

SOME FRACTAL PROPERTIES OF SETS HAVING THE MORAN STRUCTURE

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ABSTRACT. This article is devoted to sets having the Moran structure. The main attention is given to topological, metric, and fractal properties of certain sets whose elements have restrictions on using digits or combinations of digits in own representations.

1. Introduction

In 1977, the notion “fractal” was considered by B. Mandelbrot in [15]. A fractal in the wide sense is a set whose topological dimension does not coincide with the Hausdorff dimension (the fractal dimension), and in the narrow sense, it is a set that has the fractional fractal dimension.

Fractals are the most appropriate mathematical models of natural objects. The importance of fractals lies in modeling of physical and biological processes, and also, fractal is a strictly mathematical notion that unites various mathematical objects, e.g., continuous nowhere differentiable functions, singular distributions, curves and surfaces that do not have the tangent at any point, etc. (see [10, 11, 15]). Indeed, the following examples are natural examples of fractals: the rings around planets (such fractals have the property of self-similarity), the snow cover in a mountain region, linear lightning, cloud borders, forms of coast lines or rivers. In fact, one can model coast lines and rivers by continuous nowhere differentiable functions. One of the oldest mathematical examples of fractals is the Cantor set

$$C = \left\{ x : x = \sum_{n=1}^{\infty} \frac{\alpha_n}{3^n}, \alpha_n \in \{0, 2\} \right\}.$$

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This set was introduced by G. Cantor in 1883. The last fractal is the part of such well-known fractals as the Sierpinski carpet, the Koch snowflake.

Fractal sets are widely applied in computer design, algorithms of the compression to information, quantum mechanics, solid-state physics, analysis and categorizations of signals of various forms appearing in different areas (e.g., the analysis of exchange rate fluctuations in economics), etc. However, for much classes of fractals, the problem of the Hausdorff dimension calculation is difficult and the estimate of parameters on which the Hausdorff dimension of certain classes of fractal sets depends is left out of consideration.

The aim of this survey is to give some local and fractal properties of certain Moran sets. One can note that Moran sets play an important role in multifractal analysis/formalism and especially the refined multifractal formalism (for example, see the papers [1, 42, 43, 45] and references therein).

The multifractal analysis is a natural framework to finely geometrically describe the heterogeneity in the distribution at small scales of the measures on a metric space. The multifractal analysis was proved to be a very useful technique in the analysis of measures, both in theory and applications. Also, the multifractal and the fractal analysis allows one to perform a certain classification of singular measures. One can note that it was proved that singular distributions of probabilities are dominant for many classes of random variables. Possible applications in the spectral theory of self-adjoint operators serve as an additional stimulus for a further investigation of singularly continuous measures [6]. For example, one can note the following researches of singular measures: singularity of Hewitt-Stromberg measures on Bedford-McMullen carpets [2], the mutual singularity of certain measures (see [6, 8, 24, 46, 47] and references therein), dimensions of measures [13, 20, 23].

Olsen [17] introduced a general form of multifractal formalism to interpret the statistical scaling properties of singular measures where the total mass or energy is spread over regions of phase space in an irregular way. The multifractal formalism aims at expressing the dimensions (the Hausdorff and packing dimensions) of the level sets in terms of the Legendre transform of some free energy function in analogy with the usual thermodynamic theory ([1, 23, 47] and references therein).

The multifractal formalism has been proved rigorously for random and non-random self-similar measures, for self-conformal measures, for self-affine and for Moran measures (see [1, 7] and references therein). Certain researches are devoted to new multifractal formalism for which the classical multifractal formalism does not hold. For example, the paper [1] deals with a multifractal formalism based on the Hewitt-Stromberg measures and that this formalism is completely parallel to Olsen's multifractal formalism which is based on the Hausdorff and packing measures.

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Among fractal geometrical objects, Moran's types play an important role in explaining many situations, in pure mathematics as the general context of Cantor's, and in applied physics as a suitable context for studying scaling laws. These sets may be understood as attractors for dynamical systems, electrical circuits, and also smart cities where fractals are nowadays sophisticated tools in their modeling. Fractals such as Cantor and Moran's types in general are also applied in understanding physical properties at different molecular levels, such as nonmaterial composites, crystal growth, and structure, porous materials, etc. [9]. Finally, one can note some investigations in multifractal analysis of Moran sets: multifractal properties of homogeneous Moran fractals associated with Fibonacci sequence [43], multifractal properties [42, 44].

Consider space \mathbb{R}^n . In [16], P. A. P. Moran introduced the following construction of sets and calculated the Hausdorff dimension of the limit set

$$E = \bigcap_{n=1}^{\infty} \bigcup_{i_1, \dots, i_n \in A_{0,p}} \Delta_{i_1 i_2 \dots i_n}. \quad (1)$$

Here p is a fixed positive integer, $A_{0,p} = \{1, 2, \dots, p\}$, and sets $\Delta_{i_1 i_2 \dots i_n}$ are basic sets having the following properties:

- any set $\Delta_{i_1 i_2 \dots i_n}$ is closed and disjoint;
- for any $i \in A_{0,p}$ the condition $\Delta_{i_1 i_2 \dots i_n i} \subset \Delta_{i_1 i_2 \dots i_n}$ holds;
- $\lim_{n \rightarrow \infty} d(\Delta_{i_1 i_2 \dots i_n}) = 0$, where $d(\cdot)$ is the diameter of a set;
- each basic set is the closure of its interior;
- at each level the basic sets do not overlap (their interiors are disjoint);
- any basic set $\Delta_{i_1 i_2 \dots i_n i}$ is geometrically similar to $\Delta_{i_1 i_2 \dots i_n}$;
- $\frac{d(\Delta_{i_1 i_2 \dots i_n i})}{d(\Delta_{i_1 i_2 \dots i_n})} = \sigma_i$, where $\sigma_i \in (0, 1)$ for $i = \overline{1, p}$.

The Hausdorff-Besicovitch dimension α_0 of the set E is the unique root of the following equation

$$\sum_{i=1}^p \sigma_i^{\alpha_0} = 1.$$

It is easy to see that set (1) is a Cantor-like set and a self-similar fractal. The set E is called *the Moran set*.

Let us consider the second definition of the Moran set given by Hua et al. [12].

DEFINITION 1.1 (Definition of Hua et al. [40]). Let (n_k) be a sequence of positive integers, $J \in \mathbb{R}^n$ be a compact set with nonempty interior, and (Φ_k) be a sequence of positive real vectors with $\Phi_k = (\sigma_{k,1}, \sigma_{k,2}, \dots, \sigma_{k,n_k})$, where $k \in \mathbb{N}$ and

$$\sum_{j=1}^{n_k} \sigma_{k,j} < 1.$$

A set of the form

$$E = \bigcap_{k=1}^{\infty} \bigcup_{i_1, \dots, i_k \in A_{0, n_k}} \Delta_{i_1 i_2 \dots i_k},$$

where $A_{0, n_k} = \{1, 2, \dots, n_k\}$, is called *the Moran set associated with the collection F* . Here,

$$F = \bigcup_{k=0}^{\infty} F_k = \bigcup_{k=0}^{\infty} \{J_\sigma := \Delta_{i_1 i_2 \dots i_k} : k \in \mathbb{N}, i_k \in \{1, 2, \dots, n_k\}\}.$$

The collection F fulfills the Moran structure provided it satisfies the following Moran Structure Conditions (MSC):

- (1) $J_\emptyset = J$.
- (2) An arbitrary J_σ is geometrically similar to J .
- (3) For any $i, j \in \{1, 2, \dots, n_{k+1}\}$ such that $i \neq j$, the conditions

$$\Delta_{i_1 i_2 \dots i_k i} \subset \Delta_{i_1 i_2 \dots i_k}, \quad \Delta_{i_1 i_2 \dots i_k i} \cap \Delta_{i_1 i_2 \dots i_k j} = \emptyset$$

hold.

- (4) For any $j \in \{1, 2, \dots, n_{k+1}\}$,

$$\frac{d(\Delta_{i_1 i_2 \dots i_k j})}{d(\Delta_{i_1 i_2 \dots i_k})} = \sigma_{k+1, j}.$$

The elements of F_k are called *the basic elements of order k of the Moran set E* , and the elements of F are called *the basic elements of the Moran set E* .

Remark 1. Let us note that the main difference between definitions of Moran and Hua is Property 4 in MSC.

Let $M = M(J, (n_k), (\Phi_k))$ be a class of Moran sets satisfying MSC 1–4. It is known that one can define a sequence (α_k) , where α_k satisfies the equation

$$\prod_{i=1}^k \sum_{j=1}^{n_i} \sigma_{i,j}^{\alpha_k} = 1.$$

Also, suppose that

$$\begin{aligned} \alpha_* &= \liminf_{k \rightarrow \infty} \alpha_k, & \alpha^* &= \limsup_{k \rightarrow \infty} \alpha_k; \\ c_* &= \inf_{i,j} \sigma_{i,j}, & c^* &= \sup_{i,j} \sigma_{i,j}. \end{aligned}$$

FRACTAL SETS

A lot of research has been devoted to Moran-like constructions and Cantor-like sets (for example, see [4,5,12,14,18,19] and references therein). For example, in [19], the one parameter family of Cantor sets

$$\Lambda(\lambda) = \left\{ x : x = \sum_{k=1}^{\infty} i_k \lambda^k, i_k \in S \subset \{0, 1, \dots, s-1\}, s \in \mathbb{N} \text{ is a fixed number} \right\}$$

is investigated.

THEOREM 1.2 ([19]). *Suppose that the condition $s-1 < (l-1)^2$ holds. Here, l is the cardinality of the set $S = \{s_1, \dots, s_l\}$, i.e., $l = |S|$. Then, for almost all $\lambda \in [\frac{1}{s}, \frac{1}{7}]$ (with respect to Lebesgue measure) we have that*

$$\alpha_0(\Lambda(\lambda)) = \frac{\log l}{-\log \lambda}.$$

It is easy to see that we obtain the case of classical s-adic representation (2) whenever $\lambda = \frac{1}{s}$. In this case, we get

$$\alpha_0(\Lambda(\lambda)) = \log_s l.$$

The following theorem generalizes the last result.

Let $D = (d_n)$ be a fixed sequence of positive integers such that $d_n > 1$ for all $n \in \mathbb{N}$, $\varepsilon_n \in A_{d_n} = \{0, 1, \dots, d_n - 1\}$. Series of the form

$$\sum_{n=1}^{\infty} \frac{\varepsilon_n}{d_1 d_2 \cdots d_n}$$

are *Cantor series* introduced by G. Cantor in [3]. These series are generalizations of s-adic expansion (2), i.e., a Cantor series is the s-adic expansion whenever $d_n = \text{const} = s$ for all $n \in \mathbb{N}$.

THEOREM 1.3 ([14]). *Suppose that $D = (d_n)$ is a fixed sequence of positive integer numbers $d_n > 1$, $\lim_{n \rightarrow \infty} \frac{\log d_n}{\log d_1 \cdots d_n} = 0$, $I_j \subseteq \{0, 1, \dots, d_j - 1\}$, $\mathcal{I} = (I_n)$. Then,*

$$\begin{aligned} \alpha_0(\mathcal{R}_{\mathcal{I}}(D)) &= \alpha_0 \left(\left\{ x : x = \sum_{n=1}^{\infty} \frac{\varepsilon_n}{d_1 d_2 \cdots d_n}, \varepsilon_n \in I_n \right\} \right) \\ &= \liminf_{n \rightarrow \infty} \frac{\log \prod_{j=1}^n |I_j|}{\log \prod_{j=1}^n d_j}. \end{aligned}$$

The present survey is devoted to fractal sets whose elements are defined by expansions related with some cases of positive and alternating Cantor series and their images under the action of certain singular distributions. The main attention is given to topological and metric properties of these sets, and also parameters on which the Hausdorff dimension of such sets depends. The sets

considered in this paper are determined by certain restrictions on using combinations of digits in representations of their elements. Also, the main attention is given to results obtained in papers [28–30] published in Ukrainian.

Let us remark that in September 2011 and February 2012 the author presented the results of the papers [28, 29] in the reports “The main topological, metric properties of one set of numbers such that it is defined by the s -adic representation with restrictions” and “The main topological, metric properties of one set defined by the nega- s -adic and s -adic representation with a parameter, and using this set” at the seminar on fractal analysis of the Institute of Mathematics of NAS of Ukraine and the National Pedagogical Dragomanov University (archive of reports is available at

- <http://www.imath.kiev.ua/events/index.php?seminarId=21&archiv=1>).

In 2012, the results of the papers [28, 29] were presented in conference abstracts [25–27]. Also, the main results of these papers were published in English as a preprint [32].

2. Definitions

We begin with definitions of several representations of real numbers and certain series.

Let $1 < s$ be a fixed positive integer, $A = \{0, 1, \dots, s-1\}$ be an alphabet of the s -adic or nega- s -adic numeral system, and $A_0 = A \setminus \{0\} = \{1, 2, \dots, s-1\}$, and

$$L = (A_0)^\infty = (A_0) \times (A_0) \times (A_0) \times \dots$$

be the space of one-sided sequences of elements of A_0 .

An expansion of a real number $x \in [0, 1]$ in the form

$$x = \frac{\alpha_1}{s} + \frac{\alpha_2}{s^2} + \dots + \frac{\alpha_n}{s^n} + \dots, \quad (2)$$

where $\alpha_n \in A$, is called *the s -adic expansion of x* . By $x = \Delta_{\alpha_1 \alpha_2 \dots \alpha_n \dots}^s$ denote the s -adic expansion of x . The notation $\Delta_{\alpha_1 \alpha_2 \dots \alpha_n \dots}^s$ is called *the s -adic representation of x* .

Obviously, the notation $x = \Delta_{\alpha_1 \alpha_2 \dots \alpha_n \dots}^{-s}$ is called *the nega- s -adic representation of x* . Here,

$$x = \Delta_{\alpha_1 \alpha_2 \dots \alpha_n \dots}^{-s} = -\frac{\alpha_1}{s} + \frac{\alpha_2}{s^2} - \frac{\alpha_3}{s^3} + \dots + \frac{(-1)^n \alpha_n}{s^n} + \dots, \quad (3)$$

where $\alpha_n \in A$.

