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THE GENERALIZED SHIFTS AND RATIONAL NUMBERS

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ABSTRACT. This paper is devoted to conditions defined in terms of the generalized shift operator for a rational number to be representable by certain positive generalizations of q-ary expansions.

1. Introduction

The problem on conditions for a rational number to be representable by the following positive series was introduced by Georg Cantor in the paper [1] in 1869

$$\frac{\varepsilon_1}{q_1}+\frac{\varepsilon_2}{q_1q_2}+\dots+\frac{\varepsilon_k}{q_1q_2\dots q_k}+\dots, \tag{1}$$

where $Q \equiv (q_k)$ is a fixed sequence of positive integers, $q_k > 1$, and (Θ_k) is a sequence of the sets $\Theta_k \equiv \{0, 1, \dots, q_k - 1\}$, as well as $\varepsilon_k \in \Theta_k$.

Series of form (1) are called *Cantor series*. By $\Delta^Q_{\varepsilon_1\varepsilon_2\dots\varepsilon_k\dots}$ denote any number $x \in [0,1]$ having expansion (1). This notation is called *the representation of* $x \in [0,1]$ *by Cantor series* (1).

It is easy to see that Cantor series expansion (1) is the q-ary expansion

$$\frac{\varepsilon_1}{q}+\frac{\varepsilon_2}{q^2}+\dots+\frac{\varepsilon_k}{q^k}+\dots$$

of real numbers from $[0,1]$, where $\varepsilon_k \in \{0,1,\dots,q-1\}$, whenever the condition $q_k = \text{const} = q$ holds for all $k \in \mathbb{N}$ ($\mathbb{N}$ is the set of all positive integers), where $1 < q \in \mathbb{N}$.

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A number of researches are devoted to investigations of Cantor expansions from different points of view (a brief description is given in [12]) including studying the problem on Cantor series expansions of rational numbers (for example, see [1, 2, 7, 11–15, 17]). In [4], Prof. János Galambos calls the problem on representations of rational numbers by Cantor series (1) as *the fourth open problem.*

One can note that the notion of the shift operator is applicable to this problem (for example, some descriptions are given in [12, 15]). This paper is devoted to applications of the notion of the generalized shift operator to solving the problem on representations of rational numbers by positive Cantor series. The present research is the continuation of investigations presented in the papers [14, 15].

2. The shift and generalized shift operators

The shift operator σ *of expansion* (1) is a map of the following form

$$\sigma(x) = \sigma\left(\Delta^Q_{\varepsilon_1\varepsilon_2\ldots\varepsilon_k\ldots}\right) = \sum^{\infty}_{k=2}\frac{\varepsilon_k}{q_2q_3\cdots q_k} = q_1\Delta^Q_{0\varepsilon_2\ldots\varepsilon_k\ldots}.$$

It is easy to see that

$$\begin{aligned}\sigma^n(x) &= \sigma^n\left(\Delta^Q_{\varepsilon_1\varepsilon_2\ldots\varepsilon_k\ldots}\right)\\ &= \sum^{\infty}_{k=n+1}\frac{\varepsilon_k}{q_{n+1}q_{n+2}\cdots q_k} = q_1\ldots q_n\Delta^Q_{\underbrace{0\ldots0}_{n}\varepsilon_{n+1}\varepsilon_{n+2}\ldots}.\end{aligned}$$

One can note the partial case of this operator for q-ary expansions

$$\sigma^n\left(\Delta^q_{\varepsilon_1\varepsilon_2\ldots\varepsilon_k\ldots}\right) = \sum^{\infty}_{k=n+1}\frac{\varepsilon_k}{q^{k-n}} = \Delta^q_{\varepsilon_{n+1}\varepsilon_{n+2}\ldots}.$$

