sciendo

TATRA MOUNTAINS
Mathematical Publications
DOI: 10.2478/tmmp-2023-0015
Tatra Mt. Math. Publ. **84** (2023), 45–60

MODIFICATIONS OF THE SAMUELSON ECONOMIC CYCLE MODEL

Paulina Naumowicz—Miroslava Růžičková

Faculty of Mathematics, University of Białystok, Białystok, POLAND

ABSTRACT. The paper presents an analysis of modifications of the standard discrete Samuelson model with determination of the stability of the equilibrium point. The stability region of the equilibrium point depending on the parameters contained in each model is determined using the Schur-Cohn stability criterion.

1. Simple business cycle model

A market is a system of transactions aimed at selling and buying commodities. All markets, no matter how different, perform the same function: they determine the price at which the demand for a good equals the quantity of that good produced. In the simplified model, the key variables are supply and demand.

The market is not "fixed", the values of supply and demand are constantly changing. A trend is a path of output growth that ignores short-term fluctuations. Actual output values fluctuate around the trend irregularly but cyclically. The business cycle is the deviation of production from the trend. The business cycle is characterised by turning points and the periods between them:

- expansion (real GDP growth and falling unemployment);
- peak (real GDP stops growing and starts falling);
- contraction or recession (real GDP falls and unemployment rises);

2020 Mathematics Subject Classification: 39A30; 39A60; 91G15.
Keywords: difference equations, Samuelson multiplier-accelerator model, national income, equilibrium point, region of stability.
The authors has been supported by the Polish Ministry of Science and Higher Education under a subsidy for maintaining the research potential of the Faculty of Mathematics, University of Białystok, Poland.

- trough (real GDP stops falling and starts rising).

Using business cycle models, economists try to find the cause of short-term fluctuations and predict the future development of the economy. Economic policy makers use them to adjust fiscal and monetary policy in order to stabilise the economy. Poorly chosen policies or inappropriate timing can lead to destabilisation of the economy. Reliable and accurate models that reflect reality are therefore a very important tool in making decisions that affect the development of the economy. The difficulty of the task of constructing such a model lies in the irregularity of the occurrence of recessions and their unpredictable duration and course.

Samuelson's accelerator-multiplier model (1939) [5] is considered as one of the first formal business cycle models. In this model, Samuelson uses the consumption equation presented by the economist Dennis Robertson as the multiplier and the investment model developed by John Maurice Clark as the restrictive accelerator. In addition, Samuelson took into account the assumptions:

- National income Y_t in year t is equal to consumption C_t in year t, investment I_t in year t, and government expenditure G_t in year t:

$$Y_t = C_t + I_t + G_t. \tag{1}$$

- The existence of a slow production period, where consumption C_t in year t results from national income Y_{t-1} in the previous period $t-1$ and a parameter β, $0 < \beta < 1$ called the marginal propensity to consume

$$C_t = \beta Y_{t-1}, \tag{2}$$

  that is, there is no automatic and induced investment. The difference in aggregate demand stimulates manufacturing firms to adjust their production capacity in the following period.

- Investments I_t in year t depend on the change in consumption between the current year and the previous year and on a parameter called the accelerator, denoted α, $\alpha > 0$:

$$I_t = \alpha(C_t - C_{t-1}). \tag{3}$$

- Government does not interfere in economic activity, i.e., government expenditure G_t in each year t is constant

$$G_t = G. \tag{4}$$

The above conditions form the Samuelson model, which contains three equations, sometimes referred to as the income, consumption and investment equation, respectively:

$$\begin{aligned} Y_t &= C_t + I_t + G, \\ C_t &= \beta Y_{t-1}, \\ I_t &= \alpha(C_t - C_{t-1}), \quad 0 < \beta < 1, \quad \alpha > 0. \end{aligned} \tag{5}$$

System (5) can be modified to the second-order difference equation

$$Y_{t+2} - \beta(\alpha+1)Y_{t+1} + \alpha\beta Y_t = 0, \tag{6}$$

and the analysis of the stability of its equilibrium point $\hat{Y} = \frac{G}{(1-\beta)}$ can be found in many papers, for example in [6], so we will not repeat it, but only give the result of the analysis.

Theorem 1.1. *Let $\alpha > 0, 0 < \beta < 1$. Then the equilibrium point $\hat{Y} = \frac{G}{1-\beta}$ of the Samuelson model* (5) *is asymptotically stable if and only if $\alpha\beta < 1$.*

Figure 1 shows the region of stability of the equilibrium point of the Samuelson model depending on the parameters α, β.