If (k_n) is a certain fixed sequence of positive integers, then a series of the form

$$\frac{\alpha_{k_1}}{(-s)^{k_1}} + \frac{\alpha_{k_2}}{(-s)^{k_2}} + \dots + \frac{\alpha_{k_n}}{(-s)^{k_n}} + \dots, \alpha_{k_n} \in A,$$

is a *nega- s -adic series*.

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Suppose that $m_1 = k_1$, $m_2 = k_2 - k_1$, $m_3 = k_3 - k_2$, \dots , $m_n = k_n - k_{n-1}$, \dots . Then, we obtain the following series

$$\sum_{n=1}^{\infty} \frac{\alpha_{m_1+m_2+\dots+m_n}}{(-s)^{m_1+m_2+\dots+m_n}}, \quad (4)$$

where $\alpha_{m_1+m_2+\dots+m_n} \in A$.

Numbers $x \in \left[-\frac{s}{s+1}, \frac{1}{s+1}\right]$ having a representation in the form (4) have the following nega-s-adic representation

$$x = \sum_{n=1}^{\infty} \frac{\alpha_{m_1+m_2+\dots+m_n}}{(-s)^{m_1+m_2+\dots+m_n}} = \Delta_{\underbrace{0\dots 0}_{m_1-1}}^{-s} \alpha_{m_1} \underbrace{0\dots 0}_{m_2-1} \alpha_{m_1+m_2} \dots \underbrace{0\dots 0}_{m_n-1} \alpha_{m_1+m_2+\dots+m_n} \dots$$

Let (d_n) be a fixed sequence of positive integers such that $d_n > 1$ for all $n \in \mathbb{N}$, (A_n) be a sequence of the sets

$$A_n = \{0, 1, 2, \dots, d_n - 1\}, \quad \text{and} \quad L_n = A_1 \times A_2 \times A_n \times \dots$$

A series of the form

$$-\frac{\varepsilon_1}{d_1} + \frac{\varepsilon_2}{d_1 d_2} - \frac{\varepsilon_3}{d_1 d_2 d_3} + \dots + \frac{(-1)^n \varepsilon_n}{d_1 d_2 \dots d_n} + \dots, \quad (5)$$

where $\varepsilon_n \in A_n$, is called *an alternating Cantor series*.

In September 2013, see the presentation and the working paper available at:

- <https://www.researchgate.net/publication/303720347>,

- <https://www.researchgate.net/publication/316787375>,

respectively (in Ukrainian), the expansion of numbers by an alternating Cantor series was investigated as a numeral system and presented in the report "Representations of real numbers by alternating Cantor series" at the seminar on fractal analysis of the Institute of Mathematics of NAS of Ukraine and the National Pedagogical Dragomanov University. These results were published in [31].

An alternating Cantor series that is a nega-s-adic series is called *a nega-s-adic Cantor series*. That is,

$$-\frac{\varepsilon_1}{s^{m_1}} + \frac{\varepsilon_2}{s^{m_1+m_2}} - \frac{\varepsilon_3}{s^{m_1+m_2+m_3}} + \dots + \frac{(-1)^n \varepsilon_n}{s^{m_1+m_2+\dots+m_n}} + \dots, \quad \varepsilon_n \in A. \quad (6)$$

It is easy to see that the following statement is true.

LEMMA 2.1 ([30]). *Nega-s-adic series (4) is an alternating Cantor series if and only if for any $n \in \mathbb{N}$ a sequence (m_n) is a sequence of odd positive integers and $\varepsilon_n = \alpha_n \in A$ as well.*

A series of the form

$$-\frac{\alpha_1}{s^{k_1}} + \frac{\alpha_2}{s^{k_2}} - \frac{\alpha_3}{s^{k_3}} + \dots + \frac{(-1)^n \alpha_n}{s^{k_n}} + \dots, \quad \alpha_n \in A$$

is called a *mixed s-adic series*. Trivially, the last series is an alternating Cantor series.

We note that the case when sequences (α_n) and (m_n) are interdependent is interesting, e.g., when $m_n = \alpha_n \in A_0$ for an arbitrary $n \in \mathbb{N}$. In particular, we shall describe properties of the set

$$S^- = \left\{ x : x = \sum_{n=1}^{\infty} \frac{(-1)^n \alpha_n}{s^{\alpha_1 + \alpha_2 + \dots + \alpha_n}}, \quad (\alpha_n) \in L, \quad s > 2 \right\}$$

in the present article. Also, here, the following set is considered

$$M_{(-D,s)} = \left\{ x : x = \Delta_{\underbrace{0 \dots 0}_{m_1-1}}^{-s} \alpha_{m_1} \underbrace{0 \dots 0}_{m_2-1} \alpha_{m_1+m_2} \dots \underbrace{0 \dots 0}_{m_n-1} \alpha_{m_1+m_2+\dots+m_n} \dots \right\},$$

where $s > 1$ is a fixed positive integer, $\alpha_{m_1+m_2+\dots+m_n} \neq 0$ for all $n \in \mathbb{N}$, and $m_n \in \{3, 5, 7, \dots, 2i + 1, \dots\}$.

3. Fractal sets

Let us consider the Cantor set. Any element of the Cantor set has only digits 0 and 2 in own ternary representation. This set is an uncountable, perfect, and nowhere dense set of zero Lebesgue measure. Also, this is a self-similar fractal whose Hausdorff-Besicovitch dimension is equal to $\log_3 2$.

One can formulate a general theorem on values of the Hausdorff-Besicovitch dimension of a set whose elements have restrictions on using combinations of digits in own s-adic representation.

THEOREM 3.1 ([29, 32]). *Let E be a set whose elements are represented by a finite number of fixed combinations $\sigma_1, \sigma_2, \dots, \sigma_m$ of s-adic digits in the s-adic numeral system. Then, the Hausdorff-Besicovitch dimension α_0 of E satisfies the following equation*

$$N(\sigma_m^1) \left(\frac{1}{s}\right)^{\alpha_0} + N(\sigma_m^2) \left(\frac{1}{s}\right)^{2\alpha_0} + \dots + N(\sigma_m^k) \left(\frac{1}{s}\right)^{k\alpha_0} = 1,$$

where $N(\sigma_m^k)$ is a number of k -digit combinations σ_m^k from the set

$$\{\sigma_1, \sigma_2, \dots, \sigma_m\}, \quad k \in \mathbb{N}, \quad \text{and} \quad N(\sigma_m^1) + N(\sigma_m^2) + \dots + N(\sigma_m^k) = m.$$

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This theorem is interesting since fractal properties of many sets of special types follow from this theorem. For example, the following set whose elements have a functional restriction on using digits in own s-adic representation was studied in [28]

$$S = \left\{ x : x = \sum_{n=1}^{\infty} \frac{\alpha_n}{s^{\alpha_1 + \alpha_2 + \dots + \alpha_n}}, (\alpha_n) \in L \right\},$$

where $s > 2$ is a fixed positive integer. The last-mentioned set is the set of all numbers whose s-adic representations contain only the following combinations of s-adic digits

$$1, 02, 003, \dots, \underbrace{0 \dots 0}_i i, \dots, \underbrace{0 \dots 0}_{s-2} [s-1].$$

The Hausdorff-Besicovitch dimension α_0 of the set S satisfies the equation

$$\left(\frac{1}{s}\right)^{\alpha_0} + \left(\frac{1}{s}\right)^{2\alpha_0} + \left(\frac{1}{s}\right)^{3\alpha_0} + \dots + \left(\frac{1}{s}\right)^{(s-1)\alpha_0} = 1.$$

Assume that $s > 2$ is a fixed positive integer number.

Consider a class Υ_s of sets $\mathbb{S}_{(s,u)}$ represented in the form

$$\mathbb{S}_{(s,u)} = \left\{ x : x = \frac{u}{s-1} + \sum_{n=1}^{\infty} \frac{\alpha_n - u}{s^{\alpha_1 + \dots + \alpha_n}}, (\alpha_n) \in L, \alpha_n \neq u, \alpha_n \neq 0 \right\},$$

where $u = \overline{0, s-1}$, u and s are fixed for the set $\mathbb{S}_{(s,u)}$. That is, the class Υ_s contains the sets $\mathbb{S}_{(s,0)}, \mathbb{S}_{(s,1)}, \dots, \mathbb{S}_{(s,s-1)}$. We say that Υ is a class of sets such that it contains the classes $\Upsilon_3, \Upsilon_4, \dots, \Upsilon_n, \dots$

It is easy to see that the set $\mathbb{S}_{(s,u)}$ can be defined by the s-adic representation in the following form

$$\mathbb{S}_{(s,u)} = \left\{ x : x = \Delta \underbrace{u \dots u}_{\alpha_1-1} \underbrace{u \dots u}_{\alpha_2-1} \dots \underbrace{u \dots u}_{\alpha_n-1} \dots, (\alpha_n) \in L, \alpha_n \neq u, \alpha_n \neq 0 \right\}.$$

THEOREM 3.2 ([29,32]). *For an arbitrary $u \in A$ the set $\mathbb{S}_{(s,u)}$ is an uncountable, perfect, nowhere dense set of zero Lebesgue measure, and a self-similar fractal whose Hausdorff-Besicovitch dimension $\alpha_0(\mathbb{S}_{(s,u)})$ satisfies the following equation*

$$\sum_{p_i \neq u, p_i \in A_0} \left(\frac{1}{s}\right)^{p\alpha_0} = 1.$$

To prove the last statement, the auxiliary notion “cylinder” is used. This notion is useful for study of local properties of the considered sets (see the following lemma).

By $x_0 = \Delta_{c_1 \dots c_n}^{(s,u)}$ denote the equality

$$x_0 = \frac{u}{s-1} + \sum_{k=1}^{\infty} \frac{c_k - u}{s^{c_1 + \dots + c_k}}.$$

That is,

$$x_0 = \Delta_{c_1 \dots c_n}^{(s,u)} = \Delta_{\underbrace{u \dots u}_{c_1-1} \underbrace{u \dots u}_{c_2-1} \dots \underbrace{u \dots u}_{c_n-1}}$$

DEFINITION 3.3. A cylinder $\Delta_{c_1 \dots c_n}^{(s,u)}$ of rank n with base $c_1 c_2 \dots c_n$ is a set of the following form

$$\Delta_{c_1 \dots c_n}^{(s,u)} = \left\{ x : x = \left(\sum_{k=1}^n \frac{c_k - u}{s^{c_1 + \dots + c_k}} \right) + \frac{1}{s^{c_1 + \dots + c_n}} \left(\sum_{i=n+1}^{\infty} \frac{\alpha_i - u}{s^{\alpha_{n+1} + \dots + \alpha_i}} \right) + \frac{u}{s-1} \right\},$$

where c_1, c_2, \dots, c_n are fixed s -adic digits, $c_n \neq 0$, $c_n \neq u$, $\alpha_n \neq u$, $\alpha_n \neq 0$, and $2 < s \in \mathbb{N}, n \in \mathbb{N}$.

LEMMA 3.4 ([29, 32]). *Cylinders $\Delta_{c_1 \dots c_n}^{(s,u)}$ have the following properties:*

(1)

$$\inf \Delta_{c_1 \dots c_n}^{(s,u)} = \begin{cases} \tau + \frac{1}{s^{c_1 + \dots + c_n}} \left(\frac{s-1-u}{s^{s-1-1}} + \frac{u}{s-1} \right) & \text{if } u \in \{0, 1\}, \\ \tau + \frac{1}{s^{c_1 + \dots + c_n}} \frac{1}{s-1} & \text{if } u \in \{2, 3, \dots, s-1\}, \end{cases}$$

$$\sup \Delta_{c_1 \dots c_n}^{(s,u)} = \begin{cases} \tau + \frac{1}{s^{c_1 + \dots + c_n}} \frac{1}{s-1} & \text{if } u = 0, \\ \tau + \frac{1}{s^{c_1 + \dots + c_n}} \left(\frac{1}{s^{u+1-1}} + \frac{u}{s-1} \right) & \text{if } u \in \{1, 2, \dots, s-2\}, \\ \tau + \frac{1}{s^{c_1 + \dots + c_n}} \left(1 - \frac{1}{s^{s-2-1}} \right) & \text{if } u = s-1, \end{cases}$$

where

$$\tau = \sum_{k=1}^n \frac{c_k - u}{s^{c_1 + \dots + c_k}} + \sum_{k=1}^n \frac{u}{s^k}.$$

(2) if $d(\cdot)$ is the diameter of a set, then

$$d(\Delta_{c_1 \dots c_n}^{(s,u)}) = \frac{1}{s^{c_1 + \dots + c_n}} d(\mathbb{S}_{(s,u)});$$

$$(3) \quad \frac{d(\Delta_{c_1 \dots c_n c_{n+1}}^{(s,u)})}{d(\Delta_{c_1 \dots c_n}^{(s,u)})} = \frac{1}{s^{c_{n+1}}};$$

$$(4) \quad \Delta_{c_1 c_2 \dots c_n}^{(s,u)} = \bigcup_{i=1}^{s-1} \Delta_{c_1 c_2 \dots c_n i}^{(s,u)} \quad \forall c_n \in A_0, \quad n \in \mathbb{N}, \quad i \neq u.$$

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(5) *The following relationships hold:*

(a) *if $u \in \{0, 1\}$, then*

$$\inf \Delta_{c_1 \dots c_n p}^{(s,u)} > \sup \Delta_{c_1 \dots c_n [p+1]}^{(s,u)};$$

(b) *if $u \in \{2, 3, \dots, s-3\}$, then*

$$\begin{cases} \sup \Delta_{c_1 \dots c_n p}^{(s,u)} < \inf \Delta_{c_1 \dots c_n [p+1]}^{(s,u)} & \text{for all } p+1 \leq u, \\ \inf \Delta_{c_1 \dots c_n p}^{(s,u)} > \sup \Delta_{c_1 \dots c_n [p+1]}^{(s,u)} & \text{for all } u < p; \end{cases}$$

(c) *if $u \in \{s-2, s-1\}$, then*

$$\sup \Delta_{c_1 \dots c_n p}^{(s,u)} < \inf \Delta_{c_1 \dots c_n [p+1]}^{(s,u)}.$$

The fifth property of the last lemma means the following:

- for any positive integer n cylinders $\Delta_{c_1 \dots c_n}^{(s,u)}$ are right-to-left situated in the case of the set $\mathbb{S}_{(s,0)}$ or $\mathbb{S}_{(s,1)}$;
- let us have the sets $\mathbb{S}_{(s,2)}, \mathbb{S}_{(s,3)}, \dots, \mathbb{S}_{(s,s-3)}$; then cylinders $\Delta_{c_1 \dots c_n}^{(s,u)}$ ($u = \overline{2, s-3}$) are left-to-right situated for all $c_n \leq 1, c_n \leq 2, \dots, c_n \leq s-4$, respectively, and cylinders $\Delta_{c_1 \dots c_n}^{(s,u)}$ are right-to-left situated for all $c_n > 2, c_n > 3, \dots, c_n > s-3$, respectively;
- for all positive integers n cylinders $\Delta_{c_1 \dots c_n}^{(s,u)}$ are left-to-right situated in the case of the set $\mathbb{S}_{(s,s-2)}$ or $\mathbb{S}_{(s,s-1)}$;
- for any $\mathbb{S}_{(s,u)}$, $n \in \mathbb{N}$, and $c_n \neq s-1$ the following condition holds

$$\Delta_{c_1 \dots c_{n-1} c_n}^{(s,u)} \cap \Delta_{c_1 \dots c_{n-1} [c_n+1]}^{(s,u)} = \emptyset.$$

For proving the nowhere density of $\mathbb{S}_{(s,u)}$, the last property is used.

