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A UNIFIED TREATMENT OF GENERALIZED CLOSED SETS IN TOPOLOGICAL SPACES

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ABSTRACT. This paper presents a general unified approach to the notions of generalized closedness in topological spaces. The research concerning the notion of generalized closed sets in topological spaces was initiated by Norman Levine in 1970. In the succeeding years, the concepts of this type of generalizations have been investigated in many versions using the standard generalizations of topologies which has resulted in a large body of literature. However, the methods and results in the past years have become standard and lacking in innovation.

The basic notion used in this conception is the closure operator designated by a family $\mathcal{B} \subseteq \mathcal{P}(X)$, which need not be a Kuratowski operator. Here, we introduce a general conception of natural extensions of families $\mathcal{B} \subseteq \mathcal{P}(X)$, denoted by $\mathcal{B} \triangleleft \mathcal{K}$, which are determined by other families $\mathcal{K} \subseteq \mathcal{P}(X)$. Precisely,

$$\mathcal{B} \triangleleft \mathcal{K} = \left\{ A \subseteq X : \overline{A}^{\mathcal{B}} \subseteq \overline{A}^{\mathcal{K}} \right\},$$

where $\overline{(\dots)}^{\mathcal{A}}$ denotes the closure operator designated by $\mathcal{A} \subseteq \mathcal{P}(X)$.

We prove that the collection of all generalizations $\mathcal{B} \triangleleft \mathcal{K}$, where $\mathcal{B}$, $\mathcal{K} \subseteq \mathcal{P}(X)$, forms a Boolean algebra. In this theory, the family of all generalized closed sets in a topological space $X(\mathcal{T})$ is equal to $\mathcal{C} \triangleleft \mathcal{T}$, where $\mathcal{C}$ is the family of all closed subsets of X. This concept gives tools that enable the systemizing and developing of the current research area of this topic. The results obtained in this general conception easily extend and imply well-known theorems as obvious corollaries. Moreover, they also give many new results concerning relationships between various types of generalized closedness studied so far in a topological space. In particular, we prove and demonstrate in a graph that in a topological space $X(\mathcal{T})$ there exist only nine different generalizations determined by the standard generalizations of topologies. The tools introduced in this paper enabled us to show that many generalizations studied in the literature are improper.

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1. Introduction

The literature shows that a considerable amount of work has been done on different forms of generalizations of open sets (dually, closed sets as their complements). Commonly, there are two approaches to this issue. The first one relies on the investigation of subsets more general than the open ones [1–3, 7, 8, 20, 68, 88] and [29]. In the second approach, there are used operators, which are more general than the Kuratowski ones, or families satisfying weaker assumptions than a topology. Then, one investigates the types of subsets corresponding to their counterparts in topological spaces [20, 25]. Works [5, 37] and [90] concern the unification of various concepts of some of the above-mentioned generalizations. Another unifying approach has been used in [103].

In this paper, we present a unified approach to the various generalizations of the concept of generalized closed sets defined in 1970 by Norman Levine [61]. This concept has been extended in many ways and used in many topics of mathematical research but it is not embedded into a unified theory.

Levine called a subset A of a topological space $X(\mathcal{T})$ generalized closed, for short g-closed, if

$$\mathrm{cl}(A) \subseteq U \text{ whenever } A \subseteq U \text{ and } U \subseteq X \text{ is open,}$$

where $\mathrm{cl}(A)$ denotes the closure of A in $X(\mathcal{T})$.

A subset A of the space $X(\mathcal{T})$ is called generalized open [61] if $X \setminus A$ is generalized closed.

In fact, the family of all g-closed subsets in the space $X(\mathcal{T})$ is determined by the pair $(\mathcal{C}, \mathcal{T})$ of the family $\mathcal{C}$ of all closed subsets of $X(\mathcal{T})$ and the topology $\mathcal{T}$, respectively. It is easy to see that the property of being a generalized closed subset $A \subseteq X$ is expressible by

$$\mathrm{cl}(A) \subseteq \overline{A}^{\mathcal{T}},$$

where

$$\overline{A}^{\mathcal{T}} = \bigcap \{U \in \mathcal{T} : A \subseteq U\}, \quad \text{i.e., } \overline{A}^{\mathcal{T}} = \ker(A).$$

So, the family of all generalized closed subsets in the topological space $X(\mathcal{T})$ can be understood as a family determined by the pair (ϕ, ϕ^*) of the operators, where $\phi(A) = \mathrm{cl}(A)$ and $\phi^*(A) = \overline{A}^{\mathcal{T}}$.

We will use the following well-known notion.

A function $\phi : \mathcal{P}(X) \to \mathcal{P}(X)$, where $\mathcal{P}(X)$ is the collection of all subset of X, satisfying the following three properties is called an algebraic closure operator [120]:

(i) ϕ is extensive: $A \subseteq \phi(A)$ for all $A \subseteq X$,

(ii) ϕ is idempotent: $\phi\big(\phi(A)\big) = \phi(A)$ for all $A \subseteq X$,

(iii) ϕ is isotone: $A \subseteq B$ implies $\phi(A) \subseteq \phi(B)$ for all $A, B \subseteq X$.

A pair $X(\phi)$, where ϕ satisfies the above properties, is called a closure space. A subset A of X is called closed with respect to ϕ if $\phi(A) \subseteq A$. The family of all such subsets is denoted by $\mathcal{C}^{\phi}$, i.e.,

$$\mathcal{C}_{\phi} = \{A \subseteq X : \phi(A) = A\}.$$

The properties (i) and (iii) imply that the family $\mathcal{C}^{\phi}$ is closed under arbitrary intersections. Ore [92, Theorem 1] has shown that

$$\phi(A) = \bigcap \{B \in \mathcal{C}^{\phi} : A \subseteq B\}$$

for any subset $A \subseteq X$.

In this paper, we will use the closure operators designated by the families $\mathcal{B} \subseteq \mathcal{P}(X)$, which we shall denote by $\overline{A}^{\mathcal{B}}$ for $A \subseteq X$. Then, the property of being a generalized closed subsets $A \subseteq X$, i.e., $\mathrm{cl}(A) \subseteq \overline{A}^{\mathcal{T}}$, will be denoted by $A \in \mathcal{C} \triangleleft \mathcal{T}$, where $\mathcal{C}$ is a family of all closed sets in the topological space $X(\mathcal{T})$. So, the family of all generalized closed sets in $X(\mathcal{T})$ will be understood as a family designated by the pair $(\mathcal{C}, \mathcal{T})$.

More precisely, in Section 2, we define the main construction of the paper. After some simple remarks, in Definition 2.11 we define a family $\mathcal{B} \triangleleft \mathcal{K}$ as

$$\mathcal{B} \triangleleft \mathcal{K} = \left\{A \subseteq X : \overline{A}^{\mathcal{B}} \subseteq \overline{A}^{\mathcal{K}}\right\}$$

for any minimal structures $\mathcal{B}, \mathcal{K} \subseteq \mathcal{P}(X)$.

In this section, we present a unified and useful technique and tools for investigations of generalizations of type $\mathcal{B} \triangleleft \mathcal{K}$ using the closure operators designated by the families $\mathcal{B} \subseteq \mathcal{P}(X)$, which need not be Kuratowski closure operators.

Many authors have generalized and extended the notion of g-closedness by using different types of families more general than $\mathcal{C}$ or $\mathcal{T}$. Such generalizations are based on the standard generalizations of topology. Therefore, let us recall some definitions where, as usual, $\mathrm{cl}(A)$ and $\mathrm{int}(A)$ denote the closure and the interior of A in $X(\mathcal{T})$, respectively.

For a topological space $X(\mathcal{T})$, a subset $A \subseteq X$ is called α-open [88], semi--open [62], pre-open [69] (or locally dense [19]), γ-open [30] (or b-open [8]) and β-open [1] if:

$A \subseteq \mathrm{int}\big(\mathrm{cl}(\mathrm{int}(A))\big)$,

$A \subseteq \mathrm{cl}(\mathrm{int}(A))$,

$A \subseteq \mathrm{int}(\mathrm{cl}(A))$,

$A \subseteq \mathrm{cl}(\mathrm{int}(A)) \cup \mathrm{int}(\mathrm{cl}(A))$ and

$A \subseteq \mathrm{cl}\big(\mathrm{int}(\mathrm{cl}(A))\big)$, respectively.

The family of all such subsets will be denoted by $\mathcal{O}^\alpha$, $\mathcal{O}^s$, $\mathcal{O}^p$, $\mathcal{O}^\gamma$ and $\mathcal{O}^\beta$, respectively. The union of all α-open (resp. semi-open, pre-open, γ-open, β-open) sets of X contained in A is called α-interior (resp. semi-interion, pre-interior, γ-interior, β-interior) of A and is denoted by $\alpha.\mathrm{int}(A)$ (resp. $\mathrm{s.int}(A)$, $\mathrm{p.int}(A)$, $\gamma.\mathrm{int}(A)$, $\beta.\mathrm{int}(A)$).

The complement of an α-open, semi-open, pre-open, γ-open and β –open is called α-closed, semi-closed, pre-closed, γ-closed and β –closed, respectively, i.e., a subset A such that:

$$\begin{aligned}
&\mathrm{cl}\big(\mathrm{int}(\mathrm{cl}(A))\big) \subseteq A,\\
&\mathrm{int}(\mathrm{cl}(A)) \subseteq A,\\
&\mathrm{cl}(\mathrm{int}(A)) \subseteq A,\\
&\mathrm{cl}(\mathrm{int}(A)) \cap \mathrm{int}(\mathrm{cl}(A)) \subseteq A \quad \text{and}\\
&\mathrm{int}\big(\mathrm{cl}(\mathrm{int}(A))\big) \subseteq A, \quad \text{respectively.}
\end{aligned}$$

The family of all such subsets is denoted by $\mathcal{C}^\alpha$ (resp. $\mathcal{C}^s$, $\mathcal{C}^p$, $\mathcal{C}^\gamma$, $\mathcal{C}^\beta$).

The intersection of all semi-closed (resp. pre-closed, α-closed, β-closed, γ--closed) sets of X containing A is called the semi-closure [20] (resp. pre--closure [31], α-closure [68], β-closure [2] or semi-pre-closure [7], γ-closure [35] or b-closure [8]) of A and is denoted by $\mathrm{scl}(A)$ (resp. $\mathrm{pcl}(A)$, $\alpha\mathrm{cl}(A)$, $\beta\mathrm{cl}(A)$, $\gamma\mathrm{cl}(A)$).

Most investigations on the issue concerning the generalizations $\mathcal{B}\triangleleft\mathcal{K}$ are based on the standard pairs $(\mathcal{B},\mathcal{K})$ of families, where $\mathcal{B} \in \{\mathcal{C},\mathcal{C}^\alpha,\mathcal{C}^p,\mathcal{C}^s,\mathcal{C}^\gamma,\mathcal{C}^\beta\}$ and $\mathcal{K} \in \{\mathcal{T},\mathcal{O}^\alpha,\mathcal{O}^p,\mathcal{O}^s,\mathcal{O}^\gamma,\mathcal{O}^\beta\}$.

In Section 3, we show that results in this topic are corollaries that follow from the general theory proposed in Section 2. In this section, we investigate the fundamental properties of the generalizations $\mathcal{B}\triangleleft\mathcal{K}$ via the closure operators generated by the families $\mathcal{B}$ and $\mathcal{K}$.

Theorems 2.22 and 2.24 present the specificity of the generalizations of type $\mathcal{B}\triangleleft\mathcal{K}$ in the space $\mathcal{P}(X)$. In Theorem 2.25, we give a necessary and sufficient condition for generalizations $\mathcal{B}\triangleleft\mathcal{K}$ to be not essential. Theorem 2.28 offers a tool useful in an investigation of the generalizations, which are designated by the family of regular open subsets in a topological space. Theorem 2.29 concerns the iterations of the operations of generalization. At the end of the section, we show that the set Γ of all families of the form $\mathcal{B}\triangleleft\mathcal{K}$ forms a Boolean algebra.

The results obtained in Section 2 are used in Section 3. They easily imply and generalize well-known theorems as obvious corollaries give many new results concerning relationships between various types of generalized closedness studied so far in a topological space. In particular, in a diagram we demonstrate that in a topological space $X(\mathcal{T})$ there exist only nine different generalizations determined by the standard generalizations of topologies. So, the generalizations listed in the definitions (7), (8), (10), (11), (12), (13), (15) and (16) turn out to be improper.

Applying results from Section 2, we obtain many former theorems which generalized previous results and list some new properties.

In the next section, again applying properties of families $\mathcal{B} \lhd \mathcal{K}$, we investigate and generalize closedness based on the notion of regularly semi open sets. This kind of investigation was initiated by N. Palaniappan [92] in 1993.

In the final section, we study the properties of the iterative use of the operations of generalization $\mathcal{B} \lhd \mathcal{K}$, i.e., where $\mathcal{K}$ is a family of generalized open sets itself. Using Theorem 2.29, we show that many generalizations of that type studied in the literature are improper (Theorem 5.1). Namely, that listed in the forthcoming definitions (2), (10), (12), (17), (19), (21), (22), (28), (31), (33), (34) and (37).

2. Unified approach

In this section, for any pair $(\mathcal{B}, \mathcal{K})$ of minimal structures of a topological space $X(\mathcal{T})$ we define a family denoted as $\mathcal{B} \lhd \mathcal{K}$ by using the closure operators generated by $\mathcal{B}$ and $\mathcal{K}$. This construction gives a very general method of determining various types of generalized closed sets. But first, we introduce the notion of a closure operator designated by a family of subsets of X.

In the present studies, we will use families that satisfy some minimal assumptions. Let us recall a definition.

A family $\mathcal{B} \subseteq \mathcal{P}(X)$ is called a minimal structure [96] on X if $\emptyset, X \in \mathcal{B}$.

Definition 2.1. For any minimal structure $\mathcal{B} \subseteq \mathcal{P}(X)$ we define an operator $\overline{(\dots)}^{\mathcal{B}} \colon \mathcal{P}(X) \to \mathcal{P}(X)$ by

$$\overline{A}^{\mathcal{B}} = \bigcap \{B \in \mathcal{B} : A \subseteq B\}$$

for all $A \subseteq X$.

We denote by $\overline{\mathcal{B}}$ the family of all fixed points of the operator $\overline{(\dots)}^{\mathcal{B}}$, i.e.,

$$\overline{\mathcal{B}} = \left\{ A \subseteq X : \overline{A}^{\mathcal{B}} = A \right\}.$$

Remark 2.2. It is easy to check that $\overline{A}^{\mathcal{B}} = A$ for every $A \in \mathcal{B}$. If a family $\mathcal{B}$ is closed under arbitrary intersections, then the property $\overline{A}^{\mathcal{B}} = A$ is equivalent to $A \in \mathcal{B}$.

Remark 2.3. The families $\mathcal{B}$ and $\overline{\mathcal{B}}$ are equivalent in the sense that, according to the Ore result cited above, they define the same closure operator, i.e.,

(i) $\overline{A}^{\mathcal{B}} = \overline{A}^{\overline{\mathcal{B}}}$ for all $A \subseteq X$. So, as a result, we have

(ii) $\overline{\overline{\mathcal{B}}} = \overline{\mathcal{B}}$.

Of course, according to Remark 2.2, we have $\mathcal{B} \subseteq \overline{\mathcal{B}}$ and the equality $\mathcal{B} = \overline{\mathcal{B}}$ holds if and only if the family $\mathcal{B}$ is closed under arbitrary intersection.

Remark 2.4. In the case $\mathcal{B} = \mathcal{C}$ (resp. $\mathcal{B} = \mathcal{C}^\alpha$, $\mathcal{B} = \mathcal{C}^s$, $\mathcal{B} = \mathcal{C}^p$, $\mathcal{B} = \mathcal{C}^\gamma$, $\mathcal{B} = \mathcal{C}^\beta$), we have $\overline{A}^{\mathcal{B}} = \mathrm{cl}(A)$ (resp. $\overline{A}^{\mathcal{B}} = \mathrm{scl}(A)$, $\overline{A}^{\mathcal{B}} = \mathrm{pcl}(A)$, $\overline{A}^{\mathcal{B}} = \alpha\mathrm{cl}(A)$, $\overline{A}^{\mathcal{B}} = \beta\mathrm{cl}(A)$, $\overline{A}^{\mathcal{B}} = \gamma\mathrm{cl}(A)$).

It is easy to check that the following property holds.

Lemma 2.5. *For any minimal structure $\mathcal{B} \subseteq \mathcal{P}(X)$, the operator $\overline{(\dots)}^{\mathcal{B}}$ is a closure operator.*

Proof. The isotonicity and extensivity are obvious. For the proof of idempotency, let $\mathcal{P}$ and $\mathcal{P}^*$ denote the families

$$\{K \subseteq X : K \in \mathcal{B}, A \subseteq K\} \quad \text{and} \quad \left\{K \subseteq X : K \in \mathcal{B}, \overline{A}^{\mathcal{B}} \subseteq K\right\},$$

respectively.