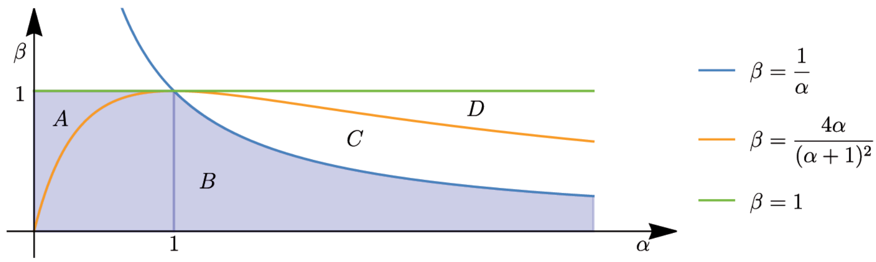

Figure 1. The stability region of the Samuelson model depending on the parameters α and β. In region A is monotonic convergence, in region B oscillatory convergence, in region C oscillatory divergence and in region D monotonic divergence.

Samuelson's model is a basic model, which means that it has several simplifications in relation to the actual behaviour of the economic situation. However, this does not mean that it is unusable; despite its simplicity, it is consistent with the empirical data. In [1], the Samuelson model was applied to the French economy in the period 1985–2018, where the authors showed that the Samuelson model can accurately and objectively measure and determine the magnitude of successive economic cycles in the French capitalist economy in the period under study. However, it cannot be denied that business cycle movements can be influenced by other variables.

Saluelson's model is still being studied in its basic form or in the form of various modifications, for example in papers [1], [3,4], [6]. There, the stability of the equilibrium point is discussed based on the nature of the roots of the characteristic equation. In [4], the original Samuelson multiplier-accelerator model is extended by introducing a discontinuity in the government expenditure variable, so that the authors consider the economy in two states, when the economy is in contraction and when the economy is growing, and examine the impact of stimulus or stabilization policies on these two states in the same model.

2. Modifications of economic cycle model

In this section, we discuss the stability of equilibrium points in several modifications of the Samuelson model. For this purpose, we will use the Schur-Cohn stability criterion based on the coefficients of the difference equation with constant coefficients

$$x_{n+k} + p_1 x_{n+k-1} + p_2 x_{n+k-2} + \cdots + p_{k-1} x_{n+1} + p_k x_n = G, \tag{7}$$

whose homogeneous part has the characteristic equation

$$P_k(r) \equiv r^k + p_1 r^{k-1} + p_2 r^{k-2} + \cdots + p_{k-1} r + p_k = 0. \tag{8}$$

We also determine the stability regions of the equilibrium point depending on the coefficients of these models.

First, we define the necessary terms.

Definition 2.1. A matrix B is said to be positive innerwise if the determinants of all of its inner matrices are positive. The inner matrices are the matrix itself and all matrices obtained by successively omitting the first and last rows and the first and last columns.

Theorem 2.2 ([2] Schur-Cohn stability criterion)**.** *The equilibrium point of equation* (7) *is asymptotically stable if and only if the following holds for* (8):

(i) $P_k(1) > 0$,

(ii) $(-1)^k P_k(-1) > 0$,

(iii) *the* $(k-1) \times (k-1)$ *matrices*

$$B^{\pm}_{k-1} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ p_1 & 1 & 0 & \cdots & 0 \\ p_2 & p_1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ p_{k-2} & p_{k-3} & p_{k-4} & \cdots & 1 \end{pmatrix} \pm \begin{pmatrix} 0 & 0 & \cdots & 0 & p_k \\ 0 & 0 & \cdots & p_k & p_{k-1} \\ 0 & 0 & \cdots & p_{k-1} & p_{k-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & p_k & \cdots & p_4 & p_3 \\ p_k & p_{k-1} & \cdots & p_3 & p_2 \end{pmatrix}$$

are positive innerwise.

Corollary 2.3. *For the second-order polynomial* $P_2(r) \equiv r^2 + p_1 r + p_2 = 0$, *the above conditions take the form:*

(i) $1 + p_1 + p_2 > 0$,

(ii) $1 - p_1 + p_2 > 0$,

(iii) $1 - p_2 > 0$.

Corollary 2.4. *For the third-order polynomial* $P_3(r) \equiv r^3 + p_1 r^2 + p_2 r + p_3 = 0$, *the above conditions take the form*:

(i) $1 + p_1 + p_2 + p_3 > 0$,

(ii) $1 - p_1 + p_2 - p_3 > 0$,

(iii) $1 + p_2 - p_3(p_1 + p_3) > 0$ *and* $1 - p_2 + p_3(p_1 - p_3) > 0$.

2.1. Multiplier-accelerator model with investment period shift

We start with the simplest modification of the standard Samuelson model, i.e., with a shift in the investment period. This has its economic rationale: investments are usually made with a certain lag, in this case we will analyse the one-period lag with respect to the classical model. The model is based on conditions (1), (2) and (4) identical to the standard model and on a modified condition (3):

- Investments I_t in year t depend on the change in consumption between the previous year and the year before and on the accelerator α, $\alpha > 0$:

$$I_t = \alpha(C_{t-1} - C_{t-2}).$$

In this case, the model takes the form

$$\begin{aligned} Y_t &= C_t + I_t + G, \\ C_t &= \beta Y_{t-1}, \\ I_t &= \alpha(C_{t-1} - C_{t-2}), \quad \alpha > 0, \quad 0 < \beta < 1 \end{aligned} \tag{9}$$

and can be modified to the third-order difference equation

$$Y_{t+3} - \beta Y_{t+2} - \alpha\beta Y_{t+1} + \alpha\beta Y_t = G. \tag{10}$$

The equilibrium point $\hat{Y} = \frac{G}{(1-\beta)}$ of equation (10) is exactly the same as in the case of the Samuelson model (5). We investigate its stability using Schur-Cohn stability criterion, i.e., according to Corollary 2.4. We investigate for which values of parameters α, β the conditions stated there are satisfied.