Consider the set of all numbers whose s-adic representations contain only combinations of s-adic digits that are used in the s-adic representations of elements of $\mathbb{S}_{(s,u)}$.

By $\tilde{\mathbb{S}}$ denote the set of all numbers whose s-adic representations contain only combinations of s-adic digits from the set

$$\left\{ 1, 02, 003, \dots, \underbrace{u \dots uc}_{c-1}, \dots, \underbrace{(s-1) \dots (s-1)(s-2)}_{s-3} \right\},$$

where $c \in A_0, u \in A, c \neq u$.

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- *self-similar fractals whose Hausdorff-Besicovitch dimension α_0 satisfies the following equation*

$$\sum_{i=1}^{s-1} \left(\frac{1}{s}\right)^{i\alpha_0} = 1.$$

Proof. Let us prove that the sets $\mathbb{S}_{(-s,0)}$ and S^- are uncountable.

Let us prove that the sets $\mathbb{S}_{(-s,0)}$ and $C[-s, A_0]$ are equivalent. That is, let us consider the mapping

$$x = \sum_{n=1}^{\infty} \frac{\alpha_n \cdot (-1)^{\alpha_1 + \alpha_2 + \dots + \alpha_n}}{s^{\alpha_1 + \alpha_2 + \dots + \alpha_n}} \xrightarrow{f} \sum_{n=1}^{\infty} \frac{\alpha_n}{(-s)^n} = f(x) = y$$

or, in other words,

$$x = \Delta_{\underbrace{0 \dots 0}_{\alpha_1 - 1} \alpha_1 \underbrace{0 \dots 0}_{\alpha_2 - 1} \alpha_2 \dots \underbrace{0 \dots 0}_{\alpha_n - 1} \alpha_n \dots}^{-s} \xrightarrow{f} \Delta_{\alpha_1 \alpha_2 \dots \alpha_n \dots}^{-s} = f(x) = y.$$

Suppose x_1 and x_2 from $\mathbb{S}_{(-s,0)}$ are such that $x_1 \neq x_2$ and

$$x_1 = \Delta_{\underbrace{0 \dots 0}_{\alpha_1 - 1} \alpha_1 \underbrace{0 \dots 0}_{\alpha_2 - 1} \alpha_2 \dots \underbrace{0 \dots 0}_{\alpha_n - 1} \alpha_n \dots}^{-s},$$

$$x_2 = \Delta_{\underbrace{0 \dots 0}_{\beta_1 - 1} \beta_1 \underbrace{0 \dots 0}_{\beta_2 - 1} \beta_2 \dots \underbrace{0 \dots 0}_{\beta_n - 1} \beta_n \dots}^{-s}$$

if $f(x_1) = f(x_2)$ is nega- s -adic irrational (i.e., this number has the unique representation), then $\alpha_n = \beta_n$ holds for all $n \in \mathbb{N}$. That is, $x_1 = x_2$. It contradicts the condition.

Assume that $f(x_1) = f(x_2)$ is nega- s -adic rational. But this is not possible because any number from $C[-s, A_0]$ does not have two expansions.

So, f is a bijection. Since $C[-s, A_0]$ is an uncountable set, we see that $\mathbb{S}_{(-s,0)}$ is an uncountable set. The proofs for S^- are similar.

Statements of this theorem follow from properties of the following notions of cylinders. The proofs are similar to those of Theorem 1 and Theorem 3 in [32] (arXiv:1703.05262). \square

DEFINITION 3.7. A cylinder $\Delta_{c_1 c_2 \dots c_n}^{(-s,0)}$ of rank n with base $c_1 c_2 \dots c_n$ is a set formed by all numbers of the set $\mathbb{S}_{(-s,0)}$ with nega- s -representations in which the first n non-zero digits coincide with c_1, c_2, \dots, c_n , respectively.

DEFINITION 3.8. A cylinder $\Delta_{c_1 c_2 \dots c_n}^-$ of rank n with base $c_1 c_2 \dots c_n$ is a subset of S^- with elements for which the following condition holds

$$\alpha_1 = c_1, \quad \alpha_2 = c_2, \dots, \quad \alpha_n = c_n,$$

where c_1, c_2, \dots, c_n is an ordered tuple of numbers.

LEMMA 3.9 ([30]). *Cylinders $\Delta_{c_1 c_2 \dots c_n}^{(-s,0)}$ have the following properties:*

(1)

$$\inf \Delta_{c_1 c_2 \dots c_n}^{(-s,0)} = \begin{cases} g_n^{(-s)} + \frac{\inf \mathbb{S}_{(-s,0)}}{(-s)^{c_1+c_2+\dots+c_n}} & \text{if } c_1 + \dots + c_n \text{ is even,} \\ g_n^{(-s)} + \frac{\sup \mathbb{S}_{(-s,0)}}{(-s)^{c_1+c_2+\dots+c_n}} & \text{if } c_1 + \dots + c_n \text{ is odd,} \end{cases}$$

$$\sup \Delta_{c_1 c_2 \dots c_n}^{(-s,0)} = \begin{cases} g_n^{(-s)} + \frac{\sup \mathbb{S}_{(-s,0)}}{(-s)^{c_1+c_2+\dots+c_n}} & \text{if } c_1 + \dots + c_n \text{ is even,} \\ g_n^{(-s)} + \frac{\inf \mathbb{S}_{(-s,0)}}{(-s)^{c_1+c_2+\dots+c_n}} & \text{if } c_1 + \dots + c_n \text{ is odd,} \end{cases}$$

where

$$g_n^{(-s)} = \sum_{i=1}^n \frac{c_i (-1)^i}{s^{c_1+c_2+\dots+c_i}}.$$

(2) Suppose $d(\cdot)$ is the diameter of a set. Then,

$$d\left(\Delta_{c_1 c_2 \dots c_n}^{(-s,0)}\right) = \frac{d(\mathbb{S}_{(-s,0)})}{s^{c_1+c_2+\dots+c_n}}.$$

(3) The main metric relationship is as follows

$$\frac{d\left(\Delta_{c_1 c_2 \dots c_n c_{n+1}}^{(-s,0)}\right)}{d\left(\Delta_{c_1 c_2 \dots c_n}^{(-s,0)}\right)} = \frac{1}{s^{c_{n+1}}}.$$

(4) For any $n \in \mathbb{N}$ the following condition holds

$$\Delta_{c_1 c_2 \dots c_n}^{(-s,0)} = \bigcup_{i=1}^{s-1} \Delta_{c_1 c_2 \dots c_n i}^{(-s,0)}.$$

(5) For cylinders $\Delta_{c_1 c_2 \dots c_n c_{n+1}}^{(-s,0)}$ of rank $(n+1)$ with base $c_1 c_2 \dots c_n c_{n+1}$ the following relationships hold:

$$\inf \Delta_{c_1 c_2 \dots c_n p}^{(-s,0)} > \sup \Delta_{c_1 c_2 \dots c_n [p+1]}^{(-s,0)} \text{ whenever } c_1 + c_2 + \dots + c_n + p \text{ is even,}$$

$$\inf \Delta_{c_1 c_2 \dots c_n [p+1]}^{(-s,0)} > \sup \Delta_{c_1 c_2 \dots c_n p}^{(-s,0)} \text{ whenever } c_1 + c_2 + \dots + c_n + p \text{ is odd.}$$

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(6) Let $T_{c_1 c_2 \dots c_n p}^{(-s,0)}$ be an interval of the form

$$T_{c_1 c_2 \dots c_n p}^{(-s,0)} = \begin{cases} \left(\sup \Delta_{c_1 c_2 \dots c_n [p+1]}^{(-s,0)}, \inf \Delta_{c_1 c_2 \dots c_n p}^{(-s,0)} \right) & \text{if } c_1 + \dots + c_n + p \text{ is even,} \\ \left(\sup \Delta_{c_1 c_2 \dots c_n p}^{(-s,0)}, \inf \Delta_{c_1 c_2 \dots c_n [p+1]}^{(-s,0)} \right) & \text{if } c_1 + \dots + c_n + p \text{ is odd,} \end{cases}$$

where $1 \leq p < s - 1$ is a positive integer. Then,

$$T_{c_1 c_2 \dots c_n p}^{(-s,0)} \cap \mathbb{S}_{(-s,0)} = \emptyset.$$

(7) For any $p \in \{1, 2, \dots, s - 2\}$ the following condition holds

$$\Delta_{c_1 c_2 \dots c_n p}^{(-s,0)} \cap \Delta_{c_1 c_2 \dots c_n [p+1]}^{(-s,0)} = \emptyset.$$

(8) If $x_0 \in \mathbb{S}_{(-s,0)}$, then

$$x_0 = \bigcap_{n=1}^{\infty} \Delta_{c_1 c_2 \dots c_n}^{(-s,0)}.$$

Proof. Properties 1–4 follow from the definitions of $\Delta_{c_1 c_2 \dots c_n}^{(-s,0)}$ and $\mathbb{S}_{(-s,0)}$.

Let us prove *Property 5*. Suppose $\Delta_{c_1 c_2 \dots c_n p}^{(-s,0)}$, $\Delta_{c_1 c_2 \dots c_n (p+1)}^{(-s,0)}$ are cylinders, where $1 \leq p < s - 1$, and

$$g_n^{(-s)} = \sum_{i=1}^n \frac{c_i}{(-s)^{c_1 + c_2 + \dots + c_i}}; \quad \varpi_n = c_1 + c_2 + \dots + c_n.$$

From the definition of $\Delta_{c_1 c_2 \dots c_n}^{(-s,0)}$, it follows that

$$\Delta_{c_1 c_2 \dots c_n p}^{(-s,0)} \subset \begin{cases} \left[g_n^{(-s)} + \frac{p}{(-s)^{\varpi_n + p}} + \frac{-(s^2 + 1)}{s(s^2 - 1)(-s)^{\varpi_n + p}}; \right. \\ \left. g_n^{(-s)} + \frac{p}{(-s)^{\varpi_n + p}} + \frac{2}{(s^2 - 1)(-s)^{\varpi_n + p}} \right], \\ \left[g_n^{(-s)} + \frac{p}{(-s)^{\varpi_n + p}} + \frac{2}{(s^2 - 1)(-s)^{\varpi_n + p}}; \right. \\ \left. g_n^{(-s)} + \frac{p}{(-s)^{\varpi_n + p}} + \frac{-(s^2 + 1)}{s(s^2 - 1)(-s)^{\varpi_n + p}} \right], \end{cases}$$

where $\varpi_n + p$ is even for the first case, and $\varpi_n + p$ is odd for the second case.

By analogy, we obtain

$$\Delta_{c_1 c_2 \dots c_n (p+1)}^{(-s,0)} \subset \begin{cases} \left[g_n^{(-s)} + \frac{p+1}{(-s)^{\varpi_n + p + 1}} + \frac{-(s^2 + 1)}{s(s^2 - 1)(-s)^{\varpi_n + p + 1}}; \right. \\ \left. g_n^{(-s)} + \frac{p+1}{(-s)^{\varpi_n + p + 1}} + \frac{2}{(s^2 - 1)(-s)^{\varpi_n + p + 1}} \right], \\ \left[g_n^{(-s)} + \frac{p+1}{(-s)^{\varpi_n + p + 1}} + \frac{2}{(s^2 - 1)(-s)^{\varpi_n + p + 1}}; \right. \\ \left. g_n^{(-s)} + \frac{p+1}{(-s)^{\varpi_n + p + 1}} + \frac{-(s^2 + 1)}{s(s^2 - 1)(-s)^{\varpi_n + p + 1}} \right], \end{cases}$$

where $(\varpi_n + p + 1)$ is even for the first case and is odd for the second case.

Let us prove the mentioned inequalities.