One can show that $\mathcal{P} \subseteq \mathcal{P}^*$. Indeed, if $K \in \mathcal{B}$ and $A \subseteq K$, then $\overline{A}^{\mathcal{B}} \subseteq \overline{K}^{\mathcal{B}}$, and according to Remark 2.2, $\overline{K}^{\mathcal{B}} = K$. So, $\overline{A}^{\mathcal{B}} \subseteq K$, i.e., $K \in \mathcal{P}^*$. Consequently, we have $\bigcap \mathcal{P}^* \subseteq \bigcap \mathcal{P}$, i.e., $\overline{\overline{A}^{\mathcal{B}}}^{\mathcal{B}} \subseteq \overline{A}^{\mathcal{B}}$. The converse inclusion is clear. □

It is convenient to use the notation $\mathcal{A}^{\mathbf{c}}$ to describe the collection

$$\{A \subseteq X : X \setminus A \in \mathcal{A}\}.$$

Lemma 2.6. *For any minimal structure $\mathcal{B} \subseteq \mathcal{P}(X)$ and a subset $A \subseteq X$, we have the following equivalence:*

$$x \in \overline{A}^{\mathcal{B}} \text{ if and only if for every } U \in \mathcal{B}^{\mathbf{c}},\ U \cap A \neq \emptyset \text{ whenever } x \in U.$$

Proof. Let $x \in \overline{A}^{\mathcal{B}}$. Assume, to the contrary, that $x \in U$ and $U \cap A = \emptyset$ for some $U \in \mathcal{B}^{\mathbf{c}}$. Then, we have $A \subseteq X \setminus U \in \mathcal{B}$ and by definition, $\overline{A}^{\mathcal{B}} \subseteq X \setminus U$. So, $U \cap \overline{A}^{\mathcal{B}} = \emptyset$, which gives $x \notin \overline{A}^{\mathcal{B}}$, and we obtain a contradiction.

Now, assume that $x \notin \overline{A}^{\mathcal{B}}$. Then, $x \notin K$ for some K such that $K \in \mathcal{B}$ and $A \subseteq K$. So, we obtain the subset $U = X \setminus K$ such that $U \in \mathcal{B}^{\mathbf{c}}$, $x \in U$ and $U \cap A = \emptyset$, which gives a contradiction and completes the proof. □

From the above property, we have the following

Corollary 2.7. *If $U \in \mathcal{B}^{\mathbf{c}}$ for some minimal structure $\mathcal{B} \subseteq \mathcal{P}(X)$, then $U \cap \overline{A}^{\mathcal{B}} \neq \emptyset$ if and only if $U \cap A \neq \emptyset$ for all $A \subseteq X$.*

Corollary 2.8. *If $\mathcal{B} \subseteq \mathcal{P}(X)$ is a minimal structure such that $\mathcal{B}^{\mathbf{c}} = \mathcal{B}$, then $x \in \overline{A}^{\mathcal{B}}$ if and only if for every $U \in \mathcal{B}$, $U \cap A \neq \emptyset$ whenever $x \in U$.*

Definition 2.9. For any minimal structures $\mathcal{B} \subseteq \mathcal{P}(X)$, we define the operator $\mathcal{B}.\operatorname{int}(\dots) : \mathcal{P}(X) \to \mathcal{P}(X)$ by

$$\mathcal{B}.\operatorname{int}(A) = \bigcup \{U : U \in \mathcal{B}^{\mathbf{c}} \quad \text{and} \quad U \subseteq A\},$$

for any $A \subseteq X$.

It is clear that $\mathcal{B}.\operatorname{int}(A) = A$ for any $A \in \mathcal{B}^{\mathbf{c}}$ and additionally, if a family $\mathcal{B} \subseteq \mathcal{P}(X)$ is closed under arbitrary intersections, then the property $\mathcal{B}.\operatorname{int}(A) = A$ is equivalent to $A \in \mathcal{B}^{\mathbf{c}}$.

In the case $\mathcal{B} = \mathcal{C}$ (resp. $\mathcal{C}^{\alpha}$, $\mathcal{C}^{s}$, $\mathcal{C}^{p}$, $\mathcal{C}^{\gamma}$, $\mathcal{C}^{\beta}$), we have $\mathcal{B}.\operatorname{int}(A) = \operatorname{int}(A)$ (resp. $\mathcal{B}.\operatorname{int}(A) = \alpha.\operatorname{int}(A)$, $\mathcal{B}.\operatorname{int}(A) = \text{s}.\operatorname{int}(A)$, $\mathcal{B}.\operatorname{int}(A) = \text{p}.\operatorname{int}(A)$, $\mathcal{B}.\operatorname{int}(A) = \gamma.\operatorname{int}(A)$, $\mathcal{B}.\operatorname{int}(A) = \beta\operatorname{int}(A)$).

Lemma 2.10. *For any family $\mathcal{B} \subseteq \mathcal{P}(X)$ and a subset $A \subseteq X$, we have*

$$\mathcal{B}.\operatorname{int}(A) = X \setminus \overline{X \setminus A}^{\mathcal{B}}.$$

P r o o f. Let $x_0 \in \mathcal{B}.\operatorname{int}(A)$, then there exists $U \in \mathcal{B}^{\mathbf{c}}$ such that $x_0 \in U$ and $U \subseteq A$. Assume, to the contrary, that $x_0 \notin X \setminus \overline{X \setminus A}^{\mathcal{B}}$. Then, $x_0 \in \overline{X \setminus A}^{\mathcal{B}}$ and, by Lemma 2.6, $U \cap (X \setminus A) \neq \emptyset$. So, $U \nsubseteq A$, which gives a contradiction.

Now, assume that $x_0 \in X \setminus \overline{X \setminus A}^{\mathcal{B}}$. Then, $x_0 \notin \overline{X \setminus A}^{\mathcal{B}}$, which according to Lemma 2.6, means that there exists a subset $U \in \mathcal{B}^{\mathbf{c}}$ such that $U \cap (X \setminus A) = \emptyset$ and $x_0 \in U$. So, we obtain $U \subseteq A$. Consequently, we have $x_0 \in \mathcal{B}.\operatorname{int}(A)$ which completes the proof. □

Now, we may give the most important definition in the whole paper.

Definition 2.11. For any pair $(\mathcal{B}, \mathcal{K})$ of minimal structures, we define

$$\mathcal{B} \lhd \mathcal{K} = \left\{ A \subseteq X : \overline{A}^{\mathcal{B}} \subseteq \overline{A}^{\mathcal{K}} \right\}.$$

Because of Remark 2.12, the family $\mathcal{B} \lhd \mathcal{K}$ will be said to be a generalization of $\mathcal{B}$ by $\mathcal{K}$.

The collections of all families of type $\mathcal{B} \lhd \mathcal{K}$ will be denoted by $\Gamma(X)$.

Remark 2.12. In a topological space, the family $\mathcal{C} \lhd \mathcal{T}$ is exactly the family of generalized closed subsets of X. And, the elements of $(\mathcal{C} \lhd \mathcal{T})^{\mathbf{c}}$ are generalized open subsets.

Remark 2.13. It is obvious that $A \in \mathcal{B} \lhd \mathcal{K}$ if and only if

$$\overline{A}^{\mathcal{B}} \subseteq U \text{ whenever } A \subseteq U \text{ and } U \in \mathcal{K}.$$

The following lemma states some useful properties of families of $\Gamma(X)$.

Lemma 2.14. *For any family* $\mathcal{B}\triangleleft\mathcal{K} \in \Gamma(X)$ *and any subset* $A \subseteq X$*, the following assertions hold:*

(i) *If* $\mathcal{K} \subseteq \mathcal{B}$*, then* $\mathcal{B} \triangleleft \mathcal{K} = \mathcal{P}(X)$*,*

(ii) $\mathcal{B} \triangleleft \mathcal{K} \subseteq \mathcal{B}^* \triangleleft \mathcal{K}$ *for any* $\mathcal{B}^* \subseteq \mathcal{P}(X)$ *such that* $\mathcal{B} \subseteq \mathcal{B}^*$*,*

(iii) $\mathcal{B} \triangleleft \mathcal{K} \supseteq \mathcal{B} \triangleleft \mathcal{K}^*$ *for any* $\mathcal{K}^* \subseteq \mathcal{P}(X)$ *such that* $\mathcal{K} \subseteq \mathcal{K}^*$*,*

(iv) $\overline{\mathcal{B}} \subseteq \mathcal{B} \triangleleft \mathcal{K}$*,*

(v) $\overline{\mathcal{B}} = \bigcap \{\mathcal{B} \triangleleft \mathcal{K} : \mathcal{K} \subseteq \mathcal{P}(X)\} = \mathcal{B} \triangleleft \mathcal{P}(X)$*,*

(vi) *If* $\mathcal{B} \subseteq \mathcal{K}$*, then* $A \in \mathcal{B} \triangleleft \mathcal{K}$ *if and only if* $\overline{A}^{\mathcal{B}} = \overline{A}^{\mathcal{K}}$*,*

(vii) $\mathcal{B} \triangleleft \mathcal{K} = \overline{\mathcal{B}} \triangleleft \mathcal{K} = \mathcal{B} \triangleleft \overline{\mathcal{K}} = \overline{\mathcal{B}} \triangleleft \overline{\mathcal{K}}$ *for all* $\mathcal{K} \subseteq \mathcal{P}(X)$*.*

P r o o f. Property (i) follows from the fact that $\mathcal{K} \subseteq \mathcal{B}$ implies $\overline{A}^{\mathcal{B}} \subseteq \overline{A}^{\mathcal{K}}$ for any $A \subseteq X$, which means that $A \in \mathcal{B} \triangleleft \mathcal{K}$ for any $A \subseteq \mathcal{P}(X)$.

For the proof of (ii), let us observe that $\overline{A}^{\mathcal{B}^*} \subseteq \overline{A}^{\mathcal{B}}$ and thus, the property $A \in \mathcal{B} \triangleleft \mathcal{K}$ implies that $\overline{A}^{\mathcal{B}^*} \subseteq \overline{A}^{\mathcal{K}}$, i.e., $A \in \mathcal{B}^* \triangleleft \mathcal{K}$.

(iii). If $\mathcal{K} \subseteq \mathcal{K}^*$, then $\overline{A}^{\mathcal{K}^*} \subseteq \overline{A}^{\mathcal{K}}$ for any $A \subseteq X$. So, for any $A \in \mathcal{B} \triangleleft \mathcal{K}^*$, we have $\overline{A}^{\mathcal{B}} \subseteq \overline{A}^{\mathcal{K}^*} \subseteq \overline{A}^{\mathcal{K}}$. Consequently, $A \in \mathcal{B} \triangleleft \mathcal{K}$.

(iv). Assume that $A \in \overline{\mathcal{B}}$. Then, $\overline{A}^{\mathcal{B}} = A$ and for every family $\mathcal{K} \subseteq \mathcal{P}(X)$, we have $A \subseteq \overline{A}^{\mathcal{K}}$. So, $\overline{A}^{\mathcal{B}} \subseteq \overline{A}^{\mathcal{K}}$, i.e., $A \in \mathcal{B} \triangleleft \mathcal{K}$, which completes the proof.

Property (v) follows immediately from (iv).

(vi). If $A \in \mathcal{B} \triangleleft \mathcal{K}$, then $\overline{A}^{\mathcal{B}} \subseteq \overline{A}^{\mathcal{K}}$ and from the fact that $\mathcal{B} \subseteq \mathcal{K}$, we have $\overline{A}^{\mathcal{K}} \subseteq \overline{A}^{\mathcal{B}}$. So, $\overline{A}^{\mathcal{K}} = \overline{A}^{\mathcal{B}}$. The converse inclusion is clear.

Property (vii) follows directly from Remark 2.3. □

Theorem 2.15. *For any nonempty set X, the collection* $\Gamma(X)$ *is a Boolean algebra under the following operations:*

$$\begin{aligned} \textit{union}: &\quad (\mathcal{B}_1 \triangleleft \mathcal{K}_1) \oplus (\mathcal{B}_2 \triangleleft \mathcal{K}_2) = (\mathcal{B}_1 \cup \mathcal{B}_2) \triangleleft (\mathcal{K}_1 \cap \mathcal{K}_2), \\ \textit{intersection}: &\quad (\mathcal{B}_1 \triangleleft \mathcal{K}_1) \odot (\mathcal{B}_2 \triangleleft \mathcal{K}_2) = (\mathcal{B}_1 \cap \mathcal{B}_2) \triangleleft (\mathcal{K}_1 \cup \mathcal{K}_2), \\ \textit{negation}: &\quad (\mathcal{B} \triangleleft \mathcal{K})' = \big(\mathcal{P}(X) \setminus (\mathcal{B} \setminus \{\emptyset, X\})\big) \triangleleft \big(\mathcal{P}(X) \setminus (\mathcal{K} \setminus \{\emptyset, X\})\big). \end{aligned}$$

P r o o f. First, let us observe that the families of types $\mathcal{B}_1 \cup \mathcal{B}_2$, $\mathcal{B}_1 \cap \mathcal{B}_2$, $\mathcal{P}(X) \setminus (\mathcal{B}_1 \setminus \{\emptyset, X\})$, where $\mathcal{B}_1$, $\mathcal{B}_2 \subseteq \mathcal{P}(X)$, contain $\emptyset$ and X. So, they fulfil the requirements of Definition 2.11 and the collection $\Gamma(X)$ is closed under the operations $\oplus$, $\odot$, $()'$.

The commutative and associative laws are automatically fulfilled.

Let us check that the family $\mathbf{0} = \{\emptyset, X\} \triangleleft \mathcal{P}(X)$ is the zero element of $\Gamma(X)$ and $\mathbf{1} = \mathcal{P}(X) \triangleleft \{\emptyset, X\}$ is the unity element of $\Gamma(X)$.

Indeed, for $\mathcal{B}$,$\mathcal{K} \subseteq \mathcal{P}(X)$, we have

$(\mathcal{B} \lhd \mathcal{K}) \oplus \mathbf{0} = (\mathcal{B} \lhd \mathcal{K}) \oplus \big(\{\emptyset, X\} \lhd \mathcal{P}(X)\big) = (\mathcal{B} \cup \{\emptyset, X\}) \lhd \big(\mathcal{K} \cap \mathcal{P}(X)\big) = \mathcal{B} \lhd \mathcal{K}$
and
$(\mathcal{B} \lhd \mathcal{K}) \odot \mathbf{1} = (\mathcal{B} \lhd \mathcal{K}) \odot \big(\mathcal{P}(X) \lhd \{\emptyset, X\}\big) = \big(\mathcal{B} \cap \mathcal{P}(X)\big) \lhd \big(\{\emptyset, X\} \cup \mathcal{K}\big) = \mathcal{B} \lhd \mathcal{K}$.