(i) $1 + p_1 + p_2 + p_3 = 1 - \beta - \alpha\beta + \alpha\beta = 1 - \beta > 0$.
The inequality is valid for all admissible values $\alpha > 0,\ 0 < \beta < 1$.

(ii) $1 - p_1 + p_2 - p_3 = 1 + \beta - \alpha\beta - \alpha\beta = 1 + \beta - 2\alpha\beta = 1 - \beta(2\alpha - 1) > 0$.
So, the inequality is valid only for values

$$2\alpha\beta - \beta < 1, \tag{11}$$

from where

$$\beta < \frac{1}{2\alpha - 1}.$$

Therefore, since the parameter β is positive, $0 < \beta < \frac{1}{2\alpha-1}$, i.e., $\alpha > \frac{1}{2}$, must hold.

(iii) $1 + p_2 - p_3(p_1 + p_3) = 1 - \alpha\beta - \alpha\beta(-\beta + \alpha\beta) > 0$,
and
$1 - p_2 + p_3(p_1 - p_3) = 1 + \alpha\beta + \alpha\beta(-\beta - \alpha\beta) > 0$.
Since $\beta - 1 < 1 - \beta$, then the inequalities can be written as

$$\alpha\beta(\alpha\beta + \beta - 1) < \alpha\beta(\alpha\beta - \beta + 1) < 1. \tag{12}$$

It is easy to show that inequality (11) implies the fulfilment of both inequalities (2.3), i.e., all the asymptotic stability conditions of the equilibrium point. Therefore, the graphs of the equations $2\alpha\beta - \beta = 1$, $\alpha = \frac{1}{2}$, $\beta = 0, \beta = 1$ represent the asymptotic stability boundary, as can be seen in Fig. 2.

Theorem 2.5. *Let $\alpha > 0, 0 < \beta < 1$. Then the equilibrium point $\hat{Y} = \frac{G}{1-\beta}$ of the modified Samuelson model* (9) *is asymptotically stable if and only if*

$$\frac{1}{2} < \alpha < \frac{1+\beta}{2\beta}.$$

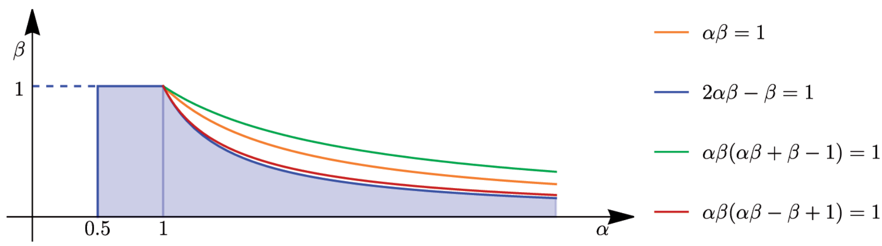

Figure 2. The stability region of the modified Samuelson model (9) depending on the parameters α and β depicted in blue.

Our investigation leads us to the following conclusion. The equilibrium point is the same as in the basic Samuelson model, but its stability region is smaller. To maintain the stability of the economy, the relationship between the consumption parameter and the accelerator in the basic model needs to be $\alpha < \frac{1}{\beta}$, while in the modified model the accelerator needs to be even smaller, i.e,

$$\alpha < \frac{1}{\beta}\frac{1+\beta}{2} < \frac{1}{\beta},$$

but not less than $\frac{1}{2}$, as mentioned above. This plays an important role in the dynamics of national income, i.e., how fast national income deviates from its original equilibrium level. A higher value of the accelerator causes national income to deviate from its equilibrium level.

2.2. Multiplier-accelerator model with tax addition

Keynes argued that the impact of any kind of taxation, direct or indirect, negatively affects national income, although in the case of government spending the role of tax collection is inevitable. Therefore, it seems appropriate to test the impact of these two types of taxes in a combined multiplier-accelerator model. We follow the simple scheme of the national income model in which the non-income tax is independent of national income and the income tax is part of it. The model is based on conditions (1), (3) and (4) identical to the standard model and a modified condition (2):