Let $\varpi_n + p = c_1 + c_2 + \cdots + c_n + p$ be an even number. Then,

$$\begin{aligned} & \inf \Delta_{c_1 c_2 \dots c_n p}^{(-s,0)} - \sup \Delta_{c_1 c_2 \dots c_n (p+1)}^{(-s,0)} \\ &= g_n^{(-s)} + \frac{p}{(-s)^{\varpi_n+p}} + \frac{-(s^2+1)}{s(s^2-1)(-s)^{\varpi_n+p}} \\ & \quad - g_n^{(-s)} - \frac{p+1}{(-s)^{\varpi_n+p+1}} - \frac{-(s^2+1)}{s(s^2-1)(-s)^{\varpi_n+p+1}} \\ &= \frac{1}{s^{\varpi_n+p}} \left(ps + p + 1 - \frac{s^3 + s^2 + s + 1}{s(s^2-1)} \right) > 0, \end{aligned}$$

because

$$\frac{s^3 + s^2 + s + 1}{s(s^2-1)} = 1 + \frac{(s+1)^2}{s(s^2-1)} = 1 + \frac{s+1}{s(s-1)} < 2.$$

Let $\varpi_n + p = c_1 + c_2 + \cdots + c_n + p$ be an odd number. Then,

$$\begin{aligned} & \inf \Delta_{c_1 c_2 \dots c_n (p+1)}^{(-s,0)} - \sup \Delta_{c_1 c_2 \dots c_n p}^{(-s,0)} \\ &= g_n^{(-s)} + \frac{p+1}{(-s)^{\varpi_n+p+1}} + \frac{-(s^2+1)}{s(s^2-1)(-s)^{\varpi_n+p+1}} \\ & \quad - g_n^{(-s)} - \frac{p}{(-s)^{\varpi_n+p}} - \frac{-(s^2+1)}{s(s^2-1)(-s)^{\varpi_n+p}} \\ &= \frac{1}{s^{\varpi_n+p+1}} \left(ps + p + 1 - \frac{s^3 + s^2 + s + 1}{s(s^2-1)} \right) > 0. \end{aligned}$$

To prove *Property 6*, it suffices to prove the following inequalities:

- under the condition that $c_1 + c_2 + \cdots + c_n + p$ is an even number

$$\begin{cases} \sup \Delta_{c_1 c_1 \dots c_n (p+1) c_{n+2}}^{(-s,0)} - \sup \Delta_{c_1 c_1 \dots c_n (p+1)}^{(-s,0)} < 0, \\ \inf \Delta_{c_1 c_1 \dots c_n p c_{n+2}}^{(-s,0)} - \inf \Delta_{c_1 c_1 \dots c_n p}^{(-s,0)} > 0; \end{cases}$$

- under the condition that $c_1 + c_2 + \cdots + c_n + p$ is an odd number

$$\begin{cases} \sup \Delta_{c_1 c_1 \dots c_n p c_{n+2}}^{(-s,0)} - \sup \Delta_{c_1 c_1 \dots c_n p}^{(-s,0)} < 0, \\ \inf \Delta_{c_1 c_1 \dots c_n (p+1) c_{n+2}}^{(-s,0)} - \inf \Delta_{c_1 c_1 \dots c_n (p+1)}^{(-s,0)} > 0. \end{cases}$$

Suppose

$$l_0(c_1, c_2, \dots, c_n, p) = \begin{cases} -\frac{s^2+1}{s(s^2-1)} & \text{whenever } c_1 + c_2 + \cdots + c_n + p \text{ is even,} \\ \frac{2}{s^2-1} & \text{whenever } c_1 + c_2 + \cdots + c_n + p \text{ is odd,} \end{cases}$$

$$l(c_1, c_2, \dots, c_n, p) = \begin{cases} \frac{2}{s^2-1} & \text{whenever } c_1 + c_2 + \cdots + c_n + p \text{ is even,} \\ -\frac{s^2+1}{s(s^2-1)} & \text{whenever } c_1 + c_2 + \cdots + c_n + p \text{ is odd.} \end{cases}$$

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Let $c_1 + c_2 + \dots + c_n + p$ be an even number. Then,

$$\begin{aligned}
 & \sup \Delta_{c_1 c_1 \dots c_n (p+1) c_{n+2}}^{(-s,0)} - \sup \Delta_{c_1 c_1 \dots c_n (p+1)}^{(-s,0)} \\
 &= g_n^{(-s)} + \frac{p+1}{(-s)^{c_1+\dots+c_n+p+1}} + \frac{c_{n+2}}{(-s)^{c_1+\dots+c_n+p+1+c_{n+2}}} \\
 & \quad + \frac{l(c_1, c_2, \dots, c_n, p+1, c_{n+2})}{(-s)^{c_1+\dots+c_n+p+1+c_{n+2}}} - g_n^{(-s)} - \frac{p+1}{(-s)^{c_1+\dots+c_n+p+1}} \\
 & \quad - \frac{l(c_1, c_2, \dots, c_n, p+1)}{(-s)^{c_1+\dots+c_n+p+1}} \\
 &= -\frac{1}{s^{c_1+\dots+c_n+p+1}} \left(\frac{c_{n+2}}{(-s)^{c_{n+2}}} + \frac{l(c_1, c_2, \dots, c_n, p+1, c_{n+2})}{(-s)^{c_{n+2}}} + \frac{s^2+1}{s(s^2-1)} \right) \\
 &= \begin{cases} -\frac{1}{s^{c_1+\dots+c_n+p+1}} \left(\frac{c_{n+2}}{s^{c_{n+2}}} - \frac{s^2+1}{s(s^2-1)s^{c_{n+2}}} + \frac{s^2+1}{s(s^2-1)} \right) < 0 & \text{if } c_{n+2} \text{ is even,} \\ -\frac{1}{s^{c_1+\dots+c_n+p+1}} \left(-\frac{c_{n+2}}{s^{c_{n+2}}} - \frac{2}{(s^2-1)s^{c_{n+2}}} + \frac{s^2+1}{s(s^2-1)} \right) < 0 & \text{if } c_{n+2} \text{ is odd,} \end{cases}
 \end{aligned}$$

because

$$\begin{aligned}
 & -\frac{c_{n+2}}{s^{c_{n+2}}} - \frac{2}{(s^2-1)s^{c_{n+2}}} + \frac{s^2+1}{s(s^2-1)} = \\
 & \quad \frac{(s^2+1)s^{c_{n+2}} + sc_{n+2} - (s^2c_{n+2} + 2)s}{(s^2-1)s^{1+c_{n+2}}} \geq 0.
 \end{aligned}$$

By analogy, we have

$$\begin{aligned}
 & \inf \Delta_{c_1 c_1 \dots c_n p c_{n+2}}^{(-s,0)} - \inf \Delta_{c_1 c_1 \dots c_n p}^{(-s,0)} \\
 &= \frac{c_{n+2}}{(-s)^{c_1+\dots+c_n+p+c_{n+2}}} + \frac{l_0(c_1, c_2, \dots, c_n, p, c_{n+2})}{(-s)^{c_1+\dots+c_n+p+c_{n+2}}} - \frac{l_0(c_1, c_2, \dots, c_n, p)}{(-s)^{c_1+\dots+c_n+p}} \\
 &= \begin{cases} \frac{1}{s^{c_1+\dots+c_n+p}} \left(\frac{c_{n+2}}{s^{c_{n+2}}} - \frac{s^2+1}{s(s^2-1)s^{c_{n+2}}} + \frac{s^2+1}{s(s^2-1)} \right) > 0 & \text{if } c_{n+2} \text{ is even,} \\ \frac{1}{s^{c_1+\dots+c_n+p}} \left(-\frac{c_{n+2}}{s^{c_{n+2}}} - \frac{2}{(s^2-1)s^{c_{n+2}}} + \frac{s^2+1}{s(s^2-1)} \right) > 0 & \text{if } c_{n+2} \text{ is odd.} \end{cases}
 \end{aligned}$$

Let $c_1 + c_2 + \dots + c_n + p$ be an odd number. Then,

$$\begin{aligned}
 & \sup \Delta_{c_1 c_1 \dots c_n p c_{n+2}}^{(-s,0)} - \sup \Delta_{c_1 c_1 \dots c_n p}^{(-s,0)} \\
 &= \frac{c_{n+2}}{(-s)^{c_1+\dots+c_n+p+c_{n+2}}} + \frac{l(c_1, c_2, \dots, c_n, p, c_{n+2})}{(-s)^{c_1+\dots+c_n+p+c_{n+2}}} - \frac{l(c_1, c_2, \dots, c_n, p)}{(-s)^{c_1+\dots+c_n+p}} \\
 &= \begin{cases} -\frac{1}{s^{c_1+\dots+c_n+p}} \left(\frac{c_{n+2}}{s^{c_{n+2}}} - \frac{s^2+1}{s(s^2-1)s^{c_{n+2}}} + \frac{s^2+1}{s(s^2-1)} \right) < 0 & \text{if } c_{n+2} \text{ is even,} \\ -\frac{1}{s^{c_1+\dots+c_n+p}} \left(-\frac{c_{n+2}}{s^{c_{n+2}}} - \frac{2}{(s^2-1)s^{c_{n+2}}} + \frac{s^2+1}{s(s^2-1)} \right) < 0 & \text{if } c_{n+2} \text{ is odd.} \end{cases}
 \end{aligned}$$

Also,

$$\begin{aligned}
 & \inf \Delta_{c_1 c_1 \dots c_n (p+1) c_{n+2}}^{(-s,0)} - \inf \Delta_{c_1 c_1 \dots c_n (p+1)}^{(-s,0)} \\
 &= \frac{c_{n+2}}{(-s)^{c_1 + \dots + c_n + p + 1 + c_{n+2}}} + \frac{l_0(c_1, c_2, \dots, c_n, p+1, c_{n+2})}{(-s)^{c_1 + \dots + c_n + p + 1 + c_{n+2}}} \\
 & \quad - \frac{l_0(c_1, c_2, \dots, c_n, p+1)}{(-s)^{c_1 + \dots + c_n + p + 1}} \\
 &= \begin{cases} \frac{1}{s^{c_1 + \dots + c_n + p}} \left(\frac{c_{n+2}}{s^{c_{n+2}}} - \frac{s^2 + 1}{s(s^2 - 1)s^{c_{n+2}}} + \frac{s^2 + 1}{s(s^2 - 1)} \right) > 0 & \text{if } c_{n+2} \text{ is even,} \\ \frac{1}{s^{c_1 + \dots + c_n + p}} \left(-\frac{c_{n+2}}{s^{c_{n+2}}} - \frac{2}{(s^2 - 1)s^{c_{n+2}}} + \frac{s^2 + 1}{s(s^2 - 1)} \right) > 0 & \text{if } c_{n+2} \text{ is odd.} \end{cases}
 \end{aligned}$$

Property 7 follows from *Property 6*.

Property 8. From properties of cylinders of $\mathbb{S}_{(-s,0)}$, it follows the following: if $x_0 \in \mathbb{S}_{(-s,0)}$, then

$$x_0 \in \Delta_{\alpha_1}^{(-s,0)} \cap \Delta_{\alpha_1 \alpha_2}^{(-s,0)} \cap \dots \cap \Delta_{\alpha_1 \alpha_2 \dots \alpha_n}^{(-s,0)} \cap \dots,$$

where

$$x_0 = \Delta_{\underbrace{0 \dots 0}_{\alpha_1 - 1}}^{-s} \alpha_1 \Delta_{\underbrace{0 \dots 0}_{\alpha_2 - 1}} \alpha_2 \dots \Delta_{\underbrace{0 \dots 0}_{\alpha_n - 1}} \alpha_n \dots$$

Also,

$$\begin{aligned}
 x_0 \in & \left[\inf \Delta_{\alpha_1}^{(-s,0)}; \sup \Delta_{\alpha_1}^{(-s,0)} \right] \cap \left[\inf \Delta_{\alpha_1 \alpha_2}^{(-s,0)}; \sup \Delta_{\alpha_1 \alpha_2}^{(-s,0)} \right] \cap \dots \\
 & \dots \cap \left[\inf \Delta_{\alpha_1 \alpha_2 \dots \alpha_n}^{(-s,0)}; \sup \Delta_{\alpha_1 \alpha_2 \dots \alpha_n}^{(-s,0)} \right] \cap \dots
 \end{aligned}$$

So, x_0 belongs to the following system of closed intervals:

$$\begin{aligned}
 & \left[\inf \Delta_{\alpha_1}^{(-s,0)}; \sup \Delta_{\alpha_1}^{(-s,0)} \right] \supset \left[\inf \Delta_{\alpha_1 \alpha_2}^{(-s,0)}; \sup \Delta_{\alpha_1 \alpha_2}^{(-s,0)} \right] \supset \dots \\
 & \dots \supset \left[\inf \Delta_{\alpha_1 \alpha_2 \dots \alpha_n}^{(-s,0)}; \sup \Delta_{\alpha_1 \alpha_2 \dots \alpha_n}^{(-s,0)} \right] \supset \dots
 \end{aligned}$$

Therefore,

$$x_0 = \bigcap_{n=1}^{\infty} \Delta_{c_1 c_2 \dots c_n}^{(-s,0)}.$$

□

LEMMA 3.10 ([30]). *Cylinders $\Delta_{c_1 c_2 \dots c_n}^-$ have the following properties:*

$$(1) \quad \Delta_{c_1 c_2 \dots c_n}^- \subset \begin{cases} \left[\sigma_{2k} + \frac{\inf S^-}{s^{c_1 + c_2 + \dots + c_{2k}}}, \sigma_{2k} + \frac{\sup S^-}{s^{c_1 + c_2 + \dots + c_{2k}}} \right] & \text{if } n = 2k, \\ \left[\sigma_{2k+1} - \frac{\sup S^-}{s^{c_1 + c_2 + \dots + c_{2k+1}}}, \sigma_{2k+1} - \frac{\inf S^-}{s^{c_1 + c_2 + \dots + c_{2k+1}}} \right] & \text{if } n = 2k + 1, \end{cases}$$

where $k \in \mathbb{N}$,

$$\sigma_n = \sum_{i=1}^n \frac{c_i}{s^{c_1 + c_2 + \dots + c_i}}, \quad \inf S^- = \frac{-s^{s-1} + s - 1}{s^s - 1}, \quad \sup S^- = \frac{-s^2 + s + 1}{s^s - 1}.$$

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$$(2) \quad d(\Delta_{c_1 c_2 \dots c_n}^-) = \frac{s^{s-1} - s^2 + 2}{(s^s - 1)s^{c_1 + c_2 + \dots + c_n}}.$$

$$(3) \quad \frac{\Delta_{c_1 c_2 \dots c_n c_{n+1}}^-}{\Delta_{c_1 c_2 \dots c_n}^-} = \frac{1}{s^{c_{n+1}}}.$$

$$(4) \quad \Delta_{c_1 c_2 \dots c_n c_{n+1}}^- \subset \Delta_{c_1 c_2 \dots c_n}^- \quad \forall c_n \in A_0, \quad n \in \mathbb{N}.$$

(5) *Cylinders* $\Delta_{c_1 c_2 \dots c_{n-1} 1}^-, \Delta_{c_1 c_2 \dots c_{n-1} 2}^-, \dots, \Delta_{c_1 c_2 \dots c_{n-1} [s-1]}^-$ are:

- *right-to-left situated whenever n is even, i.e.,*

$$\forall k \in \mathbb{N} : \sup \Delta_{c_1 c_2 \dots c_{2k-1} [c_{2k}+1]}^- < \inf \Delta_{c_1 c_2 \dots c_{2k-1} c_{2k}}^-;$$

- *left-to-right situated whenever n is odd, i.e.,*

$$\forall k \in \mathbb{N} : \sup \Delta_{c_1 c_2 \dots c_{2k} c_{2k+1}}^- < \inf \Delta_{c_1 c_2 \dots c_{2k} [c_{2k+1}+1]}^-.$$

Proof. The first, the second, and the third properties follow from the definition of a cylinder $\Delta_{c_1 c_2 \dots c_n}^-$.