Next, for any $\mathcal{B}$,$\mathcal{K} \subseteq \mathcal{P}(X)$, we have:

- $$\begin{aligned}(\mathcal{B} \lhd \mathcal{K}) \oplus (\mathcal{B} \lhd \mathcal{K})' &= (\mathcal{B} \lhd \mathcal{K}) \oplus \Big(\big(\{\emptyset, X\} \cup (\mathcal{P}(X) \setminus \mathcal{B})\big) \lhd \big(\{\emptyset, X\} \cup (\mathcal{P}(X) \setminus \mathcal{K})\big)\Big) \\ &= \Big(\mathcal{B} \cup \big(\{\emptyset, X\} \cup (\mathcal{P}(X) \setminus \mathcal{B})\big)\Big) \lhd \Big(\mathcal{K} \cap \big(\{\emptyset, X\} \cup (\mathcal{P}(X) \setminus \mathcal{K})\big)\Big) \\ &= \mathcal{P}(X) \lhd \{\emptyset, X\} = \mathbf{1},\end{aligned}$$

and

- $$\begin{aligned}(\mathcal{B} \lhd \mathcal{K}) \odot (\mathcal{B} \lhd \mathcal{K})' &= (\mathcal{B} \lhd \mathcal{K}) \odot \Big(\big(\{\emptyset, X\} \cup (\mathcal{P}(X) \setminus \mathcal{B})\big) \lhd \big(\{\emptyset, X\} \cup (\mathcal{P}(X) \setminus \mathcal{K})\big)\Big) \\ &= \big(\mathcal{B} \cap (\{\emptyset, X\} \cup (\mathcal{P}(X) \setminus \mathcal{B}))\big) \lhd \Big(\mathcal{K} \cup \big(\{\emptyset, X\} \cup (\mathcal{P}(X) \setminus \mathcal{K})\big)\Big) \\ &= \{\emptyset, X\} \lhd \mathcal{P}(X) = \mathbf{0}.\end{aligned}$$

For the proof of the distributivity, let us take $\mathcal{B}_1 \lhd \mathcal{K}_1$, $\mathcal{B}_2 \lhd \mathcal{K}_2$, $\mathcal{B}_3 \lhd \mathcal{K}_3$. Then,

- $$\begin{aligned}&(\mathcal{B}_1 \lhd \mathcal{K}_1) \odot \big((\mathcal{B}_2 \lhd \mathcal{K}_2) \oplus (\mathcal{B}_3 \lhd \mathcal{K}_3)\big) \\ &\quad= (\mathcal{B}_1 \lhd \mathcal{K}_1) \odot \big((\mathcal{B}_2 \cup \mathcal{B}_3) \lhd (\mathcal{K}_2 \cap \mathcal{K}_3)\big) \\ &\quad= \big(\mathcal{B}_1 \cap (\mathcal{B}_2 \cup \mathcal{B}_3)\big) \lhd \big(\mathcal{K}_1 \cup (\mathcal{K}_2 \cap \mathcal{K}_3)\big) \\ &\quad= \big((\mathcal{B}_1 \cap \mathcal{B}_2) \cup (\mathcal{B}_1 \cap \mathcal{B}_3)\big) \lhd \big((\mathcal{K}_1 \cup \mathcal{K}_2) \cap (\mathcal{K}_1 \cup \mathcal{K}_3)\big),\end{aligned}$$

and

- $$\begin{aligned}&\big((\mathcal{B}_1 \lhd \mathcal{K}_1) \odot (\mathcal{B}_2 \lhd \mathcal{K}_2)\big) \oplus \big((\mathcal{B}_1 \lhd \mathcal{K}_1) \odot (\mathcal{B}_3 \lhd \mathcal{K}_3)\big) \\ &\quad= \big((\mathcal{B}_1 \cap \mathcal{B}_2) \lhd (\mathcal{K}_1 \cup \mathcal{K}_2)\big) \oplus \big((\mathcal{B}_1 \cap \mathcal{B}_3) \lhd (\mathcal{K}_1 \cup \mathcal{K}_3)\big) \\ &\quad= \big((\mathcal{B}_1 \cap \mathcal{B}_2) \cup (\mathcal{B}_1 \cap \mathcal{B}_3)\big) \lhd \big((\mathcal{K}_1 \cup \mathcal{K}_2) \cap (\mathcal{K}_1 \cup \mathcal{K}_3)\big).\end{aligned}$$

For the same reason,

$$(\mathcal{B}_1 \lhd \mathcal{K}_1) \oplus \big((\mathcal{B}_2 \lhd \mathcal{K}_2) \odot (\mathcal{B}_3 \lhd \mathcal{K}_3)\big) = \big((\mathcal{B}_1 \lhd \mathcal{K}_1) \oplus (\mathcal{B}_2 \lhd \mathcal{K}_2)\big) \odot \big((\mathcal{B}_1 \lhd \mathcal{K}_1) \oplus (\mathcal{B}_3 \lhd \mathcal{K}_3)\big).$$

So, the proof is completed. □

Now, we present technical properties (Lemma 2.16, Lemma 2.18, Corollary 2.19, Lemma 2.20, Lemma 2.21), which will be used without being explicitly referred to.

Lemma 2.16. *For any family $\mathcal{B} \lhd \mathcal{K} \in \Gamma(X)$ and $A \subseteq X$, the property $A \in (\mathcal{B} \lhd \mathcal{K})^{[c]}$ is equivalent to $\mathcal{B}.\mathrm{int}(A) \supseteq \mathcal{K}.\mathrm{int}(A)$.*

P r o o f. Assume that $A \in (\mathcal{B} \lhd \mathcal{K})^{\mathbf{c}}$, i.e., according to Remark 2.13, we have $\overline{X \setminus A}^{\mathcal{B}} \subseteq U$ for any $U \in \mathcal{K}$ such that $X \setminus A \subseteq U$. Let $x \in \mathcal{K}.\mathrm{int}(A)$, then there exists a subset $P \subseteq A$ such that $x \in P$ and $P \in \mathcal{K}^{\mathbf{c}}$. We will show that $x \in \mathcal{B}.\mathrm{int}(A)$.

Since $X \setminus P \in \mathcal{K}$ and $X \setminus A \subseteq X \setminus P$, then by the assumption we get $\overline{X \setminus A}^{\mathcal{B}} \subseteq X \setminus P$. So, $P \subseteq X \setminus \overline{X \setminus A}^{\mathcal{B}}$, i.e., according to Lemma 2.10, $P \subseteq \mathcal{B}.\mathrm{int}(A)$. This means that $x \in \mathcal{B}.\mathrm{int}(A)$ and completes the first part of the proof.

Assume now that $\mathcal{B}.\mathrm{int}(A) \supseteq \mathcal{K}.\mathrm{int}(A)$, i.e., for every $P \in \mathcal{K}^{\mathbf{c}}$ such that $P \subseteq A$ we have $P \subseteq \mathcal{B}.\mathrm{int}(A)$. We will show that $A \in (\mathcal{B} \lhd \mathcal{K})^{\mathbf{c}}$, i.e., $\overline{X \setminus A}^{\mathcal{B}} \subseteq U$ for any $U \in \mathcal{K}$ such that $X \setminus A \subseteq U$.

Suppose, to the contrary, that $\overline{X \setminus A}^{\mathcal{B}} \not\subseteq U$ for some $U \in \mathcal{K}$ such that $X \setminus A \subseteq U$. Then, according to Lemma 2.6, there exists $p \notin U$ such that $V \cap (X \setminus A) \neq \emptyset$ for every $V \in \mathcal{B}^{\mathbf{c}}$ with $p \in V$. So, $p \in X \setminus U \in \mathcal{K}^{\mathbf{c}}$ and $X \setminus U \subseteq A$, which gives $p \in \mathcal{K}.\mathrm{int}(A)$. But, $V \not\subseteq A$ for every $V \in \mathcal{B}^{\mathbf{c}}$ such that $p \in V$. Consequently, $p \notin \mathcal{B}.\mathrm{int}(A)$ and the proof is completed. □

Remark 2.17. Let us note two immediate consequences of the definitions of the operators $\overline{(\dots)}^{\mathcal{B}}$ and $\mathcal{B}.\mathrm{int}(\dots)$. For any family $\mathcal{B} \subseteq \mathcal{P}(X)$ and a subset $A \subseteq X$ we have:

(i) If $A \subseteq K$ and $K \in \mathcal{B}$, then $\overline{A}^{\mathcal{B}} \subseteq K$, and

(ii) If $U \subseteq A$ and $U \in \mathcal{B}^{\mathbf{c}}$, then $U \subseteq \mathcal{B}.\mathrm{int}(A)$.

Lemma 2.18. *For any minimal structure $\mathcal{B} \subseteq \mathcal{P}(X)$ and $A \subseteq X$, the followings hold:*

(i) *The set $\overline{A}^{\mathcal{B}} \setminus A$ contains no nonempty subset $U \in \mathcal{B}^{\mathbf{c}}$,*

(ii) *The set $A \setminus \mathcal{B}.\mathrm{int}(A)$ contains no nonempty subset $U \in \mathcal{B}^{\mathbf{c}}$.*

P r o o f. Property (i) follows immediately from Lemma 2.6, and property (ii) is a consequence of the definition of the operator $\mathcal{B}.\mathrm{int}(\dots)$. □

Corollary 2.19. *For any family $\mathcal{B} \lhd \mathcal{K} \in \Gamma(X)$ and $A \in \mathcal{B} \lhd \mathcal{K}$, the set $\overline{A}^{\mathcal{B}} \setminus A$ contains no nonempty subset $U \in \mathcal{K}^{\mathbf{c}}$.*

Immediately from the above corollary, we have Theorem 2.1 [4], where $\mathcal{B} \lhd \mathcal{K} = \mathcal{C}^{\gamma} \lhd \mathcal{T}$.

Applying Lemma 2.18, we obtain the following.

Lemma 2.20. *For any minimal structure $\mathcal{B} \subseteq \mathcal{P}(X)$ and $A \subseteq X$, the followings hold:*

(i) *If $A \cup \left(X \setminus \overline{A}^{\mathcal{B}}\right) \subseteq K$ and $K \in \mathcal{B}$, then $K = X$, and*

(ii) *If $\mathcal{B}.\mathrm{int}(A) \cup (X \setminus A) \subseteq K$ and $K \in \mathcal{B}$, then $K = X$.*

Proof.

(i) Let $K \in \mathcal{B}$ and $A \cup \left(X \setminus \overline{A}^{\mathcal{B}}\right) \subseteq K$. Then $X \setminus K \subseteq X \setminus \left(A \cup \left(X \setminus \overline{A}^{\mathcal{B}}\right)\right) = (X \setminus A) \cap \overline{A}^{\mathcal{B}} = \overline{A}^{\mathcal{B}} \setminus A$ and $X \setminus K \in \mathcal{B}^{\mathbf{c}}$. So, according to Lemma 2.18 (i), $X \setminus K = \emptyset$, i.e., $K = X$.

(ii) If $\mathcal{B}.\mathrm{int}(A) \cup (X \setminus A) \subseteq K$ and $K \in \mathcal{B}$, then $X \setminus K \subseteq X \setminus \big(\mathcal{B}.\mathrm{int}(A) \cup (X \setminus A)\big) = \big(X \setminus \mathcal{B}.\mathrm{int}(A)\big) \cap A = A \setminus \mathcal{B}.\mathrm{int}(A)$ and $X \setminus K \in \mathcal{B}^{\mathbf{c}}$. Then, by Lemma 2.18 (ii), we have $X \setminus K = \emptyset$, so $K = X$. □

Lemma 2.21. *Let $\mathcal{B} \triangleleft \mathcal{K} \in \Gamma(X)$ and $A \subseteq X$. The following assertions hold:*

(i) *$A \in \mathcal{B} \triangleleft \mathcal{K}$ if and only if $Z \subseteq A \subseteq \overline{Z}^{\mathcal{B}}$ for some subset $Z \in \mathcal{B} \triangleleft \mathcal{K}$, and*

(ii) *$A \in (\mathcal{B} \triangleleft \mathcal{K})^{\mathbf{c}}$ if and only if $\mathcal{B}.\mathrm{int}(Z) \subseteq A \subseteq Z$ for some subset $Z \in (\mathcal{B} \triangleleft \mathcal{K})^{\mathbf{c}}$.*

Proof.

(i) Let $Z \subseteq A \subseteq \overline{Z}^{\mathcal{B}}$, where $Z \in B \triangleleft \mathcal{K}$, i.e., $\overline{Z}^{\mathcal{B}} \subseteq \overline{Z}^{\mathcal{K}}$. Then, $\overline{Z}^{\mathcal{K}} \subseteq \overline{A}^{\mathcal{K}}$, and according to Remark 2.3, we have $\overline{A}^{\mathcal{B}} \subseteq \overline{\overline{Z}^{\mathcal{B}}}^{\mathcal{B}} = \overline{Z}^{\mathcal{B}} \subseteq \overline{Z}^{\mathcal{K}}$. So, $\overline{A}^{\mathcal{B}} \subseteq \overline{A}^{\mathcal{K}}$, i.e., $A \in \mathcal{B} \triangleleft \mathcal{K}$. The converse implication is clear.

(ii) Let $\mathcal{B}.\mathrm{int}(Z) \subseteq A \subseteq Z$, where $Z \in (\mathcal{B} \triangleleft \mathcal{K})^{\mathbf{c}}$. Then, according to Lemma 2.10, $X \setminus \overline{X \setminus A}^{\mathcal{B}} \subseteq A \subseteq Z$ and $X \setminus Z \in \mathcal{B} \triangleleft \mathcal{K}$. So, $X \setminus Z \subseteq X \setminus A \subseteq \overline{X \setminus Z}^{\mathcal{B}}$. Now, using (i), we obtain $X \setminus A \in \mathcal{B} \triangleleft \mathcal{K}$, i.e., $A \in (\mathcal{B} \triangleleft \mathcal{K})^{\mathbf{c}}$. The converse implication is obvious. □

The above lemma implies Theorem 2.3 [4] and Theorem 3.14 [51] in the case $\mathcal{C}^{\gamma} \triangleleft \mathcal{T}$ and $\mathcal{C}^{\beta} \triangleleft \mathcal{T}$, respectively.

Theorem 2.22. *Let $\mathcal{K} \subseteq \mathcal{P}(X)$ and $x \in X$. Then*

$$X \setminus \{x\} \in \mathcal{K} \quad \textit{or} \quad X \setminus \{x\} \in \mathcal{B} \triangleleft \mathcal{K}$$

for any minimal structure $\mathcal{B} \subseteq \mathcal{P}(X)$.

Proof. Let us assume that $X \setminus \{x\} \notin \mathcal{K}$. Then, because X is the only element of $\mathcal{K}$ containing $X \setminus \{x\}$, we have $\overline{X \setminus \{x\}}^{\mathcal{K}} = X$. So, for any family $\mathcal{B} \subseteq \mathcal{P}(X)$, we get $\overline{X \setminus \{x\}}^{\mathcal{B}} \subseteq \overline{X \setminus \{x\}}^{\mathcal{K}}$, i.e., $X \setminus \{x\} \in \mathcal{B} \triangleleft \mathcal{K}$. □

Of course, for example, in the case $\mathcal{K} = \mathcal{T}$, we have $X \setminus \{x\} \in \mathcal{T}$ or $X \setminus \{x\} \in \mathcal{B} \triangleleft \mathcal{T}$. Hence, we obtain Theorem 3.11 [51] in the case $\mathcal{C}^{\beta} \triangleleft \mathcal{T}$.

The above theorem entails immediately the following corollary.

Corollary 2.23. *Let $\mathcal{B} \triangleleft \mathcal{K} \in \Gamma(X)$. Then:*

(i) *for every singleton $\{x\}$, the subset $X \setminus \{x\}$ either belongs to $\mathcal{K}$ or $\mathcal{B} \triangleleft \mathcal{K}$,*

(ii) *if $\mathcal{B} \triangleleft \mathcal{K} \subseteq \overline{\mathcal{B}}$, then $X \setminus \{x\} \in \mathcal{K}$ or $X \setminus \{x\} \in \overline{\mathcal{B}}$ for every $x \in X$.*

Theorem 2.24. *For any minimal structure $\mathcal{K} \subseteq \mathcal{P}(X)$ and $A \subseteq X$, the followings hold:*

(i) $A = \mathcal{K}.\mathrm{int}(A) \cup (\mathcal{B} \triangleleft \mathcal{K}).\mathrm{int}(A)$ *for any family* $\mathcal{B} \subseteq \mathcal{P}(X)$,

(ii) $A = \overline{A}^{\mathcal{K}} \cap \overline{A}^{\mathcal{B} \triangleleft \mathcal{K}}$ *for any family* $\mathcal{B} \subseteq \mathcal{P}(X)$.

Proof.

(i) Let $x \in A$ and $\mathcal{B} \subseteq \mathcal{P}(X)$, then according to Theorem 2.22, $X \setminus \{x\} \in \mathcal{K}$ or $X \setminus \{x\} \in \mathcal{B} \triangleleft \mathcal{K}$. This means that $\{x\} \in \mathcal{K}^{\boldsymbol{c}}$ or $\{x\} \in (\mathcal{B} \triangleleft \mathcal{K})^{\boldsymbol{c}}$. Then, directly from the definition of $\mathcal{A}.\mathrm{int}(\dots)$, we obtain $x \in \mathcal{K}.\mathrm{int}(A)$ or $x \in (\mathcal{B} \triangleleft \mathcal{K}).\mathrm{int}(A)$. So, $A \subseteq \mathcal{K}.\mathrm{int}(A) \cup \mathcal{B} \triangleleft \mathcal{K}.\mathrm{int}(A)$. The inverse inclusion is clear.

(ii) Let as assume that $x \in \overline{A}^{\mathcal{K}} \cap \overline{A}^{\mathcal{B} \triangleleft \mathcal{K}}$ for some family $\mathcal{B} \subseteq \mathcal{P}(X)$. Using Theorem 2.22, we have $\{x\} \in \mathcal{K}^{\boldsymbol{c}}$ or $\{x\} \in (\mathcal{B} \triangleleft \mathcal{K})^{\boldsymbol{c}}$. In the case $\{x\} \in \mathcal{K}^{\boldsymbol{c}}$, according to Lemma 2.6, $\{x\} \cap A \neq \emptyset$, i.e., $x \in A$. In the second case, i.e., $\{x\} \in (\mathcal{B} \triangleleft \mathcal{K})^{\boldsymbol{c}}$, in the same way, we obtain $x \in A$. So, $\overline{A}^{\mathcal{K}} \cap \overline{A}^{\mathcal{B} \triangleleft \mathcal{K}} \subseteq A$. The inverse inclusion is obvious. □

Any family $\mathcal{B} \triangleleft \mathcal{K} \in \Gamma(X)$ is a natural generalization of the family of closed (resp. α-closed, semi-closed, pre-closed, γ-closed, β-closed) sets whenever $\mathcal{B} = \mathcal{C}$ (resp. $\mathcal{C}^{\alpha}$, $\mathcal{C}^{s}$, $\mathcal{C}^{p}$, $\mathcal{C}^{\gamma}$, $\mathcal{C}^{\beta}$). The obvious question is when a family $\mathcal{B} \triangleleft \mathcal{K}$ is strictly greater than $\mathcal{B}$? Recently in [5, Theorem 2.4], it has been shown that $\mathcal{C}^{P} \triangleleft \mathcal{O}^{P} = \mathcal{C}^{P}$, $\mathcal{C}^{\gamma} \triangleleft \mathcal{O}^{\gamma} = \mathcal{C}^{\gamma}$ and $\mathcal{C}^{\beta} \triangleleft \mathcal{O}^{\beta} = \mathcal{C}^{\beta}$. Below, we formulate a general condition that guarantees the equality $\mathcal{B} \triangleleft \mathcal{K} = \mathcal{B}$.