Suppose a number $x\in[0,1]$ is represented by series (1). Then *the generalized shift operator* σ_m is a map of the following form:

$$\sigma_m(x) = \sigma_m\left(\Delta^Q_{\varepsilon_1\varepsilon_2\ldots\varepsilon_k\ldots}\right) = \sum^{m-1}_{k=1}\frac{\varepsilon_k}{q_1q_2\cdots q_k} + \sum^{\infty}_{t=m+1}\frac{\varepsilon_t}{q_1q_2\cdots q_{m-1}q_{m+1}\cdots q_t}.$$

That is, any number from $[0,1]$ can be represented by two fixed sequences (q_k) and (ε_k) (Cantor series expansions). The generalized shift operator maps the preimage into a number represented by the following two sequences

$$(q_1, q_2, \ldots, q_{m-1}, q_{m+1}, q_{m+2}, \ldots) \quad \text{and} \quad (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{m-1}, \varepsilon_{m+1}, \varepsilon_{m+2}, \ldots).$$

Properties of this operator are considered in more detail in [16].

Denote by ϑ_m the sum $\sum_{k=1}^{m}\frac{\varepsilon_k}{q_1q_2\cdots q_k}$ and by δ_m the sum

$$\varepsilon_1 q_2 q_3\cdots q_m+\varepsilon_2 q_3 q_4\cdots q_m+\dots+\varepsilon_{m-1}q_m+\varepsilon_m.$$

Then

$$\sigma_m(x)=q_m x-(q_m-1)\vartheta_{m-1}-\frac{\varepsilon_m}{q_1q_2\cdots q_{m-1}}. \tag{2}$$

For the case of positive Cantor series, the notion of the generalized shift operator is considered in more detail in [16] (see also [13], where the shift and generalized shift operators are considered for the case of alternating Cantor series).

Let us remark that the following statement is true.

Lemma 1. *For the generalized shift operator defined in terms of positive Cantor series, the following relationships hold:*

- $$\sigma_{m+1}(x)=\frac{q_{m+1}}{q_m}\sigma_m(x)-\frac{q_{m+1}-q_m}{q_m}\vartheta_{m-1}-\frac{\varepsilon_{m+1}-\varepsilon_m}{q_1q_2\cdots q_{m-1}q_m};$$
- $$\varepsilon_m=q_1q_2\cdots q_m x-q_1q_2\cdots q_{m-1}\sigma_m(x)-(q_m-1)\delta_{m-1}.$$

Proof. Let us prove *the first relationship*. Using (2), we get

$$\frac{\sigma_m(x)}{q_m}+\frac{q_m-1}{q_m}\vartheta_{m-1}+\frac{\varepsilon_m}{q_1q_2\cdots q_{m-1}q_m}=$$
$$x=\frac{\sigma_{m+1}(x)}{q_{m+1}}+\frac{q_{m+1}-1}{q_{m+1}}\vartheta_m+\frac{\varepsilon_{m+1}}{q_1q_2\cdots q_m q_{m+1}},$$

$$q_{m+1}\sigma_m(x)+q_{m+1}(q_m-1)\vartheta_{m-1}+\frac{q_{m+1}\varepsilon_m}{q_1q_2\cdots q_{m-1}}$$
$$=q_m\sigma_{m+1}(x)+q_m(q_{m+1}-1)\vartheta_m+\frac{\varepsilon_{m+1}}{q_1q_2\cdots q_{m-1}}.$$

Since $\vartheta_m=\vartheta_{m-1}+\frac{\varepsilon_m}{q_1q_2\cdots q_m}$, we obtain

$$q_{m+1}\sigma_m(x)=q_{m+1}\vartheta_{m-1}+q_m\sigma_{m+1}(x)-q_m\vartheta_{m-1}$$
$$-\frac{\varepsilon_m}{q_1q_2\cdots q_{m-1}}+\frac{\varepsilon_{m+1}}{q_1q_2\cdots q_{m-1}}.$$

Hence

$$\sigma_{m+1}(x)=\frac{q_{m+1}}{q_m}\sigma_m(x)-\frac{q_{m+1}-q_m}{q_m}\vartheta_{m-1}-\frac{\varepsilon_{m+1}-\varepsilon_m}{q_1q_2\cdots q_{m-1}q_m}.$$