Let us prove the fourth property.

1. Suppose $n = 2k, k \in \mathbb{N}$. Then, the equality $\inf \Delta_{c_1 c_2 \dots c_n c_{n+1}}^- \geq \inf \Delta_{c_1 c_2 \dots c_n}^-$ can be written in the form

$$\begin{aligned} \sum_{m=1}^{2k} \frac{(-1)^m c_m}{s^{c_1 + c_2 + \dots + c_m}} - \frac{c_{2k+1}}{s^{c_1 + c_2 + \dots + c_{2k+1}}} - \frac{\sup S^-}{s^{c_1 + c_2 + \dots + c_{2k+1}}} \\ \geq \sum_{m=1}^{2k} \frac{(-1)^m c_m}{s^{c_1 + c_2 + \dots + c_m}} + \frac{\inf S^-}{s^{c_1 + c_2 + \dots + c_{2k}}} \end{aligned}$$

or

$$-c_{2k+1} - \sup S^- \geq s^{c_{2k+1}} \inf S^-,$$

$$\frac{(s^2 - s - 1) + s^{c_{2k+1}}(s^{s-1} - s + 1) - c_{2k+1}(s^s - 1)}{s^s - 1} \geq 0.$$

It is easy to see that the last inequality is an equality under the condition

$$c_{2k+1} = 1.$$

In addition, for an even number n , let us consider the inequality

$$\sup \Delta_{c_1 c_2 \dots c_n c_{n+1}}^- \leq \sup \Delta_{c_1 c_2 \dots c_n}^-.$$

We have

$$\begin{aligned} \sum_{m=1}^{2k} \frac{(-1)^m c_m}{s^{c_1+c_2+\dots+c_m}} - \frac{c_{2k+1}}{s^{c_1+c_2+\dots+c_{2k+1}}} - \frac{\inf S^-}{s^{c_1+c_2+\dots+c_{2k+1}}} &\leq \\ &\sum_{m=1}^{2k} \frac{(-1)^m c_m}{s^{c_1+c_2+\dots+c_m}} + \frac{\sup S^-}{s^{c_1+c_2+\dots+c_{2k}}}, \\ -c_{2k+1} - \inf S^- &\leq s^{c_{2k+1}} \sup S^- \end{aligned}$$

or

$$(1 + c_{2k+1} + s^{2+c_{2k+1}} + s^{s-1}) - s - s^{1+c_{2k+1}} - c_{2k+1}s^s - s^{c_{2k+1}} \leq 0.$$

The last inequality is an equality when $c_{2k+1} = s - 1$ holds.

2. Suppose $n = 2k + 1, k \in \mathbb{N}$. Then,

$$\begin{aligned} \text{and} \quad \inf \Delta_{c_1 c_2 \dots c_n c_{n+1}}^- &\geq \inf \Delta_{c_1 c_2 \dots c_n}^- \\ \sum_{m=1}^{2k+1} \frac{(-1)^m c_m}{s^{c_1+c_2+\dots+c_m}} + \frac{c_{2k+2}}{s^{c_1+c_2+\dots+c_{2k+2}}} + \frac{\inf S^-}{s^{c_1+c_2+\dots+c_{2k+2}}} &\geq \\ &\sum_{m=1}^{2k+1} \frac{(-1)^m c_m}{s^{c_1+c_2+\dots+c_m}} - \frac{\sup S^-}{s^{c_1+c_2+\dots+c_{2k+1}}} \end{aligned}$$

are equivalent. Hence,

$$(s - 1 - c_{2k+2}) - s^{c_{2k+2}}(s^2 - s - 1) + s^{s-1}(s c_{2k+2} - 1) \geq 0.$$

If $c_{2k+2} = s - 1$, then the last inequality is an equality.

By analogy, for

$$\sup \Delta_{c_1 c_2 \dots c_n c_{n+1}}^- \leq \sup \Delta_{c_1 c_2 \dots c_n}^-,$$

we get

$$\begin{aligned} \sum_{m=1}^{2k+1} \frac{(-1)^m c_m}{s^{c_1+c_2+\dots+c_m}} + \frac{c_{2k+2}}{s^{c_1+c_2+\dots+c_{2k+2}}} + \frac{\sup S^-}{s^{c_1+c_2+\dots+c_{2k+2}}} &\leq \\ &\sum_{m=1}^{2k+1} \frac{(-1)^m c_m}{s^{c_1+c_2+\dots+c_m}} - \frac{\inf S^-}{s^{c_1+c_2+\dots+c_{2k+1}}}, \end{aligned}$$

$$\sup S^- + c_{2k+2}(s^s - 1) \leq -s^{c_{2k+2}} \inf S^-,$$

and

$$(s - s^2) + (1 - c_{2k+2}) + (s - 1)s^{c_{2k+2}} + s^{s-1}(s c_{2k+2} - s^{c_{2k+2}}) \leq 0.$$

It is true for all values of c_{2k+2} and $s > 2$, and is an equality when $c_{2k+2} = 1$.

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Let us prove *Property 5*.

•

$$\begin{aligned}
 & \forall k \in \mathbb{N} : \sup \Delta_{c_1 c_2 \dots c_{2k-1} [c_{2k}+1]}^- - \inf \Delta_{c_1 c_2 \dots c_{2k-1} c_{2k}}^- \\
 &= \frac{c_{2k} + 1}{s^{c_1+c_2+\dots+c_{2k}+1}} + \frac{\sup S^-}{s^{c_1+c_2+\dots+c_{2k}+1}} \\
 &\quad - \frac{c_{2k}}{s^{c_1+c_2+\dots+c_{2k}}} - \frac{\inf S^-}{s^{c_1+c_2+\dots+c_{2k}}} \\
 &= \frac{1}{s^{c_1+c_2+\dots+c_{2k}}} \left(\frac{1-s}{s} c_{2k} + 2 \frac{s^s - s^2 + s}{s(s^s - 1)} \right) \\
 &= \frac{s^s(2 + c_{2k} - s c_{2k}) + s(c_{2k} + 2 - 2s) - c_{2k}}{s(s^s - 1) s^{c_1+c_2+\dots+c_{2k}}} < 0.
 \end{aligned}$$

•

$$\begin{aligned}
 & \forall k \in \mathbb{N} : \sup \Delta_{c_1 c_2 \dots c_{2k} c_{2k+1}}^- - \inf \Delta_{c_1 c_2 \dots c_{2k} [c_{2k+1}+1]}^- \\
 &= \frac{c_{2k+1}}{s^{c_1+c_2+\dots+c_{2k}+c_{2k+1}}} - \frac{\inf S^-}{s^{c_1+c_2+\dots+c_{2k}+c_{2k+1}}} \\
 &\quad - \frac{1 + c_{2k+1}}{s^{1+c_1+c_2+\dots+c_{2k}+c_{2k+1}}} + \frac{\sup S^-}{s^{1+c_1+c_2+\dots+c_{2k}+c_{2k+1}}} \\
 &= \frac{1}{s^{1+c_1+c_2+\dots+c_{2k}+c_{2k+1}}} \left(\frac{s^s - 2s^2 + 2s + 1}{s^s - 1} - \frac{s^{s+1} c_{2k+1} - s c_{2k+1}}{s^s - 1} \right. \\
 &\quad \left. + \frac{s^s c_{2k+1} - c_{2k+1}}{s^s - 1} + \frac{s^s - 1}{s^s - 1} \right) \\
 &= \frac{s^s(2 + c_{2k+1} - s c_{2k+1}) + s(2 - 2s - c_{2k+1}) - c_{2k+1}}{(s^s - 1) s^{1+c_1+c_2+\dots+c_{2k}+c_{2k+1}}} < 0. \quad \square
 \end{aligned}$$

It follows from the last-mentioned lemma that the following statements are true.

COROLLARY 3.11. *For all $c_n \in \{1, 2, \dots, s-2\}$ the condition*

$$\Delta_{c_1 c_2 \dots c_{n-1} c_n}^- \cap \Delta_{c_1 c_2 \dots c_{n-1} [c_n+1]}^- = \emptyset$$

holds.

COROLLARY 3.12. *Intervals of the form*

$$\left(\sup \Delta_{c_1 c_2 \dots c_{2k-1}}^-, \inf \Delta_{c_1 c_2 \dots c_{2k-2} [c_{2k-1}+1]}^- \right)$$

and

$$\left(\sup \Delta_{c_1 c_2 \dots c_{2k-1} [c_{2k}+1]}^-, \inf \Delta_{c_1 c_2 \dots c_{2k-1} c_{2k}}^- \right),$$

where $k \in \mathbb{N}$, have the empty intersection with the set S^- .

COROLLARY 3.13. *For an arbitrary $x_0 \in S^-$ the following condition holds:*

$$x_0 = \bigcap_{n=1}^{\infty} \Delta_{c_1 c_2 \dots c_n}^-.$$

Let u be a fixed positive integer from A .

By $\mathbb{S}_{(-s,u)}$ denote the set (a subset of the segment $[-\frac{s}{s+1}, \frac{1}{s+1}]$) of all numbers x represented by the nega- s -adic expansion such that they are of the form

$$x = \sum_{n=1}^{\infty} \left(\frac{\alpha_n - u}{(-s)^{\alpha_1 + \alpha_2 + \dots + \alpha_n}} \right) - \frac{u}{s+1}, \quad \text{where } (\alpha_n) \in L.$$

This set is the following set

$$\mathbb{S}_{(-s,u)} = \left\{ x : x = \Delta_{\underbrace{u \dots u}_{\alpha_1 - 1}}^{-s} \alpha_1 \underbrace{u \dots u}_{\alpha_2 - 1} \alpha_2 \dots \underbrace{u \dots u}_{\alpha_n - 1} \alpha_n \dots \right\},$$

where $(\alpha_n) \in L$, $u \neq \alpha_n$ for all $n \in \mathbb{N}$, and u is a fixed number.

It was shown in [30] that the following statement is true.

THEOREM 3.14. *Let $\{\sigma_1, \sigma_2, \dots, \sigma_m\}$ be a fixed finite set of combinations (tuples) of nega- s -adic digits, E be a set whose elements have in own nega- s -adic representation only combinations of digits from the set $\{\sigma_1, \sigma_2, \dots, \sigma_m\}$. Then, the Hausdorff-Besicovitch dimension $\alpha_0(E)$ of the set E satisfies the equation*

$$N(\sigma_m^1) \left(\frac{1}{s} \right)^{\alpha_0} + N(\sigma_m^2) \left(\frac{1}{s} \right)^{2\alpha_0} + \dots + N(\sigma_m^k) \left(\frac{1}{s} \right)^{k\alpha_0} = 1,$$

where $N(\sigma_m^k)$ is a number of k -digit combinations from $\{\sigma_1, \sigma_2, \dots, \sigma_m\}$, $k \in \mathbb{N}$, and $N(\sigma_m^1) + N(\sigma_m^2) + \dots + N(\sigma_m^k) = m$.

Proof. Let $\{\sigma_1, \sigma_2, \dots, \sigma_m\}$ be a set of fixed combinations of nega- s -adic digits, and let the nega- s -adic representation of any number from E (E is a Cantor-like set) contain only such combinations of digits. There exist digit combinations $e_1 e_2 \dots e_r, \iota_1 \iota_2 \dots \iota_t$, where $r, t \in \mathbb{N}$ (they can be represented as one or several combinations from $\{\sigma_1, \sigma_2, \dots, \sigma_m\}$) such that

$$\inf E = \Delta_{(e_1 e_2 \dots e_r)(e_1 e_2 \dots e_r) \dots}^{-s} \quad \text{and} \quad \sup E = \Delta_{(\iota_1 \iota_2 \dots \iota_t)(\iota_1 \iota_2 \dots \iota_t) \dots}^{-s}.$$

Also, here,

$$d(E) = \sup E - \inf E, \quad \text{where } d(\cdot) \text{ is the diameter of the set.}$$

A cylinder $\Delta_{\tau_1 \tau_2 \dots \tau_n}^{(-s,E)}$ of rank n with the base $\tau_1 \tau_2 \dots \tau_n$ is a set formed by all the numbers of E with nega- s -adic representations in which the first n combinations of digits are fixed and are from $\{\sigma_1, \sigma_2, \dots, \sigma_m\}$. It is easy to see that

$$d\left(\Delta_{\tau_1 \tau_2 \dots \tau_n}^{(-s,E)}\right) = \frac{d(E)}{s^{N(\tau_1 + \tau_2 + \dots + \tau_n)}},$$

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where $N(\tau_1 + \tau_2 + \dots + \tau_n)$ is the number of digits in the combination $\tau_1 \tau_2 \dots \tau_n$.