Theorem 2.25. *For every family $\mathcal{B} \triangleleft \mathcal{K} \in \Gamma(X)$, the property*

$$X \setminus \{x\} \in \overline{\mathcal{K}} \quad \text{or} \quad X \setminus \{x\} \in \overline{\mathcal{B}} \quad \text{for every} \quad x \in X$$

implies that $\mathcal{B} \triangleleft \mathcal{K} = \overline{\mathcal{B}}$.

Proof. Let assume that $A \in \mathcal{B} \triangleleft \mathcal{K}$. We will show that $\overline{A}^{\mathcal{B}} \subseteq A$. Let $x \in \overline{A}^{\mathcal{B}}$ and assume, to the contrary, that $x \notin A$, so $A \subseteq X \setminus \{x\}$. First, assume that $X \setminus \{x\} \in \overline{\mathcal{B}}$. Then, $\overline{A}^{\mathcal{B}} \subseteq \overline{X \setminus \{x\}}^{\mathcal{B}} = X \setminus \{x\}$, so $x \notin \overline{A}^{\mathcal{B}}$, and we have a contradiction.

Now, we assume that $X \setminus \{x\} \in \overline{\mathcal{K}}$. Then, $\overline{A}^{\mathcal{K}} \subseteq \overline{X \setminus \{x\}}^{\mathcal{K}} = X \setminus \{x\}$, so $x \notin \overline{A}^{\mathcal{K}}$, and since $A \in \mathcal{B} \triangleleft \mathcal{K}$, we obtain $\overline{A}^{\mathcal{B}} \subseteq \overline{A}^{\mathcal{K}}$. Thus, $x \notin \overline{A}^{\mathcal{B}}$ which completes the proof. □

According to Corollary 2.23 and Theorem 2.25, we have the following.

Corollary 2.26. *Let $\mathcal{B} \triangleleft \mathcal{K} \in \Gamma(X)$. Then*

(i) $\mathcal{B} \triangleleft \mathcal{K} \subseteq \overline{\mathcal{B}}$ *if and only if* $X \setminus \{x\} \in \overline{\mathcal{B}}$ *or* $X \setminus \{x\} \in \mathcal{K}$ *for every* $x \in X$,

(ii) *$\mathcal{B}$ is closed under arbitrary intersection, then the property $\mathcal{B} \triangleleft \mathcal{K} = \mathcal{B}$ is equivalent to $X \setminus \{x\} \in \mathcal{B}$ or $X \setminus \{x\} \in \mathcal{K}$ for every $x \in X$.*

From the part (ii) of the above corollary, we obtain Theorem 2.5 [18] in which $\mathcal{B} \in \{\mathcal{C}, \mathcal{C}^\alpha, \mathcal{C}^s, \mathcal{C}^p, \mathcal{C}^\beta\}$ and $\mathcal{K} \in \{\mathcal{T}, \mathcal{O}^\alpha, \mathcal{O}^s, \mathcal{O}^p, \mathcal{O}^\beta\}$.

Corollary 2.27. *For any $\mathcal{B} \lhd \mathcal{K} \in \Gamma(X)$, the followings hold:*

(i) $(\mathcal{B} \lhd \mathcal{K}) \lhd \mathcal{K} = \overline{\mathcal{B} \lhd \mathcal{K}}$,

(ii) $\mathcal{K} \lhd (\mathcal{B} \lhd \mathcal{K}) = \overline{\mathcal{K}}$.

P r o o f. Property (i) follows from Corollary 2.23 and Theorem 2.25 applied to pair $(\mathcal{B} \lhd \mathcal{K}, \mathcal{K})$ instead of the pair $(\mathcal{B}, \mathcal{K})$. Whereas, in property (ii) we have used the pair $(\mathcal{K}, \mathcal{B} \lhd \mathcal{K})$ instead of the pair $(\mathcal{B}, \mathcal{K})$. □

Theorem 2.28. *For any $\mathcal{B} \lhd \mathcal{K} \in \Gamma(X)$, the followings hold*

$$(\mathcal{B} \lhd \mathcal{K}) \cap \overline{\mathcal{K}} = \overline{\mathcal{B}} \cap \overline{\mathcal{K}} \quad \textit{and} \quad (\mathcal{B} \lhd \mathcal{K}) \cap \mathcal{K} = \overline{\mathcal{B}} \cap \mathcal{K}.$$

P r o o f. If $A \in (\mathcal{B} \lhd \mathcal{K}) \cap \overline{\mathcal{K}}$, i.e., $\overline{A}^{\mathcal{B}} \subseteq \overline{A}^{\mathcal{K}}$ and $A \in \overline{\mathcal{K}}$, then according to definition of $\overline{\mathcal{K}}$, we have $\overline{A}^{\mathcal{K}} = A$. So, $\overline{A}^{\mathcal{B}} \subseteq A$, which means that $A \in \overline{\mathcal{B}}$. Consequently, we have $A \in \overline{\mathcal{B}} \cap \overline{\mathcal{K}}$. The inverse inclusion follows directly from Lemma 2.14 (iv). The proof of the second part proceeds in a similar manner and we omit it. □

Theorem 2.29. *Let $\mathcal{B}_1 \lhd \mathcal{K} \in \Gamma(X)$. Then*

(i) $\mathcal{B}_2 \lhd (\mathcal{B}_1 \lhd \mathcal{K})^{\boldsymbol{c}} \subseteq \mathcal{B}_2 \lhd \mathcal{B}_1^{\boldsymbol{c}}$ *for any* $\mathcal{B}_2 \subseteq \mathcal{P}(X)$,

(ii) $\mathcal{B}_3 \lhd (\mathcal{B}_2 \lhd \mathcal{B}_1^{\boldsymbol{c}})^{\boldsymbol{c}} \subseteq \mathcal{B}_3 \lhd \big(\mathcal{B}_2 \lhd (\mathcal{B}_1 \lhd \mathcal{K})^{\boldsymbol{c}}\big)^{\boldsymbol{c}} \subseteq \mathcal{B}_3 \lhd \mathcal{B}_2^{\boldsymbol{c}}$ *for any* $\mathcal{B}_2, \mathcal{B}_3 \subseteq \mathcal{P}(X)$,

(iii) $\mathcal{B}_4 \lhd (\mathcal{B}_3 \lhd \mathcal{B}_2^{\boldsymbol{c}})^{\boldsymbol{c}} \subseteq \mathcal{B}_4 \lhd \Big(\mathcal{B}_3 \lhd \big(\mathcal{B}_2 \lhd (\mathcal{B}_1 \lhd \mathcal{K})^{\boldsymbol{c}}\big)^{\boldsymbol{c}}\Big)^{\boldsymbol{c}} \subseteq \mathcal{B}_4 \lhd \big(\mathcal{B}_3 \lhd (\mathcal{B}_2 \lhd \mathcal{B}_1^{\boldsymbol{c}})^{\boldsymbol{c}}\big)^{\boldsymbol{c}} \subseteq \mathcal{B}_4 \lhd \mathcal{B}_3^{\boldsymbol{c}}$ *for any* $\mathcal{B}_2, \mathcal{B}_3, \mathcal{B}_4 \subseteq \mathcal{P}(X)$.

P r o o f.

(i) According to Lemma 2.14 (iv), we have $\mathcal{B}_1 \subseteq \mathcal{B}_1 \lhd \mathcal{K}$, so $\mathcal{B}_1^{\boldsymbol{c}} \subseteq (\mathcal{B}_1 \lhd \mathcal{K})^{\boldsymbol{c}}$. Consequently, by Lemma 2.14 (iii), we obtain property (i).

(ii) Applying (i), we have $\big(\mathcal{B}_2 \lhd (\mathcal{B}_1 \lhd \mathcal{K})^{\boldsymbol{c}}\big)^{\boldsymbol{c}} \subseteq (\mathcal{B}_2 \lhd \mathcal{B}_1^{\boldsymbol{c}})^{\boldsymbol{c}}$, so

$$\mathcal{B}_3 \lhd (\mathcal{B}_2 \lhd \mathcal{B}_1^{\boldsymbol{c}})^{\boldsymbol{c}} \subseteq \mathcal{B}_3 \lhd \big(\mathcal{B}_2 \lhd (\mathcal{B}_1 \lhd \mathcal{K})^{\boldsymbol{c}}\big)^{\boldsymbol{c}},$$

and again from (i), where we use $\big(\mathcal{B}_1 \lhd \mathcal{K}\big)^{\boldsymbol{c}}$ instead of $\mathcal{K}$, we obtain

$$\mathcal{B}_3 \lhd \big(\mathcal{B}_2 \lhd (\mathcal{B}_1 \lhd \mathcal{K})^{\boldsymbol{c}}\big)^{\boldsymbol{c}} \subseteq \mathcal{B}_3 \lhd \mathcal{B}_2^{\boldsymbol{c}}.$$

(iii) Analogously to the above, property (ii) implies that

$$\mathcal{B}_4 \lhd (\mathcal{B}_3 \lhd \mathcal{B}_2^{\boldsymbol{c}}) \subseteq \mathcal{B}_4 \lhd \Big(\mathcal{B}_3 \lhd \big(\mathcal{B}_2 \lhd (\mathcal{B}_1 \lhd \mathcal{K})^{\boldsymbol{c}}\big)^{\boldsymbol{c}}\Big) \subseteq \mathcal{B}_4 \lhd \big(\mathcal{B}_3 \lhd (\mathcal{B}_2 \lhd \mathcal{B}_1^{\boldsymbol{c}})^{\boldsymbol{c}}\big),$$

and from (i), using $(\mathcal{B}_2 \lhd \mathcal{B}_1^{\boldsymbol{c}})^{\boldsymbol{c}}$ instead of $\mathcal{K}$, we have

$$\mathcal{B}_4 \lhd \big(\mathcal{B}_3 \lhd (\mathcal{B}_2 \lhd \mathcal{B}_1^{\boldsymbol{c}})^{\boldsymbol{c}}\big) \subseteq \mathcal{B}_4 \lhd \mathcal{B}_3^{\boldsymbol{c}}.$$ □

3. Classical types of generalized closedness

In this section, we investigate families of type $\mathcal{B} \lhd \mathcal{K}$ in a topological space $X(\mathcal{T})$, where

$$(\mathcal{B}, \mathcal{K}) \in \left\{\mathcal{C}, \mathcal{C}^{\alpha}, \mathcal{C}^{s}, \mathcal{C}^{p}, \mathcal{C}^{\gamma}, \mathcal{C}^{\beta}\right\} \times \left\{\mathcal{T}, \mathcal{O}^{\alpha}, \mathcal{O}^{p}, \mathcal{O}^{s}, \mathcal{O}^{\gamma}, \mathcal{O}^{\beta}\right\}.$$

We will show that $\mathcal{C}$, $\mathcal{C}^{\alpha}$, $\mathcal{C}^{s}$, $\mathcal{C}^{p}$, $\mathcal{C}^{\gamma}$ and $\mathcal{C}^{\beta}$ can be understood as the families of type $\mathcal{B} \lhd \mathcal{K}$, and we will investigate the relationships between them.

First, we show that the well-known classical types of generalization of g-closedness can be considered as families of type $\mathcal{B} \lhd \mathcal{K}$ of the above case. The most known of which are listed below.

(1) The family of w-closed ($\hat{g}$-closed) set [49] ([54]) is equal to $\mathcal{C} \lhd \mathcal{O}^{s}$.
(2) The family of αg-closed set [64] is equal to $\mathcal{C}^{\alpha} \lhd \mathcal{T}$.
(3) The family of gα-closed set [65] is equal to $\mathcal{C}^{\alpha} \lhd \mathcal{O}^{\alpha}$.
(4) The family of gs-closed set [9] is equal to $\mathcal{C}^{s} \lhd \mathcal{T}$.
(5) The family of sg-closed set [13] is equal to $\mathcal{C}^{s} \lhd \mathcal{O}^{s}$.
(6) The family gp-closed set [63] is equal to $\mathcal{C}^{p} \lhd \mathcal{T}$.
(7) The family swg-closed set [63] is equal to $\mathcal{C}^{p} \lhd \mathcal{O}^{s}$.
(8) The family of pg-closed set [95] is equal to $\mathcal{C}^{p} \lhd \mathcal{O}^{p}$.
(9) The family of gb-closed set [4] is equal to $\mathcal{C}^{\gamma} \lhd \mathcal{T}$.
(10) The family of gα b-closed set [123] is equal to $\mathcal{C}^{\gamma} \lhd \mathcal{O}^{\alpha}$.
(11) The family of sgb-closed set [44] is equal to $\mathcal{C}^{\gamma} \lhd \mathcal{O}^{s}$.
(12) The family of pgb-closed set [115] is equal to $\mathcal{C}^{\gamma} \lhd \mathcal{O}^{p}$.
(13) The family of gb-closed set [124] is equal to $\mathcal{C}^{\gamma} \lhd \mathcal{O}^{\gamma}$.
(14) The family of gsp-closed set [27] is equal to $\mathcal{C}^{\beta} \lhd \mathcal{T}$.
(15) The family of bg-closed set [105] is equal to $\mathcal{C}^{\beta} \lhd \mathcal{O}^{\beta}$.
(16) The family of gpα-closed set [98] is equal to $\mathcal{C}^{p} \lhd \mathcal{O}^{\alpha}$.

Now, we start to study the relationships between families $\mathcal{B} \lhd \mathcal{K}$ in discussed cases.

We will need some properties of singletons and their complements in topological spaces. In [47], it is shown that every singleton $\{x\}$ in a topological space $X(\mathcal{T})$ is either pre-open or nowhere dense (Jankovic-Reilly decomposition [16, 17]). The nowhere denseness of $\{x\}$ implies that $X \setminus \{x\}$ is α-open. So, for every singleton $\{x\}$, the subset $X \setminus \{x\}$ is either pre-closed or α-open. It is easy to show that the pre-openness of $\{x\}$ can be exchanged by γ-openness or β-openness, so we have the following.

Remark 3.1. In a topological space $X(\mathcal{T})$, the following properties are equivalent:

(i) $X \setminus \{x\} \in \mathcal{C}^p$ or $X \setminus \{x\} \in \mathcal{O}^\alpha$,
(ii) $X \setminus \{x\} \in \mathcal{C}^\gamma$ or $X \setminus \{x\} \in \mathcal{O}^\alpha$ and
(iii) $X \setminus \{x\} \in \mathcal{C}^\beta$ or $X \setminus \{x\} \in \mathcal{O}^\alpha$ for any $x \in X$.

Similarly, every singleton $\{x\}$ of a topological space $X(\mathcal{T})$ is either open or has an empty interior [18, Lemma 2.4]. If $\{x\}$ has an empty interior, then $X \setminus \{x\}$ is pre-open. So, for every singleton $\{x\}$, the subset $X \setminus \{x\}$ is either closed or pre-open. One can easily show that the openness of $\{x\}$ may be exchanged by α-openess or semi-openness. So, the closedness of $X \setminus \{x\}$ can be exchanged by α-closedness or semi-closedness. Then, we have the following.

Remark 3.2. In a topological space $X(\mathcal{T})$, the following properties are equivalent:

(i) $X \setminus \{x\} \in \mathcal{C}$ or $X \setminus \{x\} \in \mathcal{O}^p$,
(ii) $X \setminus \{x\} \in \mathcal{C}^\alpha$ or $X \setminus \{x\} \in \mathcal{O}^p$ and
(iii) $X \setminus \{x\} \in \mathcal{C}^s$ or $X \setminus \{x\} \in \mathcal{O}^p$ for any $x \in X$.

Theorem 3.3. *In a topological space $X(\mathcal{T})$, the equalities*

(i) $\mathcal{C}^p \lhd \mathcal{O}^\alpha = \mathcal{C}^p$,
(ii) $\mathcal{C}^\gamma \lhd \mathcal{O}^\alpha = \mathcal{C}^\gamma$,
(iii) $\mathcal{C}^\beta \lhd \mathcal{O}^\alpha = \mathcal{C}^\beta$

are true.

Proof.
Part (i) follows immediately from Corollary 2.26 and from Remark 3.1 (i). Similarly, part (ii) follows from Corollary 2.26 and from Remark 3.1 (ii). Part (iii) follows from Corollary 2.26 and from Remark 3.1 (iii). □

Applying Lemma 2.14 ((iii),(iv)) and the above theorem, we obtain the following result.