Let us prove *the second relationship*. Using (2), we have

$$q_1q_2\cdots q_{m-1}\sigma_m(x)=q_1q_2\cdots q_m x-q_1q_2\cdots q_{m-1}(q_m-1)\vartheta_{m-1}-\varepsilon_m.$$

The relationship follows from the last-mentioned equality. □

3. Rational numbers

Theorem 1. *A number $x \in [0,1]$ represented by series (1) is a rational number if and only if there exist non-negative integers m_1 and m_2 such that $m_1 \neq m_2$ and the condition*

$$\{q_1q_2\cdots q_{m_1-1}\sigma_{m_1}(x)\} = \{q_1q_2\cdots q_{m_2-1}\sigma_{m_2}(x)\}$$

holds, where $\{a\}$ is the fractional part of a, $q_{-1} = q_0 = 1$, and $\sigma_0(x) = x$.

Proof. Let us prove that *the necessity* is true. Let x be a rational number, i.e., $x = \frac{a}{b}$, where $a \in \mathbb{Z}_0 = \mathbb{N} \cup \{0\}$ and $b \in \mathbb{N}$, $a < b$, and $(a,b) = 1$.

Let us consider the sequence $\big(q_1q_2\cdots q_{k-1}\sigma_k(x)\big)$. Using (2), we have

$$\begin{aligned} q_1q_2\cdots q_{k-1}\sigma_k(x) &= q_1q_2\cdots q_{k-1}q_kx - \varepsilon_k \\ &\quad - (q_k-1)(\varepsilon_1q_2q_3\cdots q_{k-1} + \cdots + \varepsilon_{k-2}q_{k-1} + \varepsilon_{k-1}) \\ &= q_1q_2\cdots q_k\frac{a}{b} + (\varepsilon_1q_2q_3\cdots q_{k-1} + \cdots + \varepsilon_{k-2}q_{k-1} + \varepsilon_{k-1}) \\ &\quad - (\varepsilon_1q_2q_3\cdots q_k + \cdots + \varepsilon_{k-1}q_k + \varepsilon_k) \\ &= \frac{aq_1q_2\cdots q_k + b\delta_{k-1} - b\delta_k}{b}. \end{aligned}$$

Since in our case

$$\sigma^k(x) = q_1q_2\cdots q_kx - (\varepsilon_1q_2q_3\cdots q_k + \cdots + \varepsilon_{k-1}q_k + \varepsilon_k) = \frac{q_1q_2\cdots q_ka - b\delta_k}{b},$$

we obtain

$$q_1q_2\cdots q_{k-1}\sigma_k(x) = \delta_{k-1} + \sigma^k(x) = \delta_{k-1} + \frac{q_1q_2\cdots q_ka - b\delta_k}{b}.$$

It is easy to see that

$$\{\sigma^k(x)\} = \begin{cases} \sigma^k(x) - 1, & \text{whenever } x = \Delta^Q_{\varepsilon_1\varepsilon_2\ldots\varepsilon_{k-1}[\varepsilon_k-1][q_{k+1}-1][q_{k+2}-1]\ldots} \\ \sigma^k(x), & \text{whenever } x \neq \Delta^Q_{\varepsilon_1\varepsilon_2\ldots\varepsilon_{k-1}[\varepsilon_k-1][q_{k+1}-1][q_{k+2}-1]\ldots}. \end{cases}$$