Since

$$E = C[-s, \{\sigma_1, \sigma_2, \dots, \sigma_m\}], \quad E \subset [\inf E; \sup E], \quad \text{and}$$

$$\frac{\Delta_{\tau_1 \tau_2 \dots \tau_n \tau_{n+1}}^{(-s, E)}}{\Delta_{\tau_1 \tau_2 \dots \tau_n}^{(-s, E)}} = \frac{1}{s^{N(\tau_{n+1})}},$$

we have

$$E = [I_{\tau_1} \cap E] \cup [I_{\tau_2} \cap E] \cap \dots \cap [I_{\tau_m} \cap E],$$

where

$$I_{\tau_i} = \left[\inf \Delta_{\tau_i}^{(-s, E)}; \sup \Delta_{\tau_i}^{(-s, E)} \right], \quad i = \overline{1, m}.$$

So,

$$\left[I_{\tau_1} \cap E \right] \overset{s^{-1}}{\sim} E, \left[I_{\tau_2} \cap E \right] \overset{s^{-1}}{\sim} E, \dots, \left[I_{\tau_{n_1}} \cap E \right] \overset{s^{-1}}{\sim} E;$$

$$\left[I_{\tau_1^2} \cap E \right] \overset{s^{-2}}{\sim} E, \left[I_{\tau_2^2} \cap E \right] \overset{s^{-2}}{\sim} E, \dots, \left[I_{\tau_{n_2}^2} \cap E \right] \overset{s^{-2}}{\sim} E;$$

.....

$$\left[I_{\tau_1^k} \cap E \right] \overset{s^{-k}}{\sim} E, \left[I_{\tau_2^k} \cap E \right] \overset{s^{-k}}{\sim} E, \dots, \left[I_{\tau_{n_k}^k} \cap E \right] \overset{s^{-k}}{\sim} E,$$

where τ_j^k is some k -digit combination from $\{\sigma_1, \sigma_2, \dots, \sigma_m\}$ ($j = \overline{1, n_k}$), and n_k is the number of k -digit combinations from $\{\sigma_1, \sigma_2, \dots, \sigma_m\}$.

Hence, the set E is a self-similar fractal whose Hausdorff dimension satisfies the equation

$$N(\sigma_m^1) \left(\frac{1}{s}\right)^{\alpha_0} + N(\sigma_m^2) \left(\frac{1}{s}\right)^{2\alpha_0} + \dots + N(\sigma_m^k) \left(\frac{1}{s}\right)^{k\alpha_0} = 1. \quad \square$$

The following statements follow from the last-mentioned theorem.

THEOREM 3.15 ([30]). *The set $\mathbb{S}_{(-s, u)}$ is:*

- an uncountable, perfect, nowhere dense sets of zero Lebesgue measure;
- a self-similar fractal, and its Hausdorff-Besicovitch dimension $\alpha_0(\mathbb{S}_{(-s, u)})$, satisfies the equation

$$\sum_{i \in A_u} \left(\frac{1}{s}\right)^{i\alpha_0} = 1, \quad \text{where } A_u = \{1, 2, \dots, s-1\} \setminus \{u\}.$$

Let us consider fractal sets whose elements are represented by nega-s-adic Cantor series.

THEOREM 3.16 ([30]). *Let $s > 1$ be a fixed positive integer, $\alpha_{m_1+m_2+\dots+m_n} \neq 0$ for all $n \in \mathbb{N}$, and $m_n \in \{3, 5, 7, \dots, 2i+1, \dots\}$. Then, the set $M_{(-D,s)}$*

$$M_{(-D,s)} = \left\{ x : x = \Delta^{-s} \underbrace{0 \dots 0}_{m_1-1} \alpha_{m_1} \underbrace{0 \dots 0}_{m_2-1} \alpha_{m_1+m_2} \dots \underbrace{0 \dots 0}_{m_n-1} \alpha_{m_1+m_2+\dots+m_n} \dots \right\}$$

is a self-similar fractal whose Hausdorff-Besicovitch dimension $\alpha_0(M_{(-D,s)})$ is equal to

$$\log_s \left(\sqrt[3]{\frac{s-1}{2} + \frac{1}{6} \sqrt{\frac{27(s-1)^2-4}{3}}} + \sqrt[3]{\frac{s-1}{2} - \frac{1}{6} \sqrt{\frac{27(s-1)^2-4}{3}}} \right).$$

Proof. From (6) and Theorem 3.14, it follows that the Hausdorff-Besicovitch dimension of the set $M_{(-D,s)}$ under

$$m_n \in \{3, 5, 7, \dots, 2i+1, \dots\}, \quad \alpha_{m_1+m_2+\dots+m_n} \neq 0,$$

and under fixed $s > 1$, satisfies the equation

$$(s-1) \left(\frac{1}{s}\right)^{3\alpha_0} + (s-1) \left(\frac{1}{s}\right)^{5\alpha_0} + (s-1) \left(\frac{1}{s}\right)^{7\alpha_0} + \dots + (s-1) \left(\frac{1}{s}\right)^{(2i+1)\alpha_0} + \dots = 1, \quad i = 1, 2, \dots$$

The last equation is equivalent to the equation

$$s^{3\alpha_0} - s^{\alpha_0} - (s-1) = 0.$$

Using Cardano's formula, we get the result. \square

COROLLARY 3.17. *If a sequence (m_n) of odd positive integers is a fixed purely periodic sequence with the period $(m_1 m_2 \dots m_t)$, then the set $M'_{(-D,s,t)}$ of all numbers represented by nega-s-adic Cantor series (6) with the corresponding sequence (m_n) is a self-similar fractal and*

$$\alpha_0(M'_{(-D,s,t)}) = \frac{t}{m_1 + m_2 + \dots + m_t}.$$

Proof. Since elements of this set have periodic nega-s-adic representation, i.e.,

$$M'_{(-D,s,t)} \ni x = \Delta^{-s} \left(\underbrace{0 \dots 0}_{m_1-1} \alpha_{m_1} \underbrace{0 \dots 0}_{m_2-1} \alpha_{m_1+m_2} \dots \underbrace{0 \dots 0}_{m_t-1} \alpha_{m_1+m_2+\dots+m_t} \right),$$

where $\{m_1, m_2, \dots, m_t\}$ is a fixed set of odd numbers and $\alpha_{m_1}, \alpha_{m_1+m_2}, \dots, \alpha_{m_1+\dots+m_t}$ are numbers from the set A , from Theorem 3.14, it follows that

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Hausdorff-Besicovitch dimension satisfies the equation

$$s^t \left(\frac{1}{s} \right)^{(m_1+m_2+\dots+m_t)\alpha_0} = 1.$$

The statement follows from the last equation. □

Finally, let us remark that restrictions on using elements of sets $\mathbb{S}_{(\pm s, u)}$ are new (they occur for the first time).

So, we considered topological, metric, and fractal properties of certain sets whose elements have restrictions on using digits in own expansions. For considered sets, the case of functional restrictions is equivalent to the case of restrictions on using combinations of digits. The simple methods for the calculation of the Hausdorff-Besicovitch dimension of such sets are described. In the case of the s -adic or nega- s -adic representations, the Hausdorff-Besicovitch dimension of a set whose elements have in own representations only combinations of digits from some fixed set of combinations of digits, depends on parameters as a number of k -digit combinations and numbers k . In addition, note that the considered sets have the Moran structure. Similar investigations did not study the case of generalizations of the s -adic or nega- s -adic representation. These investigations will be discussed by the author of the present article in a further paper.

4. Properties of images

In this section, the main attention is given to images of sets $\mathbb{S}_{(s, u)}$ and $\mathbb{S}_{(-s, u)}$ under the Salem type functions (see [34, 35, 38], the Salem function was introduced in [21]).

Let $s > 1$ be a fixed positive integer and $\alpha_n \in A = \{0, 1, \dots, s-1\}$. Let

$$P = \{p_0, p_1, \dots, p_{s-1}\}$$

be a fixed set whose elements satisfy the following properties

$$p_0 + p_1 + \dots + p_{s-1} = 1 \quad \text{and} \quad p_i > 0 \quad \text{for all } i = \overline{0, s-1}.$$

Then, let us consider the following distribution functions.

Let ζ be a random variable defined by the s -adic representation

$$\zeta = \frac{\iota_1}{s} + \frac{\iota_2}{s^2} \frac{\iota_3}{s^3} + \dots + \frac{\iota_k}{s^k} + \dots = \Delta_{\iota_1 \iota_2 \dots \iota_k \dots}^s,$$

where digits ι_k ($k = 1, 2, 3, \dots$) are random and taking the values $0, 1, \dots, s-1$ with positive probabilities p_0, p_1, \dots, p_{s-1} . That is, ι_k are independent and $P\{\iota_k = \alpha_k\} = p_{\alpha_k}$, $\alpha_k \in A$.

Let ζ be a random variable defined by the s -adic representation

$$\zeta = \Delta_{\pi_1 \pi_2 \dots \pi_k \dots}^s = \sum_{k=1}^{\infty} \frac{\pi_k}{s^k},$$

where

$$\pi_k = \begin{cases} \alpha_k & \text{if } k \text{ is odd,} \\ s-1-\alpha_k & \text{if } k \text{ is even,} \end{cases}$$

and digits π_k ($k = 1, 2, 3, \dots$) are random and taking the values $0, 1, \dots, s-1$ with positive probabilities p_0, p_1, \dots, p_{s-1} . That is, π_k are independent and $P\{\pi_k = \alpha_k\} = p_{\alpha_k}$, $P\{\pi_k = s-1-\alpha_k\} = p_{s-1-\alpha_k}$, where $\alpha_k \in A$.

Let us consider the distribution function f_{ζ} of the random variable ζ and the distribution function \tilde{F}_{ζ} of the random variable ζ :

$$f_{\zeta}(x) = \begin{cases} 0 & \text{whenever } x < 0, \\ \beta_{\alpha_1(x)} + \sum_{k=2}^{\infty} \left(\beta_{\alpha_k(x)} \prod_{j=1}^{k-1} p_{\alpha_j(x)} \right) & \text{whenever } 0 \leq x < 1, \\ 1 & \text{whenever } x \geq 1, \end{cases}$$

where $p_{\alpha_j(x)} > 0$ and

$$\beta_{\alpha_k} = \begin{cases} \sum_{i=0}^{\alpha_k(x)-1} p_i(x) & \text{whenever } \alpha_k(x) > 0 \\ 0 & \text{whenever } \alpha_k(x) = 0, \end{cases}$$

also,

$$\tilde{F}_{\zeta}(x) = \begin{cases} 0 & \text{whenever } x < 0, \\ \tilde{\beta}_{\alpha_1(x)} + \sum_{k=2}^{\infty} \left(\tilde{\beta}_{\alpha_k(x)} \prod_{j=1}^{k-1} \tilde{p}_{\alpha_j(x)} \right) & \text{whenever } 0 \leq x < 1, \\ 1 & \text{whenever } x \geq 1, \end{cases}$$

where $p_{\alpha_j(x)} > 0$,

$$\begin{aligned} x = \Delta_{\tilde{\alpha}_1 \tilde{\alpha}_2 \dots \tilde{\alpha}_k \dots}^s &= \frac{1}{s+1} - \Delta_{\alpha_1 \alpha_2 \dots \alpha_k \dots}^{-s} \\ &\equiv \frac{1}{s+1} - \sum_{k=1}^{\infty} \frac{(-1)^k \alpha_k}{s^k} = \sum_{k=1}^{\infty} \frac{\alpha_{2k-1}}{s^{2k-1}} + \sum_{k=1}^{\infty} \frac{s-1-\alpha_{2k}}{s^{2k}}, \end{aligned}$$

and

$$\tilde{p}_{\alpha_k} = \begin{cases} p_{\alpha_k} & \text{if } k \text{ is odd,} \\ p_{s-1-\alpha_k} & \text{if } k \text{ is even,} \end{cases}$$

$$\tilde{\beta}_{\alpha_k} = \begin{cases} \beta_{\alpha_k} & \text{if } k \text{ is odd,} \\ \beta_{s-1-\alpha_k} & \text{if } k \text{ is even.} \end{cases}$$

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One can note that the function

$$\tilde{F}(x) = \beta_{\alpha_1}(x) + \sum_{n=2}^{\infty} \left(\tilde{\beta}_{\alpha_n}(x) \prod_{j=1}^{n-1} \tilde{p}_{\alpha_j}(x) \right)$$

is a partial case of the function investigated in [38].

Let $x \in \mathbb{S}_{(s,u)}$. Let us consider the properties of the following images of $\mathbb{S}_{(-s,u)}$ and $\mathbb{S}_{(s,u)}$:

$$\mathbb{S}_{(P,u)} = \{y : y = f_{\xi}(x), x \in \mathbb{S}_{(s,u)}\}$$

and

$$\mathbb{S}_{(-P,u)} = \left\{ \tilde{y} : \tilde{y} = \tilde{F} \circ f_l \circ f_+(x), x \in \mathbb{S}_{(s,u)} \right\} = \{z : z = \tilde{F} \circ f_l(x), x \in \mathbb{S}_{(-s,u)}\}.$$

Here,

$$\tilde{y} = \tilde{F} \circ f_l \circ f_+(x),$$

where

$$f_+ : x = \Delta_{\alpha_1 \alpha_2 \dots \alpha_n}^s \rightarrow \Delta_{\alpha_1 \alpha_2 \dots \alpha_n}^{-s} = y$$

is not monotonic on the domain and is a nowhere differentiable function ([33]), $f_l(y) = \frac{1}{s+1} - y$, and \tilde{F} is the last-mentioned distribution function.