Corollary 3.4. *In a topological space $X(\mathcal{T})$, the following properties hold:*

(i) $\mathcal{C}^p \lhd \mathcal{O}^\beta = \mathcal{C}^p \lhd \mathcal{O}^\gamma = \mathcal{C}^p \lhd \mathcal{O}^s = \mathcal{C}^p \lhd \mathcal{O}^p = \mathcal{C}^p \lhd \mathcal{O}^\alpha = \mathcal{C}^p$,
(ii) $\mathcal{C}^\gamma \lhd \mathcal{O}^\beta = \mathcal{C}^\gamma \lhd \mathcal{O}^\gamma = \mathcal{C}^\gamma \lhd \mathcal{O}^s = \mathcal{C}^\gamma \lhd \mathcal{O}^p = \mathcal{C}^\gamma \lhd \mathcal{O}^\alpha = \mathcal{C}^\gamma$,
(iii) $\mathcal{C}^\beta \lhd \mathcal{O}^\beta = \mathcal{C}^\beta \lhd \mathcal{O}^\gamma = \mathcal{C}^\beta \lhd \mathcal{O}^s = \mathcal{C}^\beta \lhd \mathcal{O}^p = \mathcal{C}^\beta \lhd \mathcal{O}^\alpha = \mathcal{C}^\beta$.

Remark 3.5. The equalities stated above show that most of the generalizations listed in definitions (1) - (16) are improper, namely (7), (8), (10), (11), (12), (13), (14), (15) and (16) are so.

A conclusion from the above corollary is the following theorem.

Theorem 3.6 ([6], Theorem 2.4). *The following properties hold in every topological space:*

(i) $C^p \lhd O^p = C^p$,

(ii) $C^\gamma \lhd O^\gamma = C^\gamma$,

(iii) $C^\beta \lhd O^\beta = C^\beta$.

Theorem 3.7 and Corollary 3.8 below are analogous to Theorem 3.3 and Corollary 3.4, respectively.

Theorem 3.7. *The following properties hold for every topological space:*

(i) $\mathcal{C} \lhd \mathcal{O}^p = \mathcal{C}$,

(ii) $\mathcal{C}^\alpha \lhd \mathcal{O}^p = \mathcal{C}^\alpha$,

(iii) $\mathcal{C}^s \lhd \mathcal{O}^p = \mathcal{C}^s$.

Proof. Part (i) follows immediately from Corollary 2.26 and point (i) from Remark 3.2. Similarly, part (ii) follows from Corollary 2.26 and point (ii) from Remark 3.2, and part (iii) follows from Corollary 2.26 and point (iii) from Remark 3.2. □

From Lemma 2.14 (iii), (iv) and the above theorem, we obtain the following equations.

Corollary 3.8. *The following properties hold for every topological space:*

(i) $\mathcal{C} \lhd \mathcal{O}^\beta = \mathcal{C} \lhd \mathcal{O}^\gamma = \mathcal{C} \lhd \mathcal{O}^p = \mathcal{C}$,

(ii) $\mathcal{C}^\alpha \lhd \mathcal{O}^\beta = \mathcal{C}^\alpha \lhd \mathcal{O}^\gamma = \mathcal{C}^\alpha \lhd \mathcal{O}^p = \mathcal{C}^\alpha$,

(iii) $\mathcal{C}^s \lhd \mathcal{O}^\beta = \mathcal{C}^s \lhd \mathcal{O}^\gamma = \mathcal{C}^s \lhd \mathcal{O}^p = \mathcal{C}^s$.

In the following theorem, we present a general requirement of the case concerning the equality of families of type $\mathcal{B} \lhd \mathcal{K}$.

Theorem 3.9. *The following properties hold for any minimal structure $\mathcal{B}$ containing all closed sets in a topological space $X(\mathcal{T})$:*

(i) $\mathcal{B} \lhd \mathcal{O}^\alpha = \mathcal{B} \lhd \mathcal{O}^s$,

(ii) $\mathcal{B} \lhd \mathcal{O}^p = \mathcal{B} \lhd \mathcal{O}^\gamma = \mathcal{B} \lhd \mathcal{O}^\beta$.

Proof.

(i) Let $A \in \mathcal{B} \lhd \mathcal{O}^\alpha$, i.e., $\overline{A}^{\mathcal{B}} \subseteq \overline{A}^{\mathcal{O}^\alpha}$. We will show that $A \in \mathcal{B} \lhd \mathcal{O}^s$, i.e., according to Remark 2.13, assuming that $A \subseteq V$, where $V \in \mathcal{O}^s$, $\overline{A}^{\mathcal{B}} \subseteq V$. Suppose, to the contrary, that $x \in \overline{A}^{\mathcal{B}}$ and $x \notin V$, i.e., $V \subseteq X \setminus \{x\}$ for some $x \in X$ and $V \in \mathcal{O}^s$ such that $A \subseteq V$. According to Remark 3.1 (i), $X \setminus \{x\}$ is either pre-closed or α-open. Since $A \subseteq X \setminus \{x\}$ and because of the assumption in the last case, we get $\overline{A}^{\mathcal{B}} \subseteq \overline{A}^{\mathcal{O}^\alpha} \subseteq X \setminus \{x\}$. So, $x \notin \overline{A}^{\mathcal{B}}$ and we obtain a contradiction.

Now, assume that $X\setminus\{x\}$ is pre-closed, i.e., $\{x\}$ is pre-open. Since $\mathcal{C} \subseteq \mathcal{B}$, then $x \in \overline{A}^{\mathcal{B}} \subseteq \mathrm{cl}(A) \subseteq \mathrm{cl}(V) = \mathrm{cl}(\mathrm{int}(V))$. According to the assumption, $x \in \mathrm{int}(\mathrm{cl}(\{x\}))$, and consequently, $x \in \mathrm{int}(\mathrm{cl}(\{x\}))\cap\mathrm{cl}(\mathrm{int}(V))$. So, $x \in V$. The converse inclusion follows from Lemma 2.14 (iii).

(ii) We will prove that $\mathcal{B}\triangleleft\mathcal{O}^p \subseteq \mathcal{B}\triangleleft\mathcal{O}^\beta$. Assume that $A \in \mathcal{B}\triangleleft\mathcal{O}^p$, i.e., $\overline{A}^{\mathcal{B}} \subseteq \overline{A}^{\mathcal{O}^p}$. We will show that $A \in \mathcal{B}\triangleleft\mathcal{O}^\beta$, i.e., according to Remark 2.13, $\overline{A}^{\mathcal{B}} \subseteq V$ for every $V \in \mathcal{O}^\beta$ such that $A \subseteq V$. Assume, to the contrary, that there exist a subset $V \in \mathcal{O}^\beta$ and a point $x \in \overline{A}^{\mathcal{B}}$ such that $A \subseteq V$ and $x \notin V$, i.e., $V \subseteq X \setminus \{x\}$. Using Remark 3.2 (i), we know that $X \setminus \{x\}$ is either closed or pre-open. Of course, $A \subseteq X \setminus \{x\}$, so because of the assumption, in the last case, we obtain $\overline{A}^{\mathcal{B}} \subseteq \overline{A}^{\mathcal{O}^p} \subseteq X \setminus \{x\}$ and consequently $x \notin \overline{A}^{\mathcal{B}}$ which gives a contradiction.

Now, if $X \setminus \{x\}$ is closed, i.e., $\{x\}$ is open, then because of the assumption $\mathcal{C} \subseteq \mathcal{B}$, we have $x \in \overline{A}^{\mathcal{B}} \subseteq \mathrm{cl}(A) \subseteq \mathrm{cl}(V) = \mathrm{cl}\big(\mathrm{int}(\mathrm{cl}(V))\big)$. Consequently, $x \in \mathrm{int}\{x\} \cap \mathrm{cl}\big(\mathrm{int}(\mathrm{cl}(V))\big)$, so $x \in V$. The converse inclusion follows from Lemma 2.14 (iii). Finally, in view of the obvious fact that $\mathcal{B}\triangleleft\mathcal{O}^\beta \subseteq \mathcal{B}\triangleleft\mathcal{O}^\gamma \subseteq \mathcal{B}\triangleleft\mathcal{O}^p$, the proof is completed. □

From the above theorem, in the case $\mathcal{B} \in \{\mathcal{C}, \mathcal{C}^\alpha, \mathcal{C}^s, \mathcal{C}^p, \mathcal{C}^\beta\}$, we obtain Theorem 2.7 from [18].

Theorem 3.10. *The following properties hold in every topological space:*

(i) $\mathcal{C}^s \triangleleft \mathcal{O}^s \subseteq \mathcal{C}^\gamma$,

(ii) $\mathcal{C}^\alpha \triangleleft \mathcal{O}^\alpha \subseteq \mathcal{C}^p$,

(iii) $\mathcal{C}^\alpha \triangleleft \mathcal{O}^\alpha \subseteq \mathcal{C}^p \triangleleft \mathcal{O}^p$,

(iv) $\mathcal{C}^s \triangleleft \mathcal{O}^s \subseteq \mathcal{C}^\gamma \triangleleft \mathcal{O}^\gamma$,

(v) $\mathcal{C}^p \triangleleft \mathcal{O}^p \subseteq \mathcal{C}^\gamma \triangleleft \mathcal{O}^\gamma$ *and*

(vi) $\mathcal{C}^\gamma \triangleleft \mathcal{O}^\gamma \subseteq \mathcal{C}^\beta \triangleleft \mathcal{O}^\beta$.

Proof. From Lemma 2.14 (ii) and Theorem 3.6 (ii), we have $C^s \triangleleft O^s \subseteq C^\gamma \triangleleft O^s = C^\gamma$ and $C^\alpha \triangleleft O^\alpha \subseteq C^p \triangleleft O^\alpha = C^p$, which completes the proof of part (i) and (ii).

In the cases (iii)-(vi), according to Lemma 2.14 (ii), (iv) and Theorem 3.6, we have:

- $C^\alpha \triangleleft O^\alpha \subseteq C^p \triangleleft O^\alpha = C^p \subseteq C^p \triangleleft O^p$,
- $C^s \triangleleft O^s \subseteq C^\gamma \triangleleft O^s = C^\gamma \subseteq C^\gamma \triangleleft O^\gamma$,
- $C^p \triangleleft O^p \subseteq C^\gamma \triangleleft O^p = C^\gamma \subseteq C^\gamma \triangleleft O^\gamma$,
- $C^\gamma \triangleleft O^\gamma \subseteq C^\beta \triangleleft O^\gamma = C^\beta \subseteq C^\beta \triangleleft O^\beta$.

The proof is completed. □

Combining results from this section, we get the following corollary.

Corollary 3.11. *For any topological space* $(X, \mathcal{T})$*, the following equalities hold:*

(i) $\mathcal{C}^p \lhd \mathcal{K} = \mathcal{C}^p$, $\mathcal{C}^\gamma \lhd \mathcal{K} = \mathcal{C}^\gamma$ *and* $\mathcal{C}^\beta \lhd \mathcal{K} = \mathcal{C}^\beta$ *for any* $\mathcal{K} \in \{\mathcal{O}^\alpha, \mathcal{O}^s, \mathcal{O}^p, \mathcal{O}^\gamma, \mathcal{O}^\beta\}$,

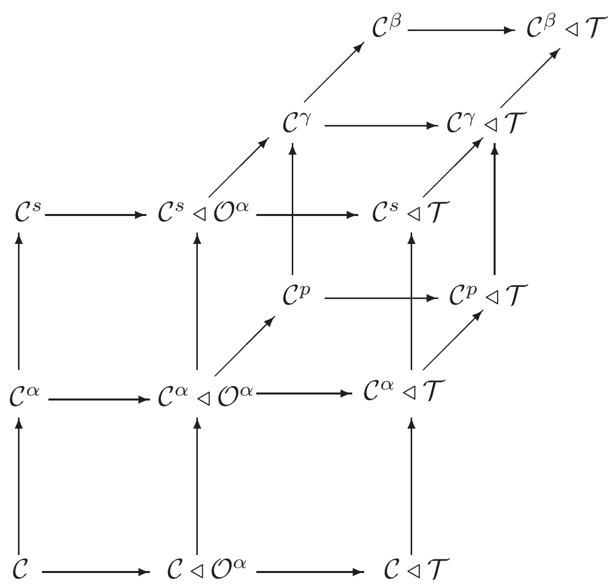
(ii) $\mathcal{C} \lhd \mathcal{K} = \mathcal{C}$, $\mathcal{C}^\alpha \lhd \mathcal{K} = \mathcal{C}^\alpha$ *and* $\mathcal{C}^s \lhd \mathcal{K} = \mathcal{C}^s$ *for any* $\mathcal{K} \in \{\mathcal{O}^p, \mathcal{O}^\gamma, \mathcal{O}^\beta\}$,

(iii) $\mathcal{C}^\gamma = \mathcal{C}^\gamma \lhd \mathcal{O}^\alpha = \mathcal{C}^\gamma \lhd \mathcal{O}^s = \mathcal{C}^\gamma \lhd \mathcal{O}^p = \mathcal{C}^\gamma \lhd \mathcal{O}^\gamma$,

(iv) $\mathcal{C}^p = \mathcal{C}^p \lhd \mathcal{O}^s = \mathcal{C}^p \lhd \mathcal{O}^p = \mathcal{C}^p \lhd \mathcal{O}^\alpha$,

(v) $\mathcal{C}^\beta = \mathcal{C}^\beta \lhd \mathcal{O}^\beta$.

Summarizing, the results proved in this section enable us to conclude that the below graph illustrates all of the investigated type and the relationships between them.



All of the above inclusions are strict as the following examples 3.13 and 3.14 below show. Before we proceed to the examples, let us note the following.

Remark 3.12. It is evident that the natural partial order in $\Gamma(X)$ to the operations $\oplus$, $\odot$, $()'$ defined in Theorem 2.15 is given by the formula

$$\mathcal{B}_1 \lhd \mathcal{K}_1 \prec \mathcal{B}_2 \lhd \mathcal{K}_2 \quad \text{if} \quad \mathcal{B}_1 \lhd \mathcal{K}_1 \odot \mathcal{B}_2 \lhd \mathcal{K}_2 = \mathcal{B}_1 \lhd \mathcal{K}_1,$$

or equivalently, if

$$\mathcal{B}_1 \lhd \mathcal{K}_1 \oplus \mathcal{B}_2 \lhd \mathcal{K}_2 = \mathcal{B}_2 \lhd \mathcal{K}_2.$$

So, it is easy to check that this order is compatible with the inclusion relation presented in the graph.

Example 3.13. Let $(\mathbb{R}, \mathcal{T})$ be the real line with the natural topology.

(i) Let us take $A = \{1, \frac{1}{2}, \frac{1}{3} \ldots\}$, then $\mathrm{cl}(A) = A \cup \{0\}$ and, of course,
$$\mathrm{cl}\big(\mathrm{int}(\mathrm{cl}(A))\big) = \emptyset,$$
so $A \in \mathcal{C}^{\alpha}$. But, for the subset $U = (0, 2)$, we have $A \subseteq U$ and $\mathrm{cl}(A) \not\subseteq U$. So, $A \notin \mathcal{C} \lhd \mathcal{T}$.

(ii) For the subset $A = (0, 1) \cup \{2\}$, we have $\mathrm{int}(\mathrm{cl}(A)) \subseteq A$, i.e., $A \in \mathcal{C}^{s}$. Let us take $U = (0, 3)$, then $A \subseteq U$ and $\mathrm{cl}\big(\mathrm{int}(\mathrm{cl}(A))\big) \not\subseteq U$. So, $A \notin \mathcal{C}^{\alpha} \lhd \mathcal{T}$.

(iii) Let $A = [(0, 1) \cap \mathbb{Q}] \cup [(1, 2) \cap \mathbb{Q}]$, where $\mathbb{Q}$ is the set of all rational numbers. Then, $\mathrm{cl}(\mathrm{int}(A)) = \emptyset$, thus $A \in \mathcal{C}^{p}$. But, $\mathrm{int}(\mathrm{cl}(A)) = (0, 2)$, and hence, taking the set $U = (0, 1) \cup (1, 2)$, we get $A \subseteq U$ and $\mathrm{int}(\mathrm{cl}(A)) \not\subseteq U$. Which means that $A \notin \mathcal{C}^{s} \lhd \mathcal{T}$.

(iv) Let us consider the set $A = (0, 1) \cup [(1, 2) \cap \mathbb{Q}]$. Of course,
$$\mathrm{int}\big(\mathrm{cl}(\mathrm{int}(A))\big) = (0, 1) \subseteq A, \quad \text{i.e., } A \in \mathcal{C}^{\beta}.$$
Now, let us note that $\mathrm{cl}(\mathrm{int}(A)) = [0, 1]$ and $\mathrm{int}(\mathrm{cl}(A)) = (0, 2)$, thus $\mathrm{cl}(\mathrm{int}(A)) \cap \mathrm{int}(\mathrm{cl}(A)) = (0, 1] \not\subseteq U$, i.e., $A \notin \mathcal{C}^{\gamma} \lhd \mathcal{T}$.