- Consumption C_t in the year t depends on the income of the previous year $t-1$, which has been reduced by the tax portion of the previous year $t-1$, and on the parameter marginal propensity to consume β, $0 < \beta < 1$:

$$C_t = \beta(Y_{t-1} - T_{t-1}), \tag{13}$$

  where the tax portion T_t for year t is the sum of the income-independent tax γ and income tax (which is part of income) with the income tax rate δ

$$T_t = \gamma + \delta Y_t. \tag{14}$$

The multiplier-accelerator model taking into account taxes thus takes the form of

$$\begin{aligned} Y_t &= C_t + I_t + G, \\ C_t &= \beta(1-\delta)Y_{t-1} - \beta\gamma, \\ I_t &= \alpha(C_t - C_{t-1}), \end{aligned} \tag{15}$$

where $\alpha > 0, 0 < \beta < 1, \gamma > 0,\ 0 < \delta < 1$, or in the form of the second order difference equation

$$G - \beta\gamma = Y_{t+2} - Y_{t+1}\beta(1-\delta)(1+\alpha) + \alpha\beta(1-\delta)Y_t \tag{16}$$

with equilibrium point

$$\hat{Y} = \frac{G - \beta\gamma}{1 - \beta(1-\delta)}. \tag{17}$$

We will investigate its stability using the Schur-Cohn stability criterion as in the previous case, but now we will use Corollary 2.3. We will investigate for which values of the parameters $\alpha, \beta, \gamma, \delta$ the conditions stated there are satisfied.

(i) $1 + p_1 + p_2 = 1 - \beta(1-\delta)(1+\alpha) + \alpha\beta(1-\delta) = 1 - \beta(1-\delta) > 0$.
The inequality holds for all admissible values $0 < \beta < 1,\ 0 < \delta < 1$.

(ii) $1 - p_1 + p_2 = 1 + \beta(1-\delta)(1+\alpha) + \alpha\beta(1-\delta) = 1 + \beta(1-\delta)(1+2\alpha) > 0$.
Since $(1+2\alpha) > 0$, $(1-\delta) > 0$ the inequality holds for all admissible values $\alpha > 0,\ 0 < \beta < 1,\ 0 < \delta < 1$.

(iii) $1 - p_2 = 1 - \alpha\beta(1-\delta) > 0$.
The inequality holds only for values $\alpha\beta(1-\delta) < 1$.

Theorem 2.6. *Let $\alpha > 0, 0 < \beta < 1, 0 < \delta < 1$. Then the equilibrium point*

$$\hat{Y} = \frac{G - \beta\gamma}{1 - \beta(1 - \delta)}$$

of the modified Samuelson model (15) *is asymptotically stable if and only if*

$$\alpha\beta(1 - \delta) < 1. \tag{18}$$

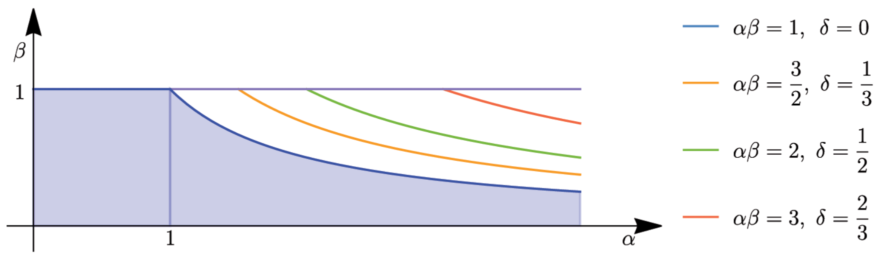

Figure 3. The stability region of the modified Samuelson model (15) for several values of δ. In the absence of income tax, $\delta = 0$, the stability region depends only on the parameters α, β and is blue. The stability region increases with increasing δ increases.

Interestingly, although the equilibrium depends on the value of γ, this non-income tax does not affect the stability region of the equilibrium. Moreover, the non-income tax must not be too high, because to achieve positive national income, the product $\beta\gamma$ must not exceed government spending G. In this case, the stability of the economy, i.e., the relationship between the consumption parameter and the accelerator, is affected by the income tax parameter δ.

When $\delta = 0$, i.e., in the absence of an income tax, in order to maintain the stability of the equilibrium point, the relationship between the consumption parameter and the accelerator needs to be, as in the basic model, $\alpha < \frac{1}{\beta}$. When $\delta \neq 0$, the condition $0 < \delta < 1$ implies $\frac{1}{1-\delta} > 1$, and since the condition

$$\alpha < \frac{1}{\beta}\frac{1}{1 - \delta}$$

is satisfied, this implies that the stability region increases as δ increases, that is, the income tax has a positive effect on government spending. For several values of δ, this can be seen in Fig. 3. The region of stability depending on the parameters α, β and δ is shown in Fig. 4.