Let us describe properties of the set $\mathbb{S}_{(P,u)}$.

THEOREM 4.1 ([39]). *The set $\mathbb{S}_{(P,u)}$ is an uncountable, perfect, and nowhere dense set of zero Lebesgue measure and also is a self-similar fractal whose Hausdorff dimension $\alpha_0(\mathbb{S}_{(P,u)})$ satisfies the following equation*

$$\sum_{i \in A_0 \setminus \{u\}} (p_i p_u^{i-1})^{\alpha_0} = 1.$$

Let c_1, c_2, \dots, c_n be an ordered tuple of integers such that $c_i \in \{0, 1, \dots, s-1\}$ for $i = \overline{1, n}$.

DEFINITION 4.2. *A cylinder of rank n with base $c_1 c_2 \dots c_n$ is a set $\Delta_{c_1 c_2 \dots c_n}^{(P,u)}$ of the form*

$$\Delta_{c_1 c_2 \dots c_n}^{(P,u)} = \left\{ x : x = \Delta_{\underbrace{u \dots u}_{c_1-1} \underbrace{u \dots u}_{c_2-1} \dots \underbrace{u \dots u}_{c_n-1} \underbrace{u \dots u}_{\alpha_{n+1}-1} \dots \underbrace{u \dots u}_{\alpha_{n+2}-1} \dots}^P, \right. \\ \left. \alpha_j = c_j, j = \overline{1, n} \right\}.$$

By $(a_1 a_2 \dots a_k)$ denote the period $a_1 a_2 \dots a_k$ in the representation of a periodic number.

The following lemma describes local properties of the set $\mathbb{S}_{(P,u)}$.

LEMMA 4.3 ([39]). *Cylinders $\Delta_{c_1 \dots c_n}^{(P,u)}$ have the following properties:*

(1)

$$\inf \Delta_{c_1 \dots c_n}^{(P,u)} = \begin{cases} \Delta_{\underbrace{0 \dots 0}_{c_1-1} \underbrace{0 \dots 0}_{c_2-1} \dots \underbrace{0 \dots 0}_{c_n-1} \underbrace{0 \dots 0}_{s-2} [s-1]}^P & \text{if } u = 0, \\ \Delta_{\underbrace{1 \dots 1}_{c_1-1} \underbrace{1 \dots 1}_{c_2-1} \dots \underbrace{1 \dots 1}_{c_n-1} \underbrace{1 \dots 1}_{s-2} [s-1]}^P & \text{if } u = 1, \\ \Delta_{\underbrace{u \dots u}_{c_1-1} \underbrace{u \dots u}_{c_2-1} \dots \underbrace{u \dots u}_{c_n-1} (1)}^P & \text{if } u \in \{2, 3, \dots, s-1\}, \end{cases}$$

$$\sup \Delta_{c_1 \dots c_n}^{(P,u)} = \begin{cases} \Delta_{\underbrace{[s-1] \dots [s-1]}_{c_1-1} \underbrace{[s-1] \dots [s-1]}_{c_n-1} \underbrace{([s-1] \dots [s-1])}_{s-3} [s-2]}^P & \text{if } u = s-1, \\ \Delta_{\underbrace{u \dots u}_{c_1-1} \underbrace{u \dots u}_{c_2-1} \dots \underbrace{u \dots u}_{c_n-1} \underbrace{(u \dots u)}_u [u+1]}^P & \text{if } u \in \{1, \dots, s-2\}, \\ \Delta_{\underbrace{0 \dots 0}_{c_1-1} \underbrace{0 \dots 0}_{c_2-1} \dots \underbrace{0 \dots 0}_{c_n-1} (1)}^P & \text{if } u = 0. \end{cases}$$

(2) *If $d(\cdot)$ is the diameter of a set, then*

$$d(\Delta_{c_1 \dots c_n}^{(P,u)}) = d(\mathbb{S}_{(P_s, u)}) p_u^{c_1 + c_2 + \dots + c_n - n} \prod_{j=1}^n p_{c_j}.$$

$$(3) \quad \frac{d(\Delta_{c_1 \dots c_n c_{n+1}}^{(P,u)})}{d(\Delta_{c_1 \dots c_n}^{(P,u)})} = p_{c_{n+1}} p_u^{c_{n+1} - 1}.$$

$$(4) \quad \Delta_{c_1 c_2 \dots c_n}^{(P,u)} = \bigcup_{i=1}^{s-1} \Delta_{c_1 c_2 \dots c_n i}^{(P,u)} \quad \forall c_n \in A_0, \quad n \in \mathbb{N}, \quad i \neq u.$$

(5) *The following relationships are satisfied:*

(a) *if $u \in \{0, 1\}$, then*

$$\inf \Delta_{c_1 \dots c_n p}^{(P,u)} > \sup \Delta_{c_1 \dots c_n [p+1]}^{(P,u)};$$

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(b) if $u \in \{2, 3, \dots, s-3\}$, then

$$\begin{cases} \sup \Delta_{c_1 \dots c_n p}^{(P,u)} < \inf \Delta_{c_1 \dots c_n [p+1]}^{(P,u)} & \text{for all } p+1 \leq u, \\ \inf \Delta_{c_1 \dots c_n p}^{(P,u)} > \sup \Delta_{c_1 \dots c_n [p+1]}^{(P,u)}, & \text{for all } u < p; \end{cases}$$

(c) if $u \in \{s-2, s-1\}$, then

$$\sup \Delta_{c_1 \dots c_n p}^{(P,u)} < \inf \Delta_{c_1 \dots c_n [p+1]}^{(P,u)} \quad (\text{in this case, the condition } p \neq s-1 \text{ holds}).$$

We considered the properties of $\mathbb{S}_{(s,u)}$ and its image $\mathbb{S}_{(P,u)}$ under the Salem function. So, the Salem function preserves the self-similarity, but, in the general case, it does not preserve the Hausdorff dimension. This map also preserves the structure of $\mathbb{S}_{(s,u)}$, but the numerical values change.

Finally, one can note the following theorem.

THEOREM 4.4 ([39]). *Let \mathbb{S} be a set whose elements are represented in terms of the s -adic representation by a finite number of fixed combinations $\tau_1, \tau_2, \dots, \tau_m$ of digits from the alphabet A .*

Let E be an image of the set \mathbb{S} under the Salem function f_ξ . Then, the Hausdorff dimension α_0 of E satisfies the following equation:

$$\sum_{j=1}^m \left(\prod_{i=0}^{s-1} p_i^{N_i(\tau_j)} \right)^{\alpha_0} = 1,$$

where

$N_i(\tau_k)$ ($k = \overline{1, m}$) is a number of the digit i in τ_k from the set $\{\tau_1, \tau_2, \dots, \tau_m\}$.

Now, we describe the properties of $\mathbb{S}_{(-P,u)}$.

Suppose $d(\cdot)$ is the diameter of a set and a cylinder $\Delta_{c_1 c_2 \dots c_n}^{(-P,u)}$ is a set whose elements are elements of $\mathbb{S}_{(-P,u)}$ and for these elements the condition $\alpha_i = c_i$ holds for all $i = \overline{1, n}$ (here c_1, c_2, \dots, c_n is a fixed tuple).

THEOREM 4.5 ([40]). *An arbitrary set $\mathbb{S}_{(-P,u)}$ is an uncountable, perfect, and nowhere dense set of zero Lebesgue measure.*

THEOREM 4.6 ([40]). *In the general case, the set $\mathbb{S}_{(-P,u)}$ is not a self-similar fractal, the Hausdorff dimension $\alpha_0(\mathbb{S}_{(-P,u)})$ of which can be calculated by the formula*

$$\alpha_0 = \liminf_{k \rightarrow \infty} \alpha_k,$$

where (α_k) is a sequence of numbers satisfying the equation

$$\left(\sum_{\substack{c_1 \text{ is odd} \\ c_1 \in \overline{A}}} (\omega_{2,c_1})^{\alpha_1} + \sum_{\substack{c_1 \text{ is even} \\ c_1 \in \overline{A}}} (\omega_{4,c_1})^{\alpha_1} \right) \times \\ \prod_{i=2}^k \left(\sum_{\substack{c_i \text{ is odd} \\ c_i \in \overline{A}}} N_{1,c_i} (\omega_{1,c_i})^{\alpha_i} + \sum_{\substack{c_i \text{ is odd} \\ c_i \in \overline{A}}} N_{2,c_i} (\omega_{2,c_i})^{\alpha_i} + \right. \\ \left. \sum_{\substack{c_i \text{ is even} \\ c_i \in \overline{A}}} N_{3,c_i} (\omega_{3,c_i})^{\alpha_i} + \sum_{\substack{c_i \text{ is even} \\ c_i \in \overline{A}}} N_{4,c_i} (\omega_{4,c_i})^{\alpha_i} \right) = 1.$$

Here, N_{j,c_i} ($j = \overline{1,4}$, $1 < i \in \mathbb{N}$) is the number of cylinders $\Delta_{c_1 c_2 \dots c_i}^{(-P,u)}$ for which

$$\frac{d \left(\Delta_{c_1 c_2 \dots c_{i-1} c_i}^{(-P,u)} \right)}{d \left(\Delta_{c_1 c_2 \dots c_{i-1}}^{(-P,u)} \right)} = \omega_{j,c_i}.$$

Also,

$$\begin{aligned} \omega_{1,c_i} &= \underbrace{p_{s-1-u} p_u \dots p_{s-1-u} p_u}_{c_i-1} p_{s-1-c_i} \frac{d(\overline{\mathbb{S}(P,u)})}{d(\mathbb{S}(P,u))} && \text{for an odd number } c_i, \\ \omega_{2,c_i} &= \underbrace{p_u p_{s-1-u} \dots p_u p_{s-1-u}}_{c_i-1} p_{c_i} \frac{d(\overline{\mathbb{S}(P,u)})}{d(\mathbb{S}(P,u))} && \text{for an odd number } c_i, \\ \omega_{3,c_i} &= \underbrace{p_{s-1-u} p_u \dots p_{s-1-u} p_u p_{s-1-u}}_{c_i-1} p_{c_i} && \text{for an even number } c_i, \\ \omega_{4,c_i} &= \underbrace{p_u p_{s-1-u} \dots p_u p_{s-1-u} p_u}_{c_i-1} p_{s-1-c_i} && \text{for an even number } c_i. \end{aligned}$$

In addition,

$$N_{1,c_i} + N_{2,c_i} = l(m+l)^{i-1} \quad \text{and} \quad N_{3,c_i} + N_{4,c_i} = m(m+l)^{i-1},$$

where l is the number of odd numbers in the set $\overline{A} = A \setminus \{0, u\}$ and m is the number of even numbers in \overline{A} .

Auxiliary values can be calculated from the following lemma.

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LEMMA 4.7 ([40]). *For the sets $\underline{\mathbb{S}}_{(P,u)}$ and $\overline{\mathbb{S}}_{(P,u)}$, the following equalities hold:*

$$\inf \underline{\mathbb{S}}_{(P,u)} = \begin{cases} \Delta_{[s-2](0[s-3])}^P & \text{if } u = 0, \\ \Delta_{[s-2]1[s-4](1[s-3])}^P & \text{if } u = 1, \\ \Delta_{([s-1-u]2)}^P & \text{if } u \in \{2, 3, \dots, s-1\}, \end{cases}$$

and

$$\sup \underline{\mathbb{S}}_{(P,u)} = \begin{cases} \Delta_{([s-1-u]2)}^P & \text{if } u \in \{0, 1\}, \\ \Delta_{[s-2](u[s-3])}^P & \text{if } u \in \{2, 3, \dots, s-1\}, \end{cases}$$

$$\inf \overline{\mathbb{S}}_{(P,u)} = \begin{cases} \Delta_{(u[s-3])}^P & \text{if } u \in \{0, 1\}, \\ \Delta_{1([s-1-u]2)}^P & \text{if } u \in \{2, 3, \dots, s-1\}, \end{cases}$$

and

$$\sup \overline{\mathbb{S}}_{(P,u)} = \begin{cases} \Delta_{1([s-1]2)}^P & \text{if } u = 0, \\ \Delta_{1[s-2]3([s-2]2)}^P & \text{if } u = 1, \\ \Delta_{(u[s-3])}^P & \text{if } u \in \{2, 3, \dots, s-1\}. \end{cases}$$

Finally, let us consider the local properties of $\mathbb{S}_{(-P,u)}$.