(v) For the subset $A = (0, 1) \cup \{2\}$, we have $\mathrm{cl}(\mathrm{int}(A)) \subseteq A$, i.e., $A \in \mathcal{C}^{s}$. Let us take $U = (0, 3)$, then $A \subseteq U$ and $\mathrm{cl}(\mathrm{int}(A)) \not\subseteq U$. So, $A \notin \mathcal{C}^{p} \lhd \mathcal{T}$.

Example 3.14. Let $\mathbb{R}$ be the real line.

(i) Let us take the topology $\mathcal{T} = \{\emptyset, \mathbb{R}\} \cup \{(-\infty, a) : a \in \mathbb{R}\}$. Then, for $A = (-\infty, 0) \cup (1, \infty)$ we have $\mathrm{int}\big(\mathrm{cl}(\mathrm{int}(A))\big) = \mathbb{R}$, so $A \notin \mathcal{C}^{\beta}$. If U is an open subset containing A, then $U = \mathbb{R}$. Thus, $\mathrm{cl}(A) \subseteq U$ which proves that $A \in \mathcal{C} \lhd \mathcal{T}$.

(ii) Let us take the topology $\mathcal{T} = \{\emptyset, \mathbb{R}, \mathbb{Q}, \mathbb{R} \setminus \mathbb{Q}\}$. If $A = \mathbb{Q} \cup K$, where $K \cap \mathbb{Q} = \emptyset$, $K \neq \emptyset$ and $K \neq \mathbb{R} \setminus \mathbb{Q}$, then $\mathrm{int}(\mathrm{cl}(A)) = \mathbb{R}$, and consequently, $A \notin \mathcal{C}^{s}$. Of course, $\mathbb{R}$ is the only subset of $\mathcal{O}^{\alpha}$ containing the set A, so, $A \in \mathcal{C} \lhd \mathcal{O}^{\alpha}$.

4. Generalizations through use of families of regular types of subsets

In [92], a new concept of generalized closedness based on the notion of regularly semi-open set is introduced, whereas in [118], another concept of generalized closedness based on the notions of regularly closed sets and regularly open sets is presented.

Let us recall that a subset A of a topological space is regularly open, regularly closed or regularly semi-open, if $A = \mathrm{int}(\mathrm{cl}(A))$, $A = \mathrm{cl}(\mathrm{int}(A))$ or $U \subseteq A \subseteq \mathrm{cl}(U)$ for some regularly open set U, respectively. In [26], it was shown that a subset is regularly semi-open if and only if it is both semi-open and semi-closed.

The family of all regularly open, regular closed and regularly semi-open subsets of X, respectively, will be denoted by $\mathcal{RO}$, $\mathcal{RC}$ or $\mathcal{RSO}$, respectively.

Many authors have investigated generalizations through families of regular subsets whose definitions are listed below. All of them can be considered as families of type $\mathcal{B} \lhd \mathcal{K}$ where $(\mathcal{B}, \mathcal{K}) \in \{\mathcal{RC}, \mathcal{C}, \mathcal{C}^\alpha, \mathcal{C}^p, \mathcal{C}^\gamma, \mathcal{C}^\beta\} \times \{\mathcal{RSO}, \mathcal{RO}, \mathcal{T}, \mathcal{O}^\gamma\}$.

(1) The family of gr-closed set [14] is equal to $\mathcal{RC} \lhd \mathcal{T}$.
(2) The family of R^*-closed set [46] is equal to $\mathcal{RC} \lhd \mathcal{RSO}$.
(3) The family of rb-closed set [76] is equal to $\mathcal{RC} \lhd \mathcal{O}^\gamma$.
(4) The family of rg-closed set [92] is equal to $\mathcal{C} \lhd \mathcal{RO}$.
(5) The family of rw-closed set [12] is equal to $\mathcal{C} \lhd \mathcal{RSO}$.
(6) The family of gαr-closed set [89], [111] is equal to $\mathcal{C}^\alpha \lhd \mathcal{RO}$.
(7) The family of rgα-closed set [122] is equal to $\mathcal{C}^\alpha \lhd \mathcal{RSO}$.
(8) The family of gpr-closed set [38] is equal to $\mathcal{C}^P \lhd \mathcal{RO}$.
(9) The family of rgw-closed set [72] is equal to $\mathcal{C}^P \lhd \mathcal{RSO}$.
(10) The family of rgb-closed set [66] is equal to $\mathcal{C}^\gamma \lhd \mathcal{RO}$.
(11) The family of γgrw-closed set [32] is equal to $\mathcal{C}^\gamma \lhd \mathcal{RSO}$.
(12) The family of gspr-closed set [109] is equal to $\mathcal{C}^\beta \lhd \mathcal{RO}$.

Many results concerning properties of this kind of generalized closedness obtained previously can be easily generalized by applying properties of $\mathcal{B} \lhd \mathcal{K}$ presented in section 2.

Applying Theorem 2.28, we can obtain generalizations of:

(1) Theorem 4.6 [111], proved for $\mathcal{B} = \mathcal{C}^\alpha$ and $\mathcal{K} = \mathcal{RO}$ holds for every $\mathcal{B}$, $\mathcal{K} \subseteq \mathcal{P}(X)$.
(2) Theorem 3.11 [12], proved for $\mathcal{B} = \mathcal{C}$ and $\mathcal{K} = \mathcal{RSO}$ holds for every $\mathcal{B}$, $\mathcal{K} \subseteq \mathcal{P}(X)$.
(3) Theorem 4.6 [112], proved for $\mathcal{B} = \mathcal{C}^\alpha$ and $\mathcal{K} = \mathcal{RO}$ holds for every $\mathcal{B}$, $\mathcal{K} \subseteq \mathcal{P}(X)$.

To justify the above corollaries, we will illustrate the first one. The other two corollaries are justified analogously. Theorem 4.6 [111] says that if A is both regular open and gαr-closed set in X, then A is α closed set.

The property of being both regular open and gαr-closed $A \subseteq X$ means, according to definition (6) above, $A \in \mathcal{RO} \cap (\mathcal{C}^\alpha \lhd \mathcal{RO})$. Using Theorem 2.28 and the fact that the family $\mathcal{C}^\alpha$ is closed under arbitrary intersection, we have $\mathcal{RO} \cap (\mathcal{C}^\alpha \triangleleft \mathcal{RO}) = \mathcal{C}^\alpha \cap \mathcal{RO}$. But $\mathcal{RO} \subseteq \mathcal{C}^\alpha$, which gives $\mathcal{RO} \cap (\mathcal{C}^\alpha \triangleleft \mathcal{RO}) \subseteq \mathcal{C}^\alpha$.

Corollary 2.19 (i) implies:

(1) Theorem 4.2 [111], proved for $\mathcal{B} = \mathcal{C}^\alpha$ and $\mathcal{K} = \mathcal{RO}$ holds for every $\mathcal{B}$, $\mathcal{K} \subseteq \mathcal{P}(X)$.
(2) Theorem 3.6 [92], proved for $\mathcal{B} = \mathcal{C}$ and $\mathcal{K} = \mathcal{RO}$ holds for every $\mathcal{B}$, $\mathcal{K} \subseteq \mathcal{P}(X)$.
(3) Theorem 3.24 [71], proved for $\mathcal{B} = \mathcal{C}^p$ and $\mathcal{K} = \mathcal{RSO}$ holds for every $\mathcal{B}$, $\mathcal{K} \subseteq \mathcal{P}2(X)$.
(4) Theorem 4.2 [112], proved for $\mathcal{B} = \mathcal{C}^\alpha$ and $\mathcal{K} = \mathcal{RO}$ holds for every $\mathcal{B}$, $\mathcal{K} \subseteq \mathcal{P}(X)$.
(5) Theorem 4.2, [66], proved for $\mathcal{B} = \mathcal{C}^\gamma$ and $\mathcal{K} = \mathcal{RO}$ holds for every $\mathcal{B}$, $\mathcal{K} \subseteq \mathcal{P}(X)$.
(6) Theorem 2.7, [122], proved for $\mathcal{B} = \mathcal{C}^\alpha$ and $\mathcal{K} = \mathcal{RSO}$ holds for every $\mathcal{B}, \mathcal{K} \subseteq \mathcal{P}(X)$.
(7) Theorem 3.25, [72], proved for $\mathcal{B} = \mathcal{C}^p$ and $\mathcal{K} = \mathcal{RSO}$ holds for every $\mathcal{B}, \mathcal{K} \subseteq \mathcal{P}(X)$.
(8) Theorem 3.5, [12], proved for $\mathcal{B} = \mathcal{C}$ and $\mathcal{K} = \mathcal{RSO}$ holds for every $\mathcal{B}, \mathcal{K} \subseteq \mathcal{P}(X)$.
(9) Theorem 3.24, [14], proved for $\mathcal{B} = \mathcal{RC}$ and $\mathcal{K} = \mathcal{T}$ holds for every $\mathcal{B}, \mathcal{K} \subseteq \mathcal{P}(X)$.
(10) Proposition 8, [106], proved for $\mathcal{B} = \mathcal{C}^\alpha$ and $\mathcal{K} = \mathcal{RSO}$ holds for every $\mathcal{B}, \mathcal{K} \subseteq \mathcal{P}(X)$.
(11) Theorem 3.22, [78], proved for $\mathcal{B} = \mathcal{C}^\alpha$ and $\mathcal{K} = \mathcal{RSO}$ holds for every $\mathcal{B}, \mathcal{K} \subseteq \mathcal{P}(X)$.
(12) Theorem 2.2, [32], proved for $\mathcal{B} = \mathcal{C}^\gamma$ and $\mathcal{K} = \mathcal{RSO}$ holds for every $\mathcal{B}, \mathcal{K} \subseteq \mathcal{P}(X)$.
(13) Theorem 3.15 [38], proved for $\mathcal{B} = \mathcal{C}^p$ and $\mathcal{K} = \mathcal{RO}$ holds for every $\mathcal{B}, \mathcal{K} \subseteq \mathcal{P}(X)$.
(14) Theorem 3.22 [116], proved for $\mathcal{B} = \mathcal{C}^\alpha$ and $\mathcal{K} = \mathcal{RSO}$ holds for every $\mathcal{B}, \mathcal{K} \subseteq \mathcal{P}(X)$.

As above, we will justify the first corollary because the other justifications are analogous. Theorem 4.2 [111] says that for any gαr-closed $A \subseteq X$, the set $\alpha\mathrm{cl}(A) \setminus A$ contains no nonempty regular closed set.

The family of all gαr-closed subset $A \subseteq X$ is equal to $\mathcal{C}^\alpha \triangleleft \mathcal{RO}$ as definition (6) above says. Of course, $\mathcal{RO}^{\mathbf{c}} = \mathcal{RC}$ and $\alpha\mathrm{cl}(A) = \overline{A}^{\mathcal{C}^\alpha}$. So, according to Corollary 2.19, $\alpha\mathrm{cl}(A) \setminus A$ contains no nonempty regular closed set.

By Lemma 2.21 (i), we obtain

(1) Theorem 3.17 [14], proved for $\mathcal{B} = \mathcal{RC}$ and $\mathcal{K} = \mathcal{T}$ holds for every $\mathcal{B},\mathcal{K} \subseteq \mathcal{P}(X)$.
(2) Theorem 3.8 [12], proved for $\mathcal{B} = \mathcal{C}$ and $\mathcal{K} = \mathcal{RSO}$ holds for every $\mathcal{B},\mathcal{K} \subseteq \mathcal{P}(X)$.
(3) Theorem 3.28 [72], proved for $\mathcal{B} = \mathcal{C}^p$ and $\mathcal{K} = \mathcal{RSO}$ holds for every $\mathcal{B},\mathcal{K} \subseteq \mathcal{P}(X)$.
(4) Theorem 2.10 [122], proved for $\mathcal{B} = \mathcal{C}^\alpha$ and $\mathcal{K} = \mathcal{RSO}$ holds for every $\mathcal{B},\mathcal{K} \subseteq \mathcal{P}(X)$.
(5) Theorem 4.4 [66], proved for $\mathcal{B} = \mathcal{C}^\gamma$ and $\mathcal{K} = \mathcal{RO}$ holds for every $\mathcal{B},\mathcal{K} \subseteq \mathcal{P}(X)$.
(6) Theorem 3.26 [116], proved for $\mathcal{B} = \mathcal{C}^\alpha$ and $\mathcal{K} = \mathcal{RSO}$ holds for every $\mathcal{B},\mathcal{K} \subseteq \mathcal{P}(X)$.
(7) Theorem 3.21 [38], proved for $\mathcal{B} = \mathcal{C}^p$ and $\mathcal{K} = \mathcal{RO}$ holds for every $\mathcal{B},\mathcal{K} \subseteq \mathcal{P}(X)$.

Theorem 3.17 [14] says that the property $A \subseteq B \subseteq R\mathrm{cl}(A)$ of a subset $B \subseteq X$, where $A \subseteq X$ is a generalized regular closed subset, implies that B is also a generalized regular closed.

The family of all generalized regular closed subsets of X, according to definition (1) above, is equal to $\mathcal{RC}^\alpha \triangleleft \mathcal{T}$, and of course, $R\mathrm{cl}(A) = \overline{A}^{\mathcal{RC}}$. So, Lemma 2.21 (i) gives $B \in \mathcal{RC} \triangleleft \mathcal{T}$. For the other cases, the proof is analogous.

As a consequence of Theorem 2.22, we have:

(1) Theorem 4.4 [111], proved for $\mathcal{B} = \mathcal{C}^\alpha$ and $\mathcal{K} = \mathcal{RO}$ holds for every $\mathcal{B},\mathcal{K} \subseteq \mathcal{P}(X)$.
(2) Theorem 3.27 [72], proved for $\mathcal{B} = \mathcal{C}^p$ and $\mathcal{K} = \mathcal{RSO}$ holds for every $\mathcal{B},\mathcal{K} \subseteq \mathcal{P}(X)$.
(3) Theorem 3.6 [12], proved for $\mathcal{B} = \mathcal{C}$ and $\mathcal{K} = \mathcal{RSO}$ holds for every $\mathcal{B},\mathcal{K} \subseteq \mathcal{P}(X)$.
(4) Theorem 3.24 [78], proved for $\mathcal{B} = \mathcal{C}^\alpha$ and $\mathcal{K} = \mathcal{RSO}$ holds for every $\mathcal{B},\mathcal{K} \subseteq \mathcal{P}(X)$.
(5) Theorem 4.7 [66], proved for $\mathcal{B} = \mathcal{C}^\gamma$ and $\mathcal{K} = \mathcal{RO}$ holds for every $\mathcal{B},\mathcal{K} \subseteq \mathcal{P}(X)$.
(6) Theorem 2.8 [122], proved for $\mathcal{B} = \mathcal{C}^\alpha$ and $\mathcal{K} = \mathcal{RSO}$ holds for every $\mathcal{B},\mathcal{K} \subseteq \mathcal{P}(X)$.
(7) Theorem 3.24 [116], proved for $\mathcal{B} = \mathcal{C}^\alpha$ and $\mathcal{K} = \mathcal{RSO}$ holds for every $\mathcal{B},\mathcal{K} \subseteq \mathcal{P}(X)$.

(8) Theorem 2.4 [32], proved for $\mathcal{B} = \mathcal{C}^{\gamma}$ and $\mathcal{K} = \mathcal{RSO}$ holds for every $\mathcal{B},\mathcal{K} \subseteq \mathcal{P}(X)$.

(9) Theorem 3.27 [71], proved for $\mathcal{B} = \mathcal{C}^{p}$ and $\mathcal{K} = \mathcal{RSO}$ holds for every $\mathcal{B},\mathcal{K} \subseteq \mathcal{P}(X)$.

(10) Theorem 4.4 [112], proved for $\mathcal{B} = \mathcal{C}^{\alpha}$ and $\mathcal{K} = \mathcal{RO}$ holds for every $\mathcal{B},\mathcal{K} \subseteq \mathcal{P}(X)$.

Theorem 4.4 [111] says that for any $x \in X$, the set $X \setminus \{x\}$ is a gαr-closed set or regular-open. This means, according to definition (6) above, that $X \setminus \{x\} \in \mathcal{C}^{\alpha} \lhd \mathcal{RO}$ or $X \setminus \{x\} \in \mathcal{RO}$, i.e., the formulation of Theorem 2.22.

5. Generalization by families of generalized open subsets

There are many types of notions of generalized closedness more complicated than those described above. It turns out that, in the generalizations of type $\mathcal{B} \lhd \mathcal{A}$ where $\mathcal{B} \times \mathcal{K} \in \{\mathcal{C}, \mathcal{C}^{\alpha}, \mathcal{C}^{s}, \mathcal{C}^{p}, \mathcal{C}^{\gamma}, \mathcal{C}^{\beta}\} \times \{\mathcal{T}, \mathcal{O}^{\alpha}, \mathcal{O}^{s}, \mathcal{O}^{p}, \mathcal{O}^{\gamma}, \mathcal{O}^{\beta}\}$, one can use various families $\mathcal{A} = (\mathcal{B} \lhd \mathcal{K})^{\boldsymbol{c}}$ of generalized open sets.