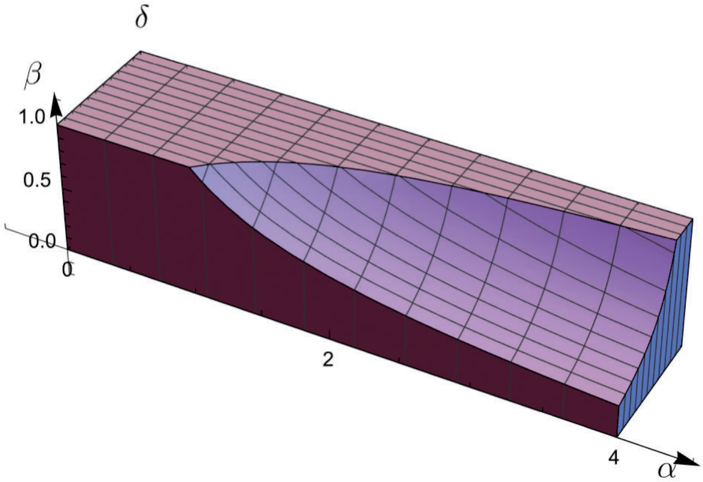

FIGURE 4. The stability region of the Samuelson model (15) with the addition of a tax depending on the parameters α, β and δ.

2.3. Multiplier-accelerator model with tax addition and investment period shift

We now include both previous modifications of Samuelson model simultaneously to obtain the system of difference equations

$$\begin{aligned} Y_t &= C_t + I_t + G, \\ C_t &= \beta(1-\delta)Y_{t-1} - \beta\gamma, \\ I_t &= \alpha(C_{t-1} - C_{t-2}), \end{aligned} \tag{19}$$

where $\alpha > 0, 0 < \beta < 1, \gamma > 0,\ 0 < \delta < 1$, or in the form of the third order difference equation

$$Y_{t+3} - \beta(1-\delta)Y_{t+2} - \alpha\beta(1-\delta)Y_{t+1} + \alpha\beta(1-\delta)Y_t = G - \gamma\beta \tag{20}$$

with equilibrium point

$$\hat{Y} = \frac{G - \gamma\beta}{1 - \beta(1-\delta)}. \tag{21}$$

The equilibrium point is the same as in the previous case. The accelerator α still has no effect on the equilibrium point as in the previous models.

We will investigate the stability of the equilibrium point using the Schur-Cohn stability criterion applying Corollary 2.4. We will investigate for which values of parameters $\alpha, \beta, \gamma, \delta$ the conditions stated there are satisfied.

(i) $1 + p_1 + p_2 + p_3 = 1 - \beta(1-\delta) - \alpha\beta(1-\delta) + \alpha\beta(1-\delta) = 1 - \beta(1-\delta) > 0.$
It can be seen that the inequality is identical to the no-shift model and holds for all admissible values $0 < \beta < 1,\ 0 < \delta < 1$.

(ii) $1 - p_1 + p_2 - p_3 = 1 + \beta(1-\delta) - \alpha\beta(1-\delta) - \alpha\beta(1-\delta) = 1 + \beta(1-\delta)(1-2\alpha) > 0.$
So, the inequality holds only for values

$$\beta(1-\delta)(2\alpha-1) < 1, \tag{22}$$

from where $\beta(1-\delta) < \dfrac{1}{2\alpha-1}$.

Therefore, since the multiplication $\beta(1-\delta)$ is positive, the inequality

$$0 < \beta(1-\delta) < \frac{1}{2\alpha-1}$$

must hold, i.e., $\alpha > \frac{1}{2}$.

(iii) $1 + p_2 - p_3(p_1 + p_3) = 1 - \alpha\beta(1-\delta)\big[\beta(1-\delta)(\alpha-1) + 1\big] > 0,$
and
$1 - p_2 + p_3(p_1 - p_3) = 1 - \alpha\beta(1-\delta)\big[\beta(1-\delta)(1+\alpha) - 1\big] > 0.$

Since $0 < \beta(1-\delta) < 1$, then $\beta(1-\delta) - 1 < 1 - \beta(1-\delta)$ and the inequalities in (iii) can be written as

$$\alpha\beta(1-\delta)\big[\alpha\beta(1-\delta) + \beta(1-\delta) - 1\big] < \alpha\beta(1-\delta)\big[\alpha\beta(1-\delta) - \beta(1-\delta) + 1\big] < 1.$$

It is easy to show that satisfying both of these inequalities follows from inequality (22). The stability region as a function of the parameters α, β and δ is shown in Fig. 5.

Theorem 2.7. *Let $\alpha > 0, 0 < \beta < 1, 0 < \delta < 1$. Then the equilibrium point $\hat{Y} = \frac{G-\beta\gamma}{1-\beta(1-\delta)}$ of the modified Samuelson model* (19) *is asymptotically stable if and only if*

$$\frac{1}{2} < \alpha < \frac{1+\beta(1-\delta)}{2\beta(1-\delta)}.$$

Even in this combined case, the equilibrium does not depend on the accelerator α but on the value of γ, although this non-income tax does not affect the stability region of the equilibrium. Moreover, as in the model with tax addition, the non-income tax must not be too high, because to achieve positive national income, the product $\beta\gamma$ must not exceed government spending G.
The region of economic stability, i.e., the relationship between the consumption parameter β and the accelerator α is affected by the income tax parameter δ so that the product $\beta(1-\delta)$ appears everywhere instead of β, as it did in model (9). When $\delta = 0$, i.e., in the absence of income tax, the maintenance of equilibrium stability is identical to the relations in model (9), i.e., $\beta(2\alpha - 1) < 1$.
If $\delta \neq 0$, instead of α in condition (18) for the stability of the equilibrium point