Hence if $x \neq \Delta^Q_{\varepsilon_1\varepsilon_2\ldots\varepsilon_{k-1}[\varepsilon_k-1][q_{k+1}-1][q_{k+2}-1]\ldots}$, then

$$\begin{aligned} [q_1q_2\cdots q_{k-1}\sigma_k(x)] &= \delta_{k-1} \\ &= \varepsilon_1q_2q_3\cdots q_{k-1} + \varepsilon_2q_3q_4\cdots q_{k-1} + \cdots + \varepsilon_{k-2}q_{k-1} + \varepsilon_{k-1} \end{aligned}$$

and

$$\{q_1q_2\cdots q_{k-1}\sigma_k(x)\} = \sigma^k(x).$$

If $x=\Delta^{Q}_{\varepsilon_1\varepsilon_2\ldots\varepsilon_{k-1}[\varepsilon_k-1][q_{k+1}-1][q_{k+2}-1]\ldots}$, then

$$
\begin{aligned}
[q_1q_2\cdots q_{k-1}\sigma_k(x)] &= 1+\delta_{k-1}\\
&= 1+\varepsilon_1q_2q_3\cdots q_{k-1}+\varepsilon_2q_3q_4\cdots q_{k-1}+\ldots+\varepsilon_{k-2}q_{k-1}+\varepsilon_{k-1}
\end{aligned}
$$

and

$$\{q_1q_2\cdots q_{k-1}\sigma_k(x)\}=0.$$

Here $[x]$ is the integer part of x and $\{x\}$ is the fractional part of x.

By analogy to arguments described in [15], we get

$$\{q_1q_2\cdots q_{k-1}\sigma_k(x)\}=\left\{\frac{q_1q_2\cdots q_ka-\delta_kb}{b}\right\}=\left\{\frac{a_k}{b}\right\},$$

where b is a fixed positive integer, $a_k\in\{0,1,\ldots,b-1,b\}$, and there exist non-negative integers m_1 and m_2 such that $m_1\neq m_2$ and $a_{m_1}=a_{m_2}$ as $k\to\infty$.

Let us prove *the sufficiency*. Suppose there exist non-negative integers m_1 and m_2 such that $m_1<m_2$ and

$$\{q_1q_2\cdots q_{m_1-1}\sigma_{m_1}(x)\}=\{q_1q_2\cdots q_{m_2-1}\sigma_{m_2}(x)\}.$$

Let us prove the case when

$$x\notin\left\{\Delta^{Q}_{\varepsilon_1\varepsilon_2\ldots\varepsilon_{m_1-1}[\varepsilon_{m_1}-1][q_{m_1+1}-1][q_{m_1+2}-1]\ldots},\ \Delta^{Q}_{\varepsilon_1\varepsilon_2\ldots\varepsilon_{m_2-1}[\varepsilon_{m_2}-1][q_{m_2+1}-1][q_{m_2+2}-1]\ldots}\right\}.$$

Since $\{x\}=x-[x]$, we have

$$
\begin{aligned}
\{q_1q_2\cdots q_{m_1-1}\sigma_{m_1}(x)\}&=q_1q_2\cdots q_{m_1-1}\sigma_{m_1}(x)-\delta_{m_1-1},\\
\{q_1q_2\cdots q_{m_2-1}\sigma_{m_2}(x)\}&=q_1q_2\cdots q_{m_2-1}\sigma_{m_2}(x)-\delta_{m_2-1},
\end{aligned}
$$

and

$$q_1q_2\cdots q_{m_1-1}\sigma_{m_1}(x)-\delta_{m_1-1}=q_1q_2\cdots q_{m_2-1}\sigma_{m_2}(x)-\delta_{m_2-1}.$$

Using (2), we obtain

$$q_1\cdots q_{m_1-1}q_{m_1}x-(q_{m_1}-1)\delta_{m_1-1}-\varepsilon_{m_1}-\delta_{m_1-1}=$$
$$q_1\cdots q_{m_2-1}q_{m_2}x-(q_{m_2}-1)\delta_{m_2-1}-\varepsilon_{m_2}-\delta_{m_2-1}.$$

Hence

$$x=\frac{q_{m_1}\delta_{m_1-1}-q_{m_2}\delta_{m_2-1}+\varepsilon_{m_1}-\varepsilon_{m_2}}{q_1q_2\cdots q_{m_1}-q_1q_2\cdots q_{m_2}}$$

is a rational number.