Many authors have defined this type of generalizations $\mathcal{B} \lhd \mathcal{A}$. Below, we present some review of such a concept.

(1) The family of $(r\omega)^{*}$-closed set [125] is equal to $\mathcal{C}^{p} \lhd (\mathcal{C} \lhd \mathcal{RSO})^{\boldsymbol{c}}$.

(2) The family of pgprw-closed set [127] is equal to $\mathcal{C}^{p} \lhd (\mathcal{C}^{\alpha} \lhd \mathcal{RSO})^{\boldsymbol{c}}$.

(3) The family of $g^{*}sr$-closed set [117] is equal to $\mathcal{RC} \lhd (\mathcal{C}^{s} \lhd \mathcal{T})^{\boldsymbol{c}}$.

(4) The family of g^{*}-closed set [57] is equal to $\mathcal{C} \lhd (\mathcal{C} \lhd \mathcal{T})^{\boldsymbol{c}}$.

(5) The family of $g\#$-closed set [58] is equal to $\mathcal{C} \lhd (\mathcal{C}^{\alpha} \lhd \mathcal{T})^{\boldsymbol{c}}$.

(6) The family of $*g$-closed set [58] is equal to $\mathcal{C} \lhd (\mathcal{C} \lhd \mathcal{O}^{s})^{\boldsymbol{c}}$.

(7) The family of middly g-closed set [94] is equal to $\mathcal{C}^{p} \lhd (\mathcal{C} \lhd \mathcal{T})^{\boldsymbol{c}}$.

(8) The family of RMG-closed set [104] is equal to $\mathcal{C}^{p} \lhd (\mathcal{C} \lhd \mathcal{RO})^{\boldsymbol{c}}$.

(9) The family of sg-closed set [121] is equal to $\mathcal{C} \lhd (\mathcal{C}^{s} \lhd \mathcal{O}^{s})^{\boldsymbol{c}}$.

(10) The family of $\beta\omega$g-closed set [84] is equal to $\mathcal{C}^{\beta} \lhd (\mathcal{C}^{\alpha} \lhd \mathcal{T})^{\boldsymbol{c}}$.

(11) The family of regular $\hat{g}$-closed set [23] is equal to $\mathcal{RC} \lhd (\mathcal{C} \lhd \mathcal{O}^{s})^{\boldsymbol{c}}$.

(12) The family of pg^{*}b-closed set [70] is equal to $\mathcal{C}^{\gamma} \lhd (\mathcal{C}^{p} \lhd \mathcal{O}^{p})^{\boldsymbol{c}}$.

(13) The family of wgr-closed set [86] is equal to $\mathcal{RC} \lhd (\mathcal{C}^{\beta} \lhd \mathcal{T})^{\boldsymbol{c}}$.

(14) The family of $\#\alpha$rg-closed set [60], [126] is equal to $\mathcal{C}^{\alpha} \lhd (\mathcal{C} \lhd \mathcal{RSO})^{\boldsymbol{c}}$.

(15) The family of g#s-closed set [55] is equal to $\mathcal{C}^{s} \lhd (\mathcal{C}^{\alpha} \lhd \mathcal{T})^{\boldsymbol{c}}$.

(16) The family of $(gsp)^{*}$-closed set [40] is equal to $\mathcal{C}^{\beta} \lhd (\mathcal{C} \lhd \mathcal{T})^{\boldsymbol{c}}$.

(17) The family of $\alpha g^{*}p$-closed set [114] is equal to $\mathcal{C}^{p} \lhd (\mathcal{C}^{\alpha} \lhd \mathcal{T})^{\boldsymbol{c}}$.

(18) The family of αg^*-closed set [114] is equal to $\mathcal{C}^\alpha \lhd (\mathcal{C}^\alpha \lhd \mathcal{T})^{\mathbf{c}}$.
(19) The family of gp^*-closed set [48] is equal to $\mathcal{C} \lhd (\mathcal{C}^p \lhd \mathcal{T})^{\mathbf{c}}$.
(20) The family of gr^*-closed set [42] is equal to $\mathcal{RC} \lhd (\mathcal{C} \lhd \mathcal{T})^{\mathbf{c}}$.
(21) The family of $\alpha\ g^*$p-closed set [102] is equal to $\mathcal{C}^\alpha \lhd (\mathcal{C}^p \lhd \mathcal{T})^{\mathbf{c}}$.
(22) The family of sg^*b-closed set [110] is equal to $\mathcal{C}^\gamma \lhd (\mathcal{C}^s \lhd \mathcal{O}^s)^{\mathbf{c}}$.
(23) The family of g^*s-closed set [97] is equal to $\mathcal{C}^s \lhd (\mathcal{C}^s \lhd \mathcal{T})^{\mathbf{c}}$.
(24) The family of g^*s-closed set [33] is equal to $\mathcal{C}^s \lhd (\mathcal{C} \lhd \mathcal{T})^{\mathbf{c}}$.
(25) The family of rps-closed set [67] is equal to $\mathcal{C}^\beta \lhd (\mathcal{C} \lhd \mathcal{RO})^{\mathbf{c}}$.
(26) The family of $s\alpha rw$-closed set [73] is equal to $\mathcal{C}^s \lhd (\mathcal{C}^\alpha \lhd (\mathcal{C} \lhd \mathcal{RSO})^{\mathbf{c}})^{\mathbf{c}}$.
(27) The family of $\#gs$-closed set [59] is equal to $\mathcal{C}^s \lhd (\mathcal{C} \lhd (\mathcal{C} \lhd \mathcal{O}^s)^{\mathbf{c}})^{\mathbf{c}}$.
(28) The family of gp**-closed set [80] is equal to $\mathcal{C} \lhd (\mathcal{C} \lhd (\mathcal{C}^p \lhd \mathcal{T})^{\mathbf{c}})^{\mathbf{c}}$.
(29) The family of sg$\omega\alpha$-closed set [87], [101] is equal to $\mathcal{C}^s \lhd (\mathcal{C}^\alpha \lhd (\mathcal{C} \lhd \mathcal{O}^s)^{\mathbf{c}})^{\mathbf{c}}$.
(30) The family of gb^*-closed set [113] is equal to $\mathcal{C}^\gamma \lhd (\mathcal{C} \lhd (\mathcal{C} \lhd \mathcal{O}^s)^{\mathbf{c}})^{\mathbf{c}}$.
(31) The family of pre g^*-closed set [45] is equal to $\mathcal{C}^p \lhd (\mathcal{C}^\alpha \lhd (\mathcal{C} \lhd \mathcal{O}^s)^{\mathbf{c}})^{\mathbf{c}}$.
(32) The family of g^*s^*-closed set [100] is equal to $\mathcal{C}^s \lhd (\mathcal{C} \lhd (\mathcal{C}, \mathcal{T})^{\mathbf{c}})^{\mathbf{c}}$.
(33) The family of αrps-closed set [39] is equal to $\mathcal{C}^\alpha \lhd (\mathcal{C}^\beta \lhd (\mathcal{C} \lhd \mathcal{RO})^{\mathbf{c}})^{\mathbf{c}}$.
(34) The family of spg$\omega\alpha$-closed set [41] is equal to $\mathcal{C}^\beta \lhd (\mathcal{C}^\alpha \lhd (\mathcal{C} \lhd \mathcal{O}^s)^{\mathbf{c}})^{\mathbf{c}}$.
(35) The family of $g\omega\alpha$-closed set [11] is equal to $\mathcal{C}^\alpha \lhd (\mathcal{C}^\alpha \lhd (\mathcal{C} \lhd \mathcal{O}^s)^{\mathbf{c}})^{\mathbf{c}}$.
(36) The family of $\tilde{g}\alpha$-closed set [25] is equal to $\mathcal{C}^\alpha \lhd (\mathcal{C}^s \lhd (\mathcal{C} \lhd (\mathcal{C} \lhd \mathcal{T})^{\mathbf{c}})^{\mathbf{c}})^{\mathbf{c}}$.
(37) The family of pg$\omega\alpha$-closed set [83] is equal to $\mathcal{C}^p \lhd (\mathcal{C}^\alpha \lhd (\mathcal{C}^\alpha \lhd (\mathcal{C} \lhd \mathcal{O}^s)^{\mathbf{c}})^{\mathbf{c}})^{\mathbf{c}}$,
(38) The family of gs^{**}-closed set [79] is equal to $\mathcal{C} \lhd (\mathcal{C}^s \lhd (\mathcal{C} \lhd (\mathcal{C} \lhd \mathcal{T})^{\mathbf{c}})^{\mathbf{c}})^{\mathbf{c}}$.
(39) The family of $\tilde{g}$s-closed set [119] is equal to $\mathcal{C}^s \lhd (\mathcal{C}^s \lhd (\mathcal{C} \lhd (\mathcal{C} \lhd \mathcal{O}^s)^{\mathbf{c}})^{\mathbf{c}})^{\mathbf{c}}$,
(40) The family of $\alpha\ g^{**}$-closed set [108] is equal to $\mathcal{C}^\alpha \lhd (\mathcal{C} \lhd (\mathcal{C} \lhd (\mathcal{C} \lhd \mathcal{T})^{\mathbf{c}})^{\mathbf{c}})^{\mathbf{c}}$.
(41) The family of $s\hat{g}$-closed set [10] is equal to $\mathcal{C}^s \lhd (\mathcal{C} \lhd \mathcal{O}^s)^{\mathbf{c}}$.

Theorem 2.29 is convenient for the investigation of relations between these types of generalizations.

In particular, one can prove that many generalizations studied in the literature are improper as shown by the following theorem.

Theorem 5.1. *For any topological space $X(\mathcal{T})$, the following conditions hold:*

(i) *gp^*-closedness (see (19)) is equivalent to closedness,*
(ii) *gp^{**}-closedness (see (28)) is equivalent to generalized closedness,*
(iii) *αg^*p-closedness (see (21)) and αrps-closedness (see (33)) are equivalent to α-closedness,*

(iv) *pg^*b-closedness (see* (12)*) and sg^*b-closedness (see* (22)*) are equivalent to γ-closedness,*

(v) *$\beta\omega g$-closedness (see* (10)*) and $spg\omega\alpha$-closedness (see* (34)*) are equivalent to β-closedness,*

(vi) *pgprw-closedness (see* (2)*), αg^*p-closedness (see* (17)*), $preg^*$-closedness (see* (31)*) and $pg\omega\alpha$-closedness (see* (37)*) are equivalent to pre-closedness.*

P r o o f.

(i) According to Theorem 2.29 (i), we have $\mathcal{C}\lhd(\mathcal{C}^p\lhd\mathcal{T})^{\mathbf{c}} \subseteq \mathcal{C}\lhd\mathcal{O}^p$, and by Theorem 3.7 (i), we get $\mathcal{C}\lhd\mathcal{O}^p = \mathcal{C}$. So, using Lemma 2.14 (iv), we obtain $\mathcal{C} = \mathcal{C}\lhd(\mathcal{C}^p\lhd\mathcal{T})^{\mathbf{c}}$.

(ii) By applying Theorem 2.29 (ii), we get

$$\mathcal{C}\lhd(\mathcal{C}\lhd\mathcal{O}^p)^{\mathbf{c}} \subseteq \mathcal{C}\lhd\left(\mathcal{C}\lhd(\mathcal{C}^p\lhd\mathcal{T})^{\mathbf{c}}\right)^{\mathbf{c}} \subseteq \mathcal{C}\lhd\mathcal{T}$$

and Theorem 3.7 (i) implies that $\mathcal{C}\lhd(\mathcal{C}\lhd\mathcal{O}^p)^{\mathbf{c}} = \mathcal{C}\lhd\mathcal{T}$, which completes the proof.

(iii) Theorem 2.29 (ii) implies that

$$\mathcal{C}^\alpha\lhd(\mathcal{C}^\beta\lhd\mathcal{T})^{\mathbf{c}} \subseteq \mathcal{C}^\alpha\lhd\left(\mathcal{C}^\beta\lhd(\mathcal{C}\lhd\mathcal{RO})^{\mathbf{c}}\right)^{\mathbf{c}} \subseteq \mathcal{C}^\alpha\lhd\mathcal{O}^\beta$$

and, since $\mathcal{C}^\alpha\lhd\mathcal{O}^\beta = \mathcal{C}^\alpha$ by Corollary 3.8 (ii), we get $\mathcal{C}^\alpha \subseteq \mathcal{C}^\alpha\lhd(\mathcal{C}^\beta\lhd\mathcal{T})^{\mathbf{c}}$ and the proof for αgp-closedness is complete. The second part can be proved analogously to the case (i).

(iv) The proof is analogous to the proof of the case (i), it is sufficient to use Corollary 3.4 (ii) instead of Theorem 3.7 (i).

(v) The proof of the case concerning (10) and of the case (34) is analogous to the proof of the case (21) and (33), respectively. In both cases, it suffices to use Corollary 3.4 (iii) instead of Corollary 3.8 (ii).

(vi) The proof of the cases concerning (2) and (17) is analogous to the case (12) of (iv), it is sufficient to use Corollary 3.4 (i) instead of Corollary 3.4 (ii). Similarly as above, the proof of the case concerning (31) is analogous to (34) of (v), where we use Corollary 3.4 (i) instead of Corollary 3.4 (iii). Now, Theorem 2.29 (iii) implies that $\mathcal{C}^p\lhd\left(\mathcal{C}^\alpha\lhd(\mathcal{C}^\alpha\lhd(\mathcal{C}\lhd\mathcal{O}^s)^{\mathbf{c}})^{\mathbf{c}}\right)^{\mathbf{c}} \subseteq \mathcal{C}^p$ and the rest of the proof proceeds as before. □

Below, we present the list of results which follow from the properties of $\mathcal{B}\lhd\mathcal{K}$ formulated in Section 2.

Lemma 2.18 (i) implies:

(1) Taking $\mathcal{B}\lhd\mathcal{K} = \mathcal{C}^\beta\lhd(\mathcal{C}\lhd\mathcal{RO})^{\mathbf{c}}$, we obtain Theorem 3.13 [67].

(2) Taking $\mathcal{B}\lhd\mathcal{K} = \mathcal{RC}\lhd(\mathcal{C}\lhd\mathcal{T})^{\mathbf{c}}$, we obtain Theorem 3.24 [42].

(3) Taking $\mathcal{B}\lhd\mathcal{K} = \mathcal{C}^p\lhd(\mathcal{C}\lhd\mathcal{RSO})^{\mathbf{c}}$, we obtain Theorem 3.28 [125].