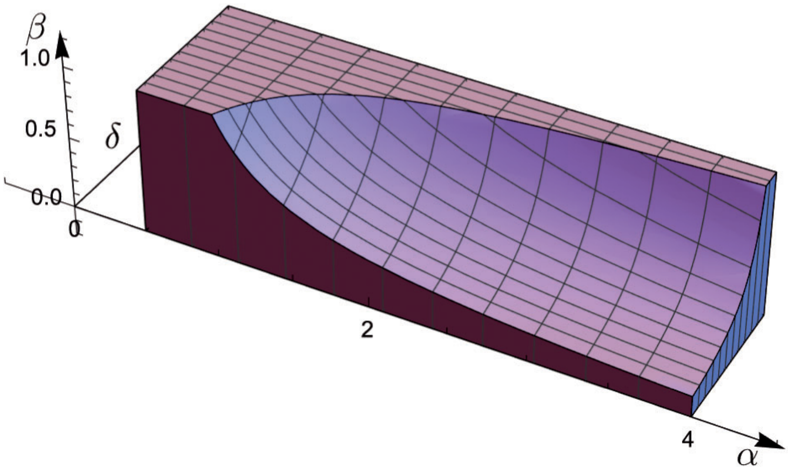

FIGURE 5. The stability region of the modified Samuelson model with tax addition and investment period shift (19) as a function of the parameters α, β and δ.

to model (15), there will be $2\alpha - 1$, and as the previous considerations show, this expression must remain positive, that is, $\alpha > \frac{1}{2}$.

2.4. Multiplier-accelerator model with foreign trade

Most modern economies are open, which means that national income is additionally influenced by foreign trade. In the standard Keynesian model of national income plus net exports, the effect of exogenous exports on national income is positive. In this model, compared to the standard model, we modify condition (1) and retain conditions (2), (3) and (4). Condition (1) will thus take the form

- National income Y_t in year t equals consumption C_t in year t, investment I_t in year t, government spending G_t in year t remains constant, $G_t = G$ and exports X_t in year t remain constant minus imports M_t in year t:

$$Y_t = C_t + I_t + G + X - M_t,$$

where imports M_t in year t are part of national income

$$M_t = mY_t, \quad 0 < m < 1.$$

Therefore, the model takes the form

$$\begin{aligned} Y_t &= C_t + I_t + G + X - mY_t, \\ C_t &= \beta Y_{t-1}, \\ I_t &= \alpha(C_t - C_{t-1}), \quad \alpha > 0, \quad 0 < \beta < 1, \quad 0 < m < 1 \end{aligned} \tag{23}$$

and can be modified to the second-order difference equation

$$Y_{t+2} - \frac{\beta(1+\alpha)}{1+m}Y_{t+1} + \frac{\alpha\beta}{1+m}Y_t = \frac{G+X}{1+m} \tag{24}$$

with the equilibrium point

$$\hat{Y} = \frac{G+X}{1+m-\beta}.$$

The equilibrium point therefore depends on the export of X and also on imports, i.e., on the multiplier m, but the accelerator α again plays no role. If foreign trade were to completely disappear, i.e., $X = 0$ and $m = 0$, then it is obvious that the model and its equilibrium point take their standard form. The equilibrium point will be the same as in the standard model even if foreign trade does not disappear, but if imports equal exports, i.e., if $X = M_t$. The most favourable situation occurs when exports exceed imports $X > M_t$, that is, when net exports are positive.

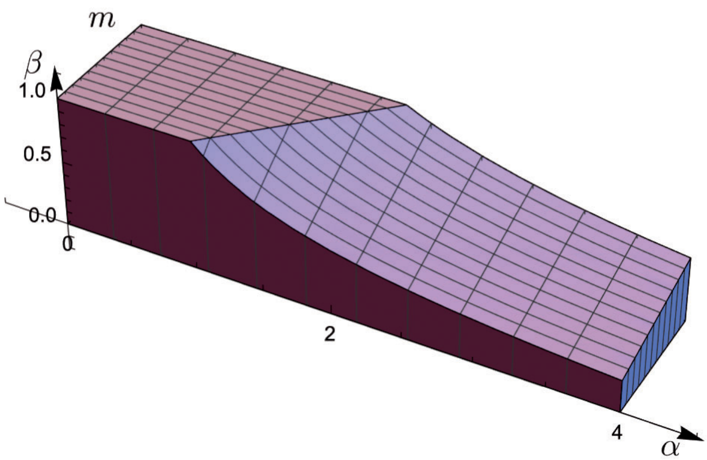

FIGURE 6. The stability region of the modified Samuelson model with foreign trade (19) as a function of the parameters α, β and δ.

The stability of the equilibrium point will be investigated using the Schur-Cohn stability criterion, i.e., according to Corollary 2.3.