Assume

$$\tilde{\alpha}_n = \begin{cases} \alpha_n & \text{whenever } n \text{ is odd,} \\ s-1-\alpha_n & \text{whenever } n \text{ is even} \end{cases}$$

and

$$\tilde{u} = \begin{cases} u & \text{whenever } u \text{ is situated at an odd position in the representation,} \\ s-1-u & \text{whenever } u \text{ is situated at an even position in the representation.} \end{cases}$$

LEMMA 4.8. *Cylinders $\Delta_{c_1 \dots c_n}^{(-P,u)}$ have the following properties:*

(1)

$$\inf \Delta_{c_1 \dots c_n}^{(-P,u)} = \begin{cases} \tau_n + \left(\prod_{j=1}^n \tilde{p}_{c_j, c_1 + \dots + c_j} \right) \left(\prod_{\substack{i=1, c_1 + \dots + c_{n-1} \\ i \notin C_{n-1}}} \tilde{p}_{u, i} \right) \inf \overline{\mathbb{S}}_{(P,u)} & \text{if } c_1 + \dots + c_n \text{ is even,} \\ \tau_n + \left(\prod_{j=1}^n \tilde{p}_{c_j, c_1 + \dots + c_j} \right) \left(\prod_{\substack{i=1, c_1 + \dots + c_{n-1} \\ i \notin C_{n-1}}} \tilde{p}_{u, i} \right) \inf \underline{\mathbb{S}}_{(P,u)} & \text{if } c_1 + \dots + c_n \text{ is odd,} \end{cases}$$

$$\sup \Delta_{c_1 \dots c_n}^{(-P, u)} = \begin{cases} \tau_n + \left(\prod_{j=1}^n \tilde{p}_{c_j, c_1 + \dots + c_j} \right) \left(\prod_{\substack{i=1, c_1 + \dots + c_n - 1 \\ i \notin C_{n-1}}} \tilde{p}_{u, i} \right) \sup \overline{\mathbb{S}}_{(P, u)} \\ \text{if } c_1 + \dots + c_n \text{ is even,} \\ \tau_n + \left(\prod_{j=1}^n \tilde{p}_{c_j, c_1 + \dots + c_j} \right) \left(\prod_{\substack{i=1, c_1 + \dots + c_n - 1 \\ i \notin C_{n-1}}} \tilde{p}_{u, i} \right) \sup \underline{\mathbb{S}}_{(P, u)} \\ \text{if } c_1 + \dots + c_n \text{ is odd,} \end{cases}$$

where

$$\tau_n = \Delta_{\underbrace{\tilde{u} \dots \tilde{u}}_{c_1 - 1} \underbrace{\tilde{c}_1 \tilde{u} \dots \tilde{u}}_{c_2 - 1} \dots \underbrace{\tilde{c}_n \tilde{u} \dots \tilde{u}}_{c_n - 1} (0).$$

(2) If $d(\cdot)$ is the diameter of a set, then

$$d\left(\Delta_{c_1 \dots c_n}^{(-P, u)}\right) = \begin{cases} \left(\prod_{j=1}^n \tilde{p}_{c_j, c_1 + \dots + c_j} \right) \left(\prod_{\substack{i=1, c_1 + \dots + c_n - 1 \\ i \notin C_{n-1}}} \tilde{p}_{u, i} \right) d\left(\overline{\mathbb{S}}_{(P, u)}\right) \\ \text{if } c_1 + \dots + c_n \text{ is even } d\left(\overline{\mathbb{S}}_{(P, u)}\right), \\ \left(\prod_{j=1}^n \tilde{p}_{c_j, c_1 + \dots + c_j} \right) \left(\prod_{\substack{i=1, c_1 + \dots + c_n - 1 \\ i \notin C_{n-1}}} \tilde{p}_{u, i} \right) d\left(\underline{\mathbb{S}}_{(P, u)}\right) d\left(\overline{\mathbb{S}}_{(P, u)}\right) \\ \text{if } c_1 + \dots + c_n \text{ is odd. } d\left(\overline{\mathbb{S}}_{(P, u)}\right). \end{cases}$$

$$(3) \quad \frac{d\left(\Delta_{c_1 \dots c_n c_{n+1}}^{(-P, u)}\right)}{d\left(\Delta_{c_1 \dots c_n}^{(-P, u)}\right)} = \begin{cases} p_{s-1-c_{n+1}} \left(\prod_{i=c_1+c_2+\dots+c_{n+1}}^{c_1+\dots+c_{n+1}-1} \tilde{p}_{u, i} \right) \\ \text{if } c_1 + \dots + c_n, c_{n+1} \text{ are even,} \\ p_{c_{n+1}} \left(\prod_{i=c_1+c_2+\dots+c_{n+1}}^{c_1+\dots+c_{n+1}-1} \tilde{p}_{u, i} \right) \\ \text{if } c_1 + \dots + c_n \text{ is odd, } c_{n+1} \text{ is even,} \\ p_{s-1-c_{n+1}} \left(\prod_{i=c_1+c_2+\dots+c_{n+1}}^{c_1+\dots+c_{n+1}-1} \tilde{p}_{u, i} \right) \left(\overline{\mathbb{S}}_{(P, u)} / \underline{\mathbb{S}}_{(P, u)} \right) \\ \text{if } c_1 + \dots + c_n, c_{n+1} \text{ are odd,} \\ p_{c_{n+1}} \left(\prod_{i=c_1+c_2+\dots+c_{n+1}}^{c_1+\dots+c_{n+1}-1} \tilde{p}_{u, i} \right) \left(\underline{\mathbb{S}}_{(P, u)} / \overline{\mathbb{S}}_{(P, u)} \right) \\ \text{if } c_1 + \dots + c_n \text{ is even, } c_{n+1} \text{ is odd.} \end{cases}$$

$$(4) \quad \Delta_{c_1 c_2 \dots c_n}^{(-P, u)} = \bigcup_{c \in \overline{A}} \Delta_{c_1 c_2 \dots c_n c}^{(-P, u)} \quad \forall c_n \in \overline{A}, n \in \mathbb{N}.$$

(5) *The following relationships are satisfied:*

(a) *if $u \in \{0, 1\}$, then*

$$\left\{ \begin{array}{l} \inf \Delta_{c_1 \dots c_n [c+1]}^{(-P, u)} > \sup \Delta_{c_1 \dots c_n c}^{(-P, u)} \\ \quad \text{whenever } c_1 + \dots + c_n + c \text{ is even,} \\ \inf \Delta_{c_1 \dots c_n c}^{(-P, u)} > \sup \Delta_{c_1 \dots c_n [c+1]}^{(-P, u)} \\ \quad \text{whenever } c_1 + \dots + c_n + c \text{ is odd,} \end{array} \right. \quad (c \neq s - 1);$$

(b) *if $u \in \{2, 3, \dots, s - 3\}$, then for an odd $c_1 + \dots + c_n + c$*

$$\left\{ \begin{array}{l} \sup \Delta_{c_1 \dots c_n c}^{(-P, u)} < \inf \Delta_{c_1 \dots c_n [c+1]}^{(P, u)} \quad \text{for all } c + 1 \leq u, \\ \inf \Delta_{c_1 \dots c_n c}^{(-P, u)} > \sup \Delta_{c_1 \dots c_n [c+1]}^{(-P, u)} \quad \text{for all } u < c; \end{array} \right.$$

if $u \in \{2, 3, \dots, s - 3\}$, then for an even $c_1 + \dots + c_n + c$

$$\left\{ \begin{array}{l} \inf \Delta_{c_1 \dots c_n [c+1]}^{(-P, u)} > \sup \Delta_{c_1 \dots c_n c}^{(-P, u)} \quad \text{for all } u < c, \\ \inf \Delta_{c_1 \dots c_n c}^{(-P, u)} > \sup \Delta_{c_1 \dots c_n [c+1]}^{(-P, u)} \quad \text{for all } c + 1 \leq u; \end{array} \right.$$

(c) *if $u \in \{s - 2, s - 1\}$, then*

$$\left\{ \begin{array}{l} \inf \Delta_{c_1 \dots c_n [c+1]}^{(-P, u)} > \sup \Delta_{c_1 \dots c_n c}^{(-P, u)} \quad \text{whenever } c_1 + \dots + c_n + c \text{ is odd,} \\ \inf \Delta_{c_1 \dots c_n c}^{(-P, u)} > \sup \Delta_{c_1 \dots c_n [c+1]}^{(-P, u)} \quad \text{whenever } c_1 + \dots + c_n + c \text{ is even.} \end{array} \right.$$

Remark 2. One can note that if for $\mathbb{S}_{(-s, u)}$ and $\mathbb{S}_{(s, u)}$, topological, metric, and fractal properties (without some properties of cylinders) are similar, then fractal and some local properties of $\mathbb{S}_{(-P, u)}$ and $\mathbb{S}_{(P, u)}$ are different. For example, $\mathbb{S}_{(P, u)}$ is a self-similar fractal (i.e., this is a Moran set by Moran's definition, [16]) but $\mathbb{S}_{(-P, u)}$ is a non-self-similar set having the Moran structure (i.e., this is a Moran set by the definition of Hua et al. (see the definition in [12])).

5. Certain examples

Let us consider the case of the sets $\mathbb{S}_{(P_3, 0)}$ and $\mathbb{S}_{(-P_3, 0)}$. That is, the set $\mathbb{S}_{(P_3, 0)}$ is a set of the form

$$S_{(P_3, 0)} := \left\{ x : x = \Delta \underbrace{0 \dots 0}_{\alpha_1 - 1} \underbrace{0 \dots 0}_{\alpha_2 - 1} \dots \underbrace{0 \dots 0}_{\alpha_n - 1} \alpha_n \dots, \alpha_n \in \{1, 2\}, n = 1, 2, 3, \dots \right\}.$$

In the other words, our set is the image of a certain set under the Salem function f_ξ , and this certain set is the set whose elements are represented in terms of the 3-adic (ternary) representation by using combinations of ternary digits only from $\{1, 02\}$. So, applying Theorem 4.1, we obtain that the Hausdorff dimension of $\mathbb{S}_{(P_3,0)}$ satisfies the equation

$$(p_1)^{\alpha_0} + (p_0 p_2)^{\alpha_0} = 1. \quad (7)$$

Let us consider the set $\mathbb{S}_{(-P_3,0)}$. That is,

$$S_{(-P_3,0)} := \left\{ x : x = \Delta_{\underbrace{0\dots 0}_{\alpha_1-1} \underbrace{0\dots 0}_{\alpha_2-1} \dots \underbrace{0\dots 0}_{\alpha_n-1} \alpha_n \dots}, \alpha_n \in \{1, 2\}, n = 1, 2, 3, \dots \right\}.$$

Since ([40, 41])

$$\Delta_{\alpha_1 \alpha_2 \dots \alpha_n \dots}^{P_3} \equiv \Delta_{\alpha_1 [2-\alpha_2] \alpha_3 \dots \alpha_{2k-1} [2-\alpha_{2k}] \dots}^{-P_3}$$

and

$$\Delta_{\alpha_1 \alpha_2 \dots \alpha_n \dots}^{-P_3} \equiv \Delta_{\alpha_1 [2-\alpha_2] \alpha_3 \dots \alpha_{2k-1} [2-\alpha_{2k}] \dots}^{P_3},$$

we have that this set is a subset of the set

$$\left\{ x : x = \Delta_{\delta_1 \delta_2 \dots \delta_n \dots}^{P_3}, \delta_n \in \{00, 1, 22\} \right\}.$$

Actually it is a subset, because, for example,

$$1 = \Delta_{2222222\dots}^{P_3} \notin \mathbb{S}_{(-P_3,0)}.$$

Using Lemma 4.7, we have

$$\inf \mathbb{S}_{(-P_3,0)} = \inf \overline{\mathbb{S}_{(P_3,0)}} = \Delta_{020202\dots}^{-P_3} = \Delta_{000000\dots}^{P_3} = 0,$$

$$\sup \mathbb{S}_{(-P_3,0)} = \sup \overline{\mathbb{S}_{(P_3,0)}} = \Delta_{1020202\dots}^{-P_3} = \Delta_{1222222\dots}^{P_3} = p_0 + p_1$$

and

$$\inf \underline{\mathbb{S}_{(P_3,0)}} = \Delta_{1000000\dots}^{P_3} = \beta_1 = p_0,$$

$$\sup \underline{\mathbb{S}_{(P_3,0)}} = \Delta_{2222222\dots}^{P_3} = 1.$$

Hence, using Theorem 4.6 and Lemma 4.8, we obtain

$$(\text{here } d(\cdot) = \sup(\cdot) - \inf(\cdot)),$$

$$d(\overline{\mathbb{S}_{(P_3,0)}}) = \beta_2 = p_0 + p_1,$$

$$d(\underline{\mathbb{S}_{(P_3,0)}}) = 1 - \beta_1 = 1 - p_0,$$

as well as

$$\omega_1 = p_1 \frac{p_0 + p_1}{p_1 + p_2}, \quad \omega_2 = p_1 \frac{p_1 + p_2}{p_0 + p_1}, \quad \omega_3 = p_2^2, \quad \omega_4 = p_0^2.$$

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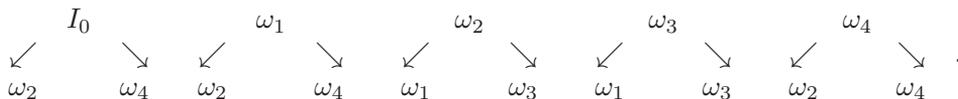
In addition, for any step $k \in \mathbb{N}$, the following relationships hold:

$$N_{1,c_i} + N_{2,c_i} + N_{3,c_i} + N_{4,c_i} = 2^k$$

and

$$N_{1,c_i} = N_{2,c_i} = N_{3,c_i} = N_{4,c_i} = 2^{k-2}.$$

The local structure of our set can be characterized by the following scheme:



Here,

$$I_0 = [\inf \mathbb{S}_{(-P_3,0)}, \sup \mathbb{S}_{(-P_3,0)}].$$

So, the Hausdorff dimension $\dim_H(\mathbb{S}_{(-P_3,0)})$ of $\mathbb{S}_{(-P_3,0)}$ is equal to

$$\alpha_* = \liminf_{k \rightarrow \infty} \gamma_k,$$

where (γ_k) is a sequence of numbers satisfying the equation

$$\left((\omega_2)^{\gamma_1} + (\omega_4)^{\gamma_1} \right) \prod_{i=2}^k \left(2^{i-2} \left((\omega_1)^{\gamma_i} + (\omega_2)^{\gamma_i} + (\omega_3)^{\gamma_i} + (\omega_4)^{\gamma_i} \right) \right) = 1.$$

EXAMPLE. Suppose

$$p_0 = \frac{1}{6}, \quad p_1 = \frac{2}{6} = \frac{1}{3}, \quad \text{and} \quad p_2 = \frac{3}{6} = \frac{1}{2}.$$

Then, the set $\mathbb{S}_{(P_3,0)}$ is a self-similar fractal whose Hausdorff dimension is approximately equal to 0.408985; but the set $\mathbb{S}_{(-P_3,0)}$ is not a self-similar fractal, its Hausdorff dimension is approximately equal to 0.422592.

EXAMPLE. Suppose

$$p_0 = p_2 = 0.25 \quad \text{and} \quad p_1 = 0.5.$$

Then, the sets $\mathbb{S}_{(P_3,0)}$ and $\mathbb{S}_{(-P_3,0)}$ are self-similar fractals and their Hausdorff dimensions are approximately equal 0.46496.

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