(4) Taking $\mathcal{B} \triangleleft \mathcal{K} = \mathcal{C} \triangleleft (\mathcal{C} \triangleleft \mathcal{T})^{\boldsymbol{c}}$, we obtain Theorem 3.14 [57].
(5) Taking $\mathcal{B} \triangleleft \mathcal{K} = \mathcal{C}^s \triangleleft (\mathcal{C}^s \triangleleft (\mathcal{C} \triangleleft (\mathcal{C} \triangleleft \mathcal{O}^s)^{\boldsymbol{c}})^{\boldsymbol{c}})^{\boldsymbol{c}}$, we obtain Theorem 3.19 [119].
(6) Taking $\mathcal{B} \triangleleft \mathcal{K} = \mathcal{RC} \triangleleft (\mathcal{C} \triangleleft \mathcal{O}^s)^{\boldsymbol{c}}$, we obtain Theorem 4.4, [24].
(7) Taking $\mathcal{B} \triangleleft \mathcal{K} = \mathcal{C}^\alpha \triangleleft (\mathcal{C}^s \triangleleft (\mathcal{C} \triangleleft (\mathcal{C} \triangleleft \mathcal{T})^{\boldsymbol{c}})^{\boldsymbol{c}})^{\boldsymbol{c}}$, we obtain Theorem 2.14 (a) [25].
(8) Taking $\mathcal{B} \triangleleft \mathcal{K} = \mathcal{C}^p \triangleleft (\mathcal{C}, \mathcal{T})^{\boldsymbol{c}}$, we obtain Theorem 3.12 [52].
(9) Taking $\mathcal{B} \triangleleft \mathcal{K} = \mathcal{C}^s \triangleleft (\mathcal{C}^\alpha \triangleleft (\mathcal{C} \triangleleft \mathcal{O}^s)^{\boldsymbol{c}})^{\boldsymbol{c}}$, we obtain Theorem 3.21 [87].
(10) Taking $\mathcal{B} \triangleleft \mathcal{K} = \mathcal{C}^\alpha \triangleleft (\mathcal{C} \triangleleft \mathcal{RSO})^{\boldsymbol{c}}$, we obtain Theorem 3.23 [60].
(11) Taking $\mathcal{B} \triangleleft \mathcal{K} = \mathcal{RC} \triangleleft (\mathcal{C}^\beta \triangleleft \mathcal{T})^{\boldsymbol{c}}$, we obtain Theorem 3.7 [86].
(12) Taking $\mathcal{B} \triangleleft \mathcal{K} = \mathcal{RC} \triangleleft (\mathcal{C} \triangleleft \mathcal{O}^s)^{\boldsymbol{c}}$, we obtain Theorem 4.4 [23].
(13) Taking $\mathcal{B} \triangleleft \mathcal{K} = \mathcal{C}^\gamma \triangleleft (\mathcal{C} \triangleleft (\mathcal{C} \triangleleft \mathcal{O}^s)^{\boldsymbol{c}})^{\boldsymbol{c}}$, we obtain Theorem 4.10 [113].
(14) Taking $\mathcal{B} \triangleleft \mathcal{K} = \mathcal{C} \triangleleft (\mathcal{C}^s \triangleleft \mathcal{O}^s)^{\boldsymbol{c}}$, we obtain Theorem 4.2 [121].
(15) Taking $\mathcal{B} \triangleleft \mathcal{K} = \mathcal{C}^s \triangleleft (\mathcal{C} \triangleleft (\mathcal{C} \triangleleft \mathcal{O}^s)^{\boldsymbol{c}})^{\boldsymbol{c}}$, we obtain Theorem 3.21 [101].
(16) Taking $\mathcal{B} \triangleleft \mathcal{K} = (\mathcal{C} \triangleleft \mathcal{O}^s) \triangleleft \mathcal{O}^\alpha$, we obtain Theorem 3.9 [93].
(17) Taking $\mathcal{B} \triangleleft \mathcal{K} = \mathcal{C}^s \triangleleft (\mathcal{C} \triangleleft (\mathcal{C} \triangleleft \mathcal{T})^{\boldsymbol{c}})^{\boldsymbol{c}}$, we obtain Theorem 3.33 [100].
(18) Taking $\mathcal{B} \triangleleft \mathcal{K} = \mathcal{C}^\alpha \triangleleft (\mathcal{C}^\alpha \triangleleft (\mathcal{C} \triangleleft \mathcal{O}^s)^{\boldsymbol{c}})^{\boldsymbol{c}}$, we obtain Theorem 3.21 [11].
(19) Taking $\mathcal{B} \triangleleft \mathcal{K} = \mathcal{C}^s \triangleleft (\mathcal{C}^s \triangleleft \mathcal{T})^{\boldsymbol{c}}$, we obtain Theorem 3.16 [97].
(20) Taking $\mathcal{B} \triangleleft \mathcal{K} = \mathcal{C}^s \triangleleft (\mathcal{C} \triangleleft \mathcal{T})^{\boldsymbol{c}}$, we obtain Lemma 3.1 [33].
(21) Taking $\mathcal{B} \triangleleft \mathcal{K} = \mathcal{C}^\alpha \triangleleft (\mathcal{C} \triangleleft (\mathcal{C} \triangleleft (\mathcal{C} \triangleleft \mathcal{T})^{\boldsymbol{c}})^{\boldsymbol{c}})^{\boldsymbol{c}}$, we obtain Theorem 3.26 [108].

The following theorems follow from Lemma 2.21 (i)

(1) Taking $\mathcal{B} \triangleleft \mathcal{K} = \mathcal{C}^\beta \triangleleft (\mathcal{C} \triangleleft \mathcal{RO})^{\boldsymbol{c}}$, we obtain Theorem 3.15 [67].
(2) Taking $\mathcal{B} \triangleleft \mathcal{K} = \mathcal{RC} \triangleleft (\mathcal{C} \triangleleft \mathcal{T})^{\boldsymbol{c}}$, we obtain Theorem 3.21 [42].
(3) Taking $\mathcal{B} \triangleleft \mathcal{K} = \mathcal{C}^s \triangleleft (\mathcal{C}^\alpha \triangleleft (\mathcal{C} \triangleleft \mathcal{O}^s)^{\boldsymbol{c}})^{\boldsymbol{c}}$, we obtain Theorem 3.23 [87].
(4) Taking $\mathcal{B} \triangleleft \mathcal{K} = \mathcal{C}^\alpha \triangleleft (\mathcal{C} \triangleleft \mathcal{RSO})^{\boldsymbol{c}}$, we obtain Theorem 3.28 [60].
(5) Taking $\mathcal{B} \triangleleft \mathcal{K} = \mathcal{C} \triangleleft (\mathcal{C}^s \triangleleft (\mathcal{C} \triangleleft (\mathcal{C} \triangleleft \mathcal{T})^{\boldsymbol{c}})^{\boldsymbol{c}})^{\boldsymbol{c}}$, we obtain Theorem 4.3 [79].
(6) Taking $\mathcal{B} \triangleleft \mathcal{K} = \mathcal{RC} \triangleleft (\mathcal{C}^\beta \triangleleft \mathcal{T})^{\boldsymbol{c}}$, we obtain Theorem 3.9 [86].
(7) Taking $\mathcal{B} \triangleleft \mathcal{K} = \mathcal{RC} \triangleleft (\mathcal{C} \triangleleft \mathcal{O}^s)^{\boldsymbol{c}}$, we obtain Theorem 4.7 [23].
(8) Taking $\mathcal{B} \triangleleft \mathcal{K} = \mathcal{C}^p \triangleleft (\mathcal{C}, \mathcal{T})^{\boldsymbol{c}}$, we obtain Theorem 3.14 [52].
(9) Taking $\mathcal{B} \triangleleft \mathcal{K} = \mathcal{C}^\alpha \triangleleft (\mathcal{C}^s \triangleleft (\mathcal{C} \triangleleft (\mathcal{C} \triangleleft \mathcal{T})^{\boldsymbol{c}})^{\boldsymbol{c}})^{\boldsymbol{c}}$, we obtain Theorem 2.14 (b), [25].
(10) Taking $\mathcal{B} \triangleleft \mathcal{K} = \mathcal{RC} \triangleleft (\mathcal{C} \triangleleft \mathcal{O}^s)^{\boldsymbol{c}}$, we obtain Theorem 4.7 [24].
(11) Taking $\mathcal{B} \triangleleft \mathcal{K} = \mathcal{C} \triangleleft (\mathcal{C} \triangleleft \mathcal{T})^{\boldsymbol{c}}$, we obtain Theorem 3.15 [57].
(12) Taking $\mathcal{B} \triangleleft \mathcal{K} = \mathcal{C}^s \triangleleft (\mathcal{C}^s \triangleleft (\mathcal{C} \triangleleft (\mathcal{C} \triangleleft \mathcal{O}^s)^{\boldsymbol{c}})^{\boldsymbol{c}})^{\boldsymbol{c}}$, we obtain Theorem 3.25 [119].
(13) Taking $\mathcal{B} \triangleleft \mathcal{K} = \mathcal{C}^\gamma \triangleleft (\mathcal{C} \triangleleft (\mathcal{C} \triangleleft \mathcal{O}^s)^{\boldsymbol{c}})^{\boldsymbol{c}}$, we obtain Theorem 4.11 [113].
(14) Taking $\mathcal{B} \triangleleft \mathcal{K} = \mathcal{C} \triangleleft (\mathcal{C}^s \triangleleft \mathcal{O}^s)^{\boldsymbol{c}}$, we obtain Theorem 4.4 [121].

(15) Taking $\mathcal{B} \lhd \mathcal{K} = \mathcal{C}^s \lhd (\mathcal{C} \lhd (\mathcal{C} \lhd \mathcal{O}^s)^{\boldsymbol{c}})^{\boldsymbol{c}}$, we obtain Theorem 3.23 [101].

(16) Taking $\mathcal{B} \lhd \mathcal{K} = (\mathcal{C} \lhd \mathcal{O}^s) \lhd \mathcal{O}^\alpha$, we obtain Theorem 3.10 [93].

(17) Taking $\mathcal{B} \lhd \mathcal{K} = \mathcal{C}^s \lhd (\mathcal{C} \lhd (\mathcal{C} \lhd \mathcal{T})^{\boldsymbol{c}})^{\boldsymbol{c}}$, we obtain Theorem 3.35 [100].

(18) Taking $\mathcal{B} \lhd \mathcal{K} = \mathcal{C}^\alpha \lhd (\mathcal{C}^\alpha \lhd (\mathcal{C} \lhd \mathcal{O}^s)^{\boldsymbol{c}})^{\boldsymbol{c}}$, we obtain Theorem 3.23 [11].

(19) Taking $\mathcal{B} \lhd \mathcal{K} = \mathcal{C}^\alpha \lhd (\mathcal{C} \lhd (\mathcal{C} \lhd (\mathcal{C} \lhd \mathcal{T})^{\boldsymbol{c}})^{\boldsymbol{c}})^{\boldsymbol{c}}$, we obtain Theorem 2.10 [108].

By Lemma 2.21 (ii), we have

(1) Taking $\mathcal{B} \lhd \mathcal{K} = \mathcal{C}^s \lhd (\mathcal{C} \lhd (\mathcal{C} \lhd \mathcal{O}^s)^{\boldsymbol{c}})^{\boldsymbol{c}}$, we obtain Theorem 3.38 [101].

(2) Taking $\mathcal{B} \lhd \mathcal{K} = \mathcal{C}^\alpha \lhd (\mathcal{C} \lhd (\mathcal{C} \lhd (\mathcal{C} \lhd \mathcal{T})^{\boldsymbol{c}})^{\boldsymbol{c}})^{\boldsymbol{c}}$, we obtain Theorem 3.3 [108].

The following results are consequences of Theorem 2.22:

(1) Taking $\mathcal{B} \lhd \mathcal{K} = \mathcal{C}^\beta \lhd (\mathcal{C} \lhd \mathcal{RO})^{\boldsymbol{c}}$, we obtain Theorem 3.16 [67].

(2) Taking $\mathcal{B} \lhd \mathcal{K} = \mathcal{RC} \lhd (\mathcal{C} \lhd \mathcal{T})^{\boldsymbol{c}}$, we obtain Theorem 3.26 [42].

(3) Taking $\mathcal{B} \lhd \mathcal{K} = \mathcal{C}^\gamma \lhd (\mathcal{C} \lhd (\mathcal{C} \lhd \mathcal{O}^s)^{\boldsymbol{c}})^{\boldsymbol{c}}$, we obtain Theorem 4.14 [113].

(4) Taking $\mathcal{B} \lhd \mathcal{K} = \mathcal{C}^s \lhd (\mathcal{C}^s \lhd (\mathcal{C} \lhd (\mathcal{C} \lhd \mathcal{O}^s)^{\boldsymbol{c}})^{\boldsymbol{c}})^{\boldsymbol{c}}$, we obtain Theorem 3.27 [119].

(5) Taking $\mathcal{B} \lhd \mathcal{K} = \mathcal{C}^\alpha \lhd (\mathcal{C}^s \lhd (\mathcal{C} \lhd (\mathcal{C} \lhd \mathcal{T})^{\boldsymbol{c}})^{\boldsymbol{c}})^{\boldsymbol{c}}$, we obtain Theorem 2.18 (i) [25].

(6) Taking $\mathcal{B} \lhd \mathcal{K} = \mathcal{C}^s \lhd (\mathcal{C}^\alpha \lhd (\mathcal{C} \lhd \mathcal{O}^s)^{\boldsymbol{c}})^{\boldsymbol{c}}$, we obtain Theorem 3.27 [87].

(7) Taking $\mathcal{B} \lhd \mathcal{K} = \mathcal{C}^\alpha \lhd \mathcal{C}^\alpha \lhd (\mathcal{C} \lhd \mathcal{RSO})^{\boldsymbol{c}}$, we obtain Theorem 3.27 [60].

(8) Taking $\mathcal{B} \lhd \mathcal{K} = \mathcal{C}^\alpha \lhd (\mathcal{C}^s \lhd (\mathcal{C} \lhd (\mathcal{C} \lhd \mathcal{T})^{\boldsymbol{c}})^{\boldsymbol{c}})^{\boldsymbol{c}}$, we obtain Theorem 2.18(iii) [25].

(9) Taking $\mathcal{B} \lhd \mathcal{K} = \mathcal{C}^p \lhd (\mathcal{C} \lhd \mathcal{RSO})^{\boldsymbol{c}}$, we obtain Theorem 3.26 [125].

(10) Taking $\mathcal{B} \lhd \mathcal{K} = \mathcal{RC} \lhd (\mathcal{C}^s \lhd \mathcal{T})^{\boldsymbol{c}}$, we obtain Theorem 3.16 [117].

(11) Taking $\mathcal{B} \lhd \mathcal{K} = \mathcal{C} \lhd (\mathcal{C}^s \lhd \mathcal{O}^s)^{\boldsymbol{c}}$, we obtain Theorem 4.8 [121].

(12) Taking $\mathcal{B} \lhd \mathcal{K} = \mathcal{C}^s \lhd (\mathcal{C} \lhd (\mathcal{C} \lhd \mathcal{O}^s)^{\boldsymbol{c}})^{\boldsymbol{c}}$, we obtain Theorem 3.27 [59].

(13) Taking $\mathcal{B} \lhd \mathcal{K} = \mathcal{C}^\alpha \lhd (\mathcal{C}^\alpha \lhd (\mathcal{C} \lhd \mathcal{O}^s)^{\boldsymbol{c}})^{\boldsymbol{c}}$, we obtain Theorem 3.25 [11].

(14) Taking $\mathcal{B} \lhd \mathcal{K} = \mathcal{C}^s \lhd (\mathcal{C}^s \lhd \mathcal{T})^{\boldsymbol{c}}$, we obtain Theorem 3.21 [97].

(15) Taking $\mathcal{B} \lhd \mathcal{K} = \mathcal{C}^s \lhd (\mathcal{C} \lhd \mathcal{T})^{\boldsymbol{c}}$, we obtain Theorem 4.2 [33].

(16) Taking $\mathcal{B} \lhd \mathcal{K} = \mathcal{C}^\alpha \lhd (\mathcal{C} \lhd (\mathcal{C} \lhd (\mathcal{C} \lhd \mathcal{T})^{\boldsymbol{c}})^{\boldsymbol{c}})^{\boldsymbol{c}}$, we obtain Theorem 2.11 [108].

From Theorem 2.28 we have:

(1) Taking $\mathcal{B} \lhd \mathcal{K} = \mathcal{C}^s \lhd (\mathcal{C} \lhd (\mathcal{C} \lhd \mathcal{O}^s)^{\boldsymbol{c}})^{\boldsymbol{c}}$, we obtain Theorem 3.26 [101].

(2) Taking $\mathcal{B} \lhd \mathcal{K} = \mathcal{C}^\alpha \lhd (\mathcal{C}^\alpha \lhd (\mathcal{C} \lhd \mathcal{O}^s)^{\boldsymbol{c}})^{\boldsymbol{c}}$, we obtain Theorem 3.30 [11].

(3) Taking $\mathcal{B} \lhd \mathcal{K} = \mathcal{C}^s \lhd (\mathcal{C} \lhd \mathcal{T})^{\boldsymbol{c}}$, we obtain Lemma 3.2 [33].

Our study leaves an open question. Before it, let us use the following convenient notation:

For a given family $\mathcal{B} \subset \mathcal{P}(X)$, we denote by $[\mathcal{B}](\mathcal{K})$ the family $\mathcal{B} \lhd \mathcal{K}^{[c]}$ and, in general, $[\mathcal{B}]^{n+1}(\mathcal{K}) = [\mathcal{B}]\big([\mathcal{B}]^n(\mathcal{K})\big)$ for all positive integer n, where $[\mathcal{B}]^0(\mathcal{K}) = \mathcal{K}$ and $[\mathcal{B}]^1(\mathcal{K}) = [\mathcal{B}](\mathcal{K})$.

Using Theorem 2.29, it is easy to show that for any family $\mathcal{B} \subset \mathcal{P}(X)$ the following

$$\mathcal{B} \subset [\mathcal{B}]^n(\mathcal{B}) \subset [\mathcal{B}](\mathcal{B}) \quad \text{for any } \ n \in N$$

holds.

In particular, for any $n \in N$ we have $\mathcal{C} \subset [\mathcal{C}]^n(\mathcal{C}) \subset \mathcal{C} \triangleleft \mathcal{T}$, so it remains the following open question

Question 1. Under what conditions on a topological space $X(\mathcal{T})$, such that $\mathcal{C} \neq \mathcal{C} \triangleleft \mathcal{T}$, does the sequence $\mathcal{C}, [\mathcal{C}]^1(\mathcal{C}), [\mathcal{C}]^2(\mathcal{C}), [\mathcal{C}]^3(\mathcal{C}), \ldots$ satisfy condition $[\mathcal{C}]^n(\mathcal{C}) = [\mathcal{C}]^{n+1}(\mathcal{C})$ for every $n \geq 1$?

6. Conclusion

Some topological properties that are considered on the base of generalized closed sets have been found to be useful in the study of certain objects of digital topology and subsequently possible application in computers [50, 53, 81] and quantum mechanics. So, a general unified approach to their study can be turned out helpful to the development of the use of this theory, because it delivers convenient tools. In particular, one can check which of the generalizations studied in the literature are improper.

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