(i) $1 + p_1 + p_2 = 1 - \frac{\beta(1+\alpha)}{1+m} + \frac{\alpha\beta}{1+m} > 0$, thus $\frac{1+m-\beta}{1+m} > 0$. Since $1 + m > \beta$, the inequality holds for all admissible values of $0 < \beta < 1$, $0 < m < 1$.

(ii) $1 - p_1 + p_2 = 1 + \frac{\beta(1+\alpha)}{1+m} + \frac{\alpha\beta}{1+m} > 0$. It is obvious, the inequality holds for all admissible values of $\alpha > 0,\ 0 < \beta < 1,\ 0 < m < 1$.

(iii) $1 - p_2 = 1 - \dfrac{\alpha\beta}{1+m} > 0.$

The inequality only holds for values of $\alpha\beta < 1 + m$.

The stability region as a function of the parameters α, β and m is shown in figure 6.

Theorem 2.8. *Let $\alpha > 0, 0 < \beta < 1, 0 < \delta < 1$. Then the equilibrium point*

$$\hat{Y} = \frac{G - \beta\gamma}{1 - \beta(1-\delta)}$$

of the modified Samuelson model (23) *is asymptotically stable if and only if*

$$\alpha\beta < 1 + m. \tag{25}$$

2.5. Multiplier-accelerator model with foreign trade and taxes

The previous two modifications can be combined into one by adding conditions for taxes and foreign trade to the standard model. Thus, we modify conditions (1), (2) and keep conditions (3), (4). Condition (1) will be as in model (19).

- National income Y_t in year t equals consumption C_t in year t, investment I_t in year t, government spending G_t in year t remains constant, $G_t = G$, and exports X_t in year t remain constant minus imports M_t in year t:

  $$Y_t = C_t + I_t + G + X - M_t,$$

  where imports M_t in year t are part of national income

  $$M_t = mY_t, \quad 0 < m < 1.$$

Condition (2) will be as in model (15).

- Consumption C_t in the year t depends on the income for the previous year $t-1$, which has been reduced by the tax portion of the previous year $t-1$ and on the marginal propensity to consume parameter β, $0 < \beta < 1$:

  $$C_t = \beta(Y_{t-1} - T_{t-1}), \tag{26}$$

  where tax part T_t for year t is the sum of the income-independent tax γ and the income tax (which is part of income) with an income tax rate δ

  $$T_t = \gamma + \delta Y_t. \tag{27}$$

The multiplier-accelerator model, taking into account foreign trade and taxes, thus takes the form

$$\begin{aligned} Y_t &= C_t + I_t + G + X - mY_t,\\ C_t &= \beta(1-\delta)Y_{t-1} - \beta\gamma, \\ I_t &= \alpha(C_t - C_{t-1}), \end{aligned} \tag{28}$$

where $\alpha > 0, 0 < \beta < 1, \gamma > 0,\ 0 < \delta < 1,\ 0 < m < 1.$

The system (28) can be modified to the second-order difference equation

$$Y_{t+2} - \frac{(1-\delta)\beta(1+\alpha)}{1+m} Y_{t+1} + \frac{\alpha\beta(1-\delta)}{1+m} Y_t = \frac{G+X-\beta\gamma}{1+m} \tag{29}$$

with the equilibrium point

$$\hat{Y} = \frac{G+X-\beta\gamma}{1+m-\beta(1-\delta)}.$$

As in the previous models, the accelerator α does not affect the equilibrium point, but is affected by income tax, non-income tax, imports and exports. The tax and foreign trade augmented model reverts to the tax augmented form when imports equal exports, i.e., when $X = M_t$.

We investigate the stability of the equilibrium point using Schur-Cohn stability criterion, i.e., according to Corollary 2.3.

(i) $1 + p_1 + p_2 = 1 - \frac{\beta(1-\delta)(1+\alpha)}{1+m} + \frac{\alpha\beta(1-\delta)}{1+m} > 0,$
thus

$$\frac{1+m-\beta(1-\delta)}{1+m} > 0.$$

Since $\beta(1-\delta) < 1 < 1+m$, the inequality holds for all admissible values

$$0 < \beta < 1, \quad 0 < \delta < 1, \quad 0 < m < 1.$$

(ii) $1 - p_1 + p_2 = 1 + \frac{\beta(1-\delta)(1+\alpha)}{1+m} + \frac{\alpha\beta(1-\delta)}{1+m} > 0,$
thus

$$1 + \frac{\beta(1-\delta)(1+2\alpha)}{1+m} > 0.$$

It is obvious, the inequality holds for all admissible values

$$\alpha > 0, \quad 0 < \beta < 1, \quad 0 < \delta < 1, \quad 0 < m < 1.$$

(iii) $1 - p_2 = 1 - \frac{\alpha\beta(1-\delta)}{1+m} > 0.$
The inequality holds only for values $\alpha\beta(1-\delta) < 1+m$.

The region of stability depending on the parameters α, β and δ is shown in Fig. 7 for certain values of m.