The proof is analogues for the case when

$$x\in\left\{\Delta^{Q}_{\varepsilon_1\varepsilon_2\ldots\varepsilon_{m_1-1}[\varepsilon_{m_1}-1][q_{m_1+1}-1][q_{m_1+2}-1]\ldots},\ \Delta^{Q}_{\varepsilon_1\varepsilon_2\ldots\varepsilon_{m_2-1}[\varepsilon_{m_2}-1][q_{m_2+1}-1][q_{m_2+2}-1]\ldots}\right\}.$$

□

One can note that certain numbers from $[0,1]$ have two different representations by Cantor series (1), i.e.,

$$\Delta^Q_{\varepsilon_1\varepsilon_2\ldots\varepsilon_{m-1}\varepsilon_m 000\ldots} = \Delta^Q_{\varepsilon_1\varepsilon_2\ldots\varepsilon_{m-1}[\varepsilon_m-1][q_{m+1}-1][q_{m+2}-1]\ldots} = \sum_{i=1}^{m}\frac{\varepsilon_i}{q_1q_2\cdots q_i}.$$

Such numbers are called *Q-rational.* The other numbers in $[0,1]$ are called *Q-irrational.*

Let $c_1, c_2, \ldots, c_m$ be an ordered tuple of integers such that

$$c_i \in \{0,1,\ldots,q_i-1\} \quad \text{for all} \quad i=\overline{1,m}.$$

Then *a cylinder $\Lambda^Q_{c_1c_2\ldots c_m}$ of rank m with base $c_1c_2\ldots c_m$* is a set of the form

$$\Lambda^Q_{c_1c_2\ldots c_m} \equiv \{x : x = \Delta^Q_{c_1c_2\ldots c_m\varepsilon_{m+1}\varepsilon_{m+2}\ldots\varepsilon_{m+k}\ldots}\}.$$

Theorem 2. *Suppose a number x represented by series* (1) *and*

$$x \neq \Delta^Q_{\varepsilon_1\ldots\varepsilon_{m-1}[\varepsilon_m-1][q_{m+1}-1][q_{m+2}-1]\ldots} \quad \text{for any} \quad m\in\mathbb{N}.$$

Then x is a rational number $\frac{a}{b}$ (here $a,b\in\mathbb{N}$, $a<b$, and $(a,b)=1$) if and only if the condition

$$\varepsilon_{m+1} = \left[\frac{(q_{m+1}+1)q_1q_2\cdots q_m a - b(q_1q_2\cdots q_m\sigma_{m+1}(x) + q_{m+1}\delta_m)}{b}\right]$$

holds for any $m\in\mathbb{N}$, where $\varepsilon_1 = \left[\frac{a}{b}q_1\right]$, $[x]$ is the integer part of x, and

$$\delta_m = \varepsilon_1q_2q_3\cdots q_m + \varepsilon_2q_3q_4\cdots q_m + \cdots + \varepsilon_{m-1}q_m + \varepsilon_m.$$

P r o o f. N e c e s s i t y. Let x be a rational number $\frac{a}{b}$. Then for any $m\in\mathbb{N}$ there exists a cylinder $\Lambda^Q_{\varepsilon_1\varepsilon_2\ldots\varepsilon_m}$ such that $x\in\Lambda^Q_{\varepsilon_1\varepsilon_2\ldots\varepsilon_m}$. That is,

$$x \in \left[\frac{\delta_m}{q_1q_2\cdots q_m}, \frac{\delta_m+1}{q_1q_2\cdots q_m}\right].$$

