t}$, $B(t) \approx \frac{4}{3}t^{3/2}$, $C(t) \approx t^{3/2}$, $R(t) \approx 0.018t^{3/2}$, $\phi(t) = \frac{4t-1}{8t^2}$ and

$$\psi(t) = \frac{\sqrt{2}t - 1}{2\sqrt{2}t^2}.$$

Thus (2.16) becomes

$$\int_4^\infty \frac{d}{t^\beta}\left(\frac{2(\sqrt{2}t-4)}{\sqrt{2}t^2}\right)^{/3}\left(\frac{2}{3\sqrt{2}}t^{3/2}\right)^{1/3}\mathrm{d}t \approx \int_4^\infty \frac{d_1}{t^{\beta-1/6}}\,\mathrm{d}t = \infty,$$

where $d_1 > 0$ is a constant. The condition (2.17) becomes

$$\int_4^\infty \frac{d}{t^\beta}\left(\frac{2(t-1)}{t^2}\right)^{1/3}\left(0.018t^{3/2}\right)^{1/3}\mathrm{d}t \approx \int_4^\infty \frac{d_2}{t^{\beta-1/6}}\,\mathrm{d}t = \infty,$$

where $d_2 > 0$ is a constant. Therefore, by Corollary 2.6, equation (3.1) is oscillatory.

EXAMPLE 3.2. Consider the half-linear Emden-Fowler neutral differential equation

$$\left(t^{3/2}\left(\left(y(t)+ty\left(\frac{t}{2}\right)\right)''\right)^3\right)' + \frac{d}{t^{3/2}}y^3\left(\frac{t}{3}\right) = 0, t \geq 1, \tag{3.2}$$

where $d > 0$ is a constant.

Here $a(t) = t^{3/2}$, $p(t) = t$, $f(t) = \frac{d}{t^{3/2}}$, $\sigma(t) = \frac{t}{2}$, $g(t) = \frac{t}{3}$, $\alpha = \lambda = 3$. Then $A(t) \approx 2\sqrt{t}$, $B(t) \approx \frac{4}{3}t^{3/2}$, $C(t) \approx t^{3/2}$, $\phi(t) = \frac{4t-1}{8t^2}$, $\psi(t) = \frac{\sqrt{2}t-1}{2\sqrt{2}t^2}$, and $R_1(t,s) = \frac{4\sqrt{2}}{9\sqrt{3}}t^{3/2} + \frac{8\sqrt{2}}{9\sqrt{3}}s^{3/2} - \frac{4\sqrt{2}}{8\sqrt{3}}t^{3/2}s$. Conditions (2.26) and (2.27) become

$$\lim_{t\to\infty}\sup t^{3/2}\int_t^\infty \frac{d_1}{s^2}\,\mathrm{d}s = \lim_{t\to\infty}\sup t^{1/2} = \infty$$

and

$$\lim_{t\to\infty}\sup \int_{2/3t}^t f(s)\phi^3(s/4)R_1^3(t,s)\,\mathrm{d}s = \lim_{t\to\infty}\left(d_1\sqrt{t} - \frac{d_2}{\sqrt{t}} + \frac{d_3}{t} - \frac{d_4}{t^2}\right) = \infty,$$

respectively. Hence by Theorem 2.10, equation (3.2) is oscillatory.

EXAMPLE 3.3. Consider the third-order superlinear neutral differential equation

$$\left(\sqrt{t}\big(y(t)+ty(t-2)\big)''\right)' + \exp(3^t)x^3(t-4) = 0, t \geq 2. \tag{3.3}$$

Here

$$\begin{array}{lllll} a(t)=\sqrt{t}, & p(t)=t, & f(t)=\exp(3^t), & \sigma(t)=t-2, & g(t)=t-4, \\ \alpha=1, & \lambda=3, & \delta_1=4, & \delta_2=3, & \delta_3=2 \end{array}$$

.

Then

$$A(t) \approx 2\sqrt{t}, \quad B(t) \approx \frac{4}{3}t^{3/2}, \quad C(t) \approx t^{3/2}, \quad R(t) \approx 0.018t^{3/2}, \quad \phi(t) = \frac{4t-1}{8t^2}$$

and

$$\psi(t) = \frac{\sqrt{2}t-1}{2\sqrt{2}t^2}.$$

The condition (2.18) becomes

$$\lim_{t\to\infty} \inf d_1 3^{-t/2} 3^t \log(t^{3/2}) > 0,$$

where $d_1 > 0$ is a constant. The condition (2.19) becomes

$$\lim_{t\to\infty} \inf d_2 3^{-t} 3^t \log(t^{3/2}) > 0,$$

where $d_2 > 0$ is a constant. Therefore, by Corollary 2.7, equation (3.3) is oscillatory.

4. Conclusion

In this paper, we have established some new sufficient conditions for the oscillation of (E) when α and λ satisfy different conditions. The results obtained in this paper generalize those in [2,5,6,7,9,11,12]. Further some of the results obtained in the literature applied to our examples since $a(t) \neq 1$ and $\alpha \neq 1$. Hence the criteria established in this paper are new and interesting for any researcher working in this area. Moreover, the results for the case $a(t) \equiv 1$, $\alpha = 1$ and $\lambda > 1$ answer the open problem posed in [2].

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