Theorem 2.9. *Let $\alpha > 0, 0 < \beta < 1, 0 < \delta < 1$. Then the equilibrium point*

$$\hat{Y} = \frac{G-\beta\gamma}{1-\beta(1-\delta)}$$

of the modified Samuelson model (28) *is asymptotically stable if and only if*

$$\alpha\beta(1-\delta) < 1+m. \tag{30}$$

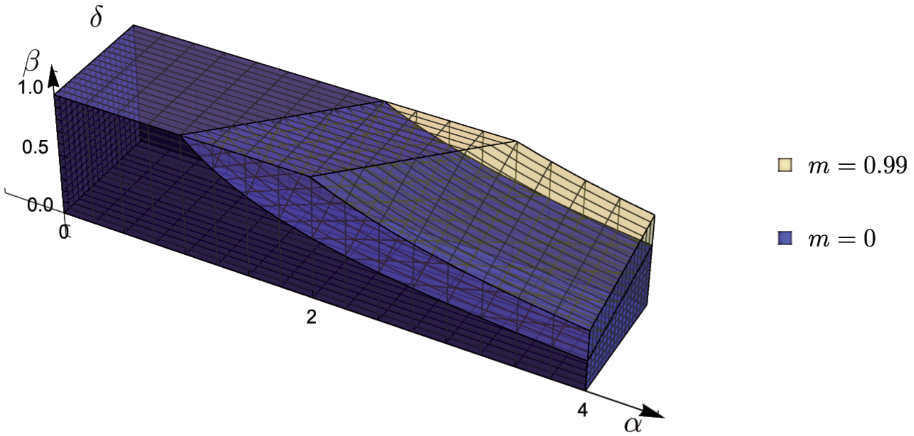

FIGURE 7. The stability region of the modified Samuelson model with foreign trade and taxes (28) depending on the parameters α, β and δ for certain values of m.

3. Conclusions

Analysis of modifications of the basic Samuelson model showed that the relationship between the multiplier β and the accelerator α remains similar to that of the basic model. The equilibrium point in neither model is affected by the accelerator as in the basic model, but the equilibrium point is affected by the non-income tax γ and the income tax rate δ in the tax added model, and by the exports X and the imports coefficient m in the foreign trade model. The time shift in investment did not affect the equilibrium point, but caused a change in the boundary of its stability region by the condition $\alpha > \frac{1}{2}$.

Our investigation further implies that in models with tax added, the non-income tax γ must not be too high, because to achieve positive national income, the product $\beta\gamma$ must not exceed government spending G. The income tax δ affects the stability region in such a way that the multiplier β is everywhere replaced by the product $\beta(1-\delta)$. Thus, we can conclude that investment, government expenditure and export through their multipliers have a favourable multiplier effect on national income.

Acknowledgement. The authors has been supported by the Polish Ministry of Science and Higher Education under a subsidy for maintaining the research potential of the Faculty of Mathematics, University of Białystok, Poland.

REFERENCES

[1] ABD AL-RZAQ S. A.—ABDULHASAN, A. J.: *Using the Samuelson model to assess the French economy's business cycle*: A Mathematical Study, Turkish J. Comput. Math. Education (TURCOMAT) **12** (2021), no. 7, 3242–3249. https://doi.org/10.17762/turcomat.v12i7.4401.

[2] ELAYDI, S.: *An introduction to Difference Equations*, 3rd ed. In: *Undergraduate Texts in Mathematics*. Springer, New York, 2005.

[3] LUO, Y.: *Discrete time or continuous time, That is the question: the case of Samuelson's multiplier-accelerator model*, Munich Personal RePEc Archive MPRA (posted October 25, 2016 14:13 UTC) https://mpra.ub.uni-muenchen.de/74738/1/MPRA_paper_74738.pdf

[4] PIIROINEN, P. T.—RAGHAVENDRA, S.: *A Nonsmooth Extension of Samuelson's Multiplier-Accelerator Model*, International J. of Bifurcation and Chaos **29** (2019), no. 10, 15 pp. https://doi.org/10.1142/S0218127419300271

[5] SAMUELSON, P. A.: *Interactions between the multiplier analysis and the principle of acceleration*, *Review of Economics and Statistics* **21** (1939), no. 2, 75–78. DOI: 10.2307/1927758. https://www.jstor.org/stable/1927758?origin=crossref

[6] PENEVA TODOROVA, T. P.—KUTROLLI, M.: , *An expanded multiplier-accelerator model*, Journal of Economics and Political Economy **6** (2019), no. 4, 368–382. https://www.researchgate.net/profile/Tamara-Todorova/publication/338829465_An_expanded_multiplier-accelerator_model/links/5e2d5a7b4585150ee783b8b8/An-expanded-multiplier-accelerator-model.pdf

Received January 6, 2023

Faculty of Mathematics
University of Białystok
K. Ciołkowskiego 1M
15–245 Białystok
POLAND

E-mail: paulina.naumowicz1@gmail.com
m.ruzickova@uwb.edu.pl