Since

$$\Delta^Q_{\varepsilon_1\ldots\varepsilon_{m-1}\varepsilon_m[q_{m+1}-1][q_{m+2}-1]\ldots} = \Delta^Q_{\varepsilon_1\varepsilon_2\ldots\varepsilon_{m-1}[\varepsilon_m+1]000\ldots}, \quad \text{where} \quad \varepsilon_m \neq q_m-1,$$

we do not use representations of the form $\Delta^Q_{\varepsilon_1\varepsilon_2\ldots\varepsilon_m[\varepsilon_m-1][q_{m+1}-1][q_{m+2}-1]\ldots}$ and assume that

$$\frac{\delta_m}{q_1q_2\cdots q_m} \leq x < \frac{\delta_m+1}{q_1q_2\cdots q_m},$$

$$\delta_m \leq q_1q_2\cdots q_m\frac{a}{b} < \delta_m+1.$$

Since

$$\sigma_{m+1}(x) = \sum_{k=1}^{m} \frac{\varepsilon_k}{q_1 q_2 \cdots q_k} + \frac{\sigma^{m+1}(x)}{q_1 q_2 \cdots q_m},$$

$$\sigma^{m+1}(x) = q_1 \cdots q_m q_{m+1} x - \delta_{m+1},$$

and

$$\delta_{m+1} = \varepsilon_{m+1} + q_{m+1}\delta_m,$$

we have

$$0 \le \frac{a}{b} q_1 q_2 \cdots q_m - q_1 q_2 \cdots q_m \sigma_{m+1}(x) + \frac{a}{b} q_1 q_2 \cdots q_{m+1} - q_{m+1}\delta_m - \varepsilon_{m+1} < 1,$$

$$\varepsilon_{m+1} \le \frac{a}{b} q_1 q_2 \cdots q_m (q_{m+1}+1) - q_1 q_2 \cdots q_m \sigma_{m+1}(x) - q_{m+1}\delta_m < \varepsilon_{m+1} + 1.$$

Hence

$$\begin{aligned}\varepsilon_{m+1} &= \left[\frac{(q_{m+1}+1) q_1 q_2 \cdots q_m a - b(q_1 q_2 \cdots q_m \sigma_{m+1}(x) + q_{m+1}\delta_m)}{b}\right] \\ &= [z_{m+1}],\end{aligned}$$

where $\varepsilon_1 = \left[\frac{a}{b} q_1\right]$. □

S u f f i c i e n c y. If $\varepsilon_{m+1} = [z_{m+1}]$, then

$$\begin{aligned} x &= \vartheta_{m+1} + \frac{\sigma^{m+1}(x)}{q_1 q_2 \cdots q_{m+1}} \\ &= \frac{\delta_{m+1}}{q_1 q_2 \cdots q_{m+1}} + \frac{\sigma^{m+1}(x)}{q_1 q_2 \cdots q_{m+1}} \\ &= \frac{\varepsilon_{m+1} + q_{m+1}\delta_m}{q_1 q_2 \cdots q_{m+1}} + \frac{\sigma^{m+1}(x)}{q_1 q_2 \cdots q_{m+1}} \\ &= \frac{[z_{m+1}] + q_{m+1}\delta_m}{q_1 q_2 \cdots q_{m+1}} + \frac{\sigma^{m+1}(x)}{q_1 q_2 \cdots q_{m+1}} \\ &= \frac{z_{m+1} - \{z_{m+1}\}}{q_1 q_2 \cdots q_{m+1}} + \frac{q_{m+1}\delta_m}{q_1 q_2 \cdots q_{m+1}} + \frac{\sigma^{m+1}(x)}{q_1 q_2 \cdots q_{m+1}} \\ &= \frac{a}{b} q_1 q_2 \cdots q_{m+1} + \sigma^m(x) - \{z_{m+1}\} \\ &= \frac{a}{b} \end{aligned}$$

because

$$\{z_{m+1}\} = \{\varepsilon_{m+1} + \sigma^m(x)\}, \quad q_1 q_2 \cdots q_m \sigma_{m+1}(x) = \delta_m + \sigma^{m+1}(x),$$

and also

$$q_1 q_1 \cdots q_m x - \delta_m = \sigma^m(x).$$ □

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