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GRADED CLASSICAL WEAKLY PRIME SUBMODULES OVER NON-COMMUTATIVE GRADED RINGS

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ABSTRACT. The goal of this article is to propose and examine the notion of graded classical weakly prime submodules over non-commutative graded rings which is a generalization of the concept of graded classical weakly prime submodules over commutative graded rings. We investigate the structure of these types of submodules in various categories of graded modules.

1. Introduction

Throughout this article, all rings considered are associative and have a nonzero unity. Similarly, all modules are unital left modules. Let G be a multiplicative group with identity e, and let A be a ring with nonzero unity 1. A ring A is called G-graded if

$$A = \bigoplus_{g \in G} A_g \quad \text{with} \quad A_g A_h \subseteq A_{gh} \quad \text{for all } g, h \in G,$$

where A_g is an additive subgroup of A for each $g \in G$. Here $A_g A_h$ denotes the set of all finite sums of elements $a_g b_h$, where $a_g \in A_g$ and $b_h \in A_h$. This structure is denoted as $G(A)$. The elements of A_g are called homogeneous of degree g. If $a \in A$, it can be uniquely written as $a = \sum_{g \in G} a_g$, where a_g is the component

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of a in A_g, and $a_g = 0$ for all but finitely many g. The additive subgroup A_e is a subring of A and $1 \in A_e$. The set of all homogeneous elements of A is $\bigcup_{g\in G} A_g$ and is denoted by $h(A)$. Let P be a left ideal of a G-graded ring A. Then P is called a graded left ideal if

$$P = \bigoplus_{g\in G}(P \cap A_g), \quad \text{i.e., for } a \in P, a = \sum_{g\in G} a_g,$$

where $a_g \in P$ for all $g \in G$. Not all left ideals of a graded ring are necessarily graded left ideals (see Example 1.1 in [3]).

Let A be a G-graded ring. A left A-module M is called G-graded if

$$M = \bigoplus_{g\in G} M_g \quad \text{with } A_g M_h \subseteq M_{gh} \quad \text{for all} \quad g, h \in G,$$

where M_g is an additive subgroup of M for all $g \in G$. The elements of M_g are called homogeneous of degree g. If $m \in M$, then m can be written uniquely as $m = \sum_{g\in G} m_g$, where m_g is the component of m in M_g and $m_g = 0$ for all but finitely many g. Each M_g is an A_e-module. The set of all homogeneous elements of M is $\bigcup_{g\in G} M_g$ and is denoted by $h(M)$. Let K be an A-submodule of M. K is called a graded submodule of M if

$$K = \bigoplus_{g\in G}(K \cap M_g), \quad \text{i.e., for } x \in K, x = \sum_{g\in G} x_g$$

where $x_g \in K$ for all $g \in G$. As with graded ideals, not all A-submodules of a graded A-module are necessarily graded submodules. For further details and terminology, see [10, 14].

The concept of graded prime ideals in commutative graded rings was introduced in [17]. A proper graded ideal P of a commutative graded ring A is said to be graded prime if, whenever $x, y \in h(A)$ such that $xy \in P$, then either $x \in P$ or $y \in P$. This notion and its generalizations are central in graded commutative algebra, as they provide valuable tools for studying the properties of graded commutative rings. Several generalizations of graded prime ideals have been proposed and explored. For instance, Atani introduced the concept of graded weakly prime ideals in [5]. A proper graded ideal P of a commutative graded ring A is said to be a graded weakly prime ideal if, whenever $x, y \in h(A)$ and $0 \neq xy \in P$, then $x \in P$ or $y \in P$. By ([5], Theorem 2.12), the following statements are equivalent for a graded ideal P of $G(A)$ with $P \neq A$, where A is a commutative graded ring:

(1) P is a graded weakly prime ideal of $G(A)$.

(2) For each $g, h \in G$, the inclusion $0 \neq IJ \subseteq P$ with A_e-submodules I of A_g and J of A_h implies that $I \subseteq P$ or $J \subseteq P$.

For graded rings that are not necessarily commutative, it is clear that (2) does not imply (1). In [3], Alshehry and Abu-Dawwas defined a graded weakly prime ideal for non-commutative graded rings as follows: a graded left ideal P of $G(A)$ with $P \neq A$ is said to be a graded weakly prime ideal of $G(A)$ if for each $g, h \in G$, the inclusion $0 \neq IJ \subseteq P$ with A_e-submodules I of A_g and J of A_h implies that $I \subseteq P$ or $J \subseteq P$. Equivalently, whenever $x, y \in h(A)$ such that $0 \neq xAy \subseteq P$, then either $x \in P$ or $y \in P$ ([3, Proposition 2.3]). In [2], the standard definition of a graded prime ideal P for a graded non-commutative ring A is that $P \neq A$ and whenever I and J are graded left ideals of A such that $IJ \subseteq P$, then either $I \subseteq P$ or $J \subseteq P$. Accordingly, in [3], they defined a graded left ideal of a graded ring A to be a graded weakly prime as follows: a proper graded left ideal P of A is said to be a graded weakly prime ideal of A if whenever I and J are graded left ideals of A such that $0 \neq IJ \subseteq P$, then either $I \subseteq P$ or $J \subseteq P$.

Let K be a graded A-submodule of a left A-module M. Then $(K :_A M) = \{a \in A : aM \subseteq K\}$ is a graded two-sided ideal of A [8]. K is said to be faithful if $\mathrm{Ann}_A(K) = (0 :_A K) = 0$. In [2], a proper graded A-submodule K of a graded A-module M over a non-commutative graded ring A is said to be graded prime if whenever L is a graded A-submodule of M and I is a graded ideal of A such that $IL \subseteq K$, then either $L \subseteq K$ or $I \subseteq (K :_A M)$. If A is commutative, this definition is equivalent to: a proper graded A-submodule K of a graded A-module M is said to be graded prime if whenever $a \in h(A)$ and $x \in h(M)$ are such that $ax \in K$, then either $x \in K$ or $a \in (K :_A M)$. In a similar way, a proper graded A-submodule K of a graded A-module M over a non-commutative graded ring A is said to be graded weakly prime if whenever L is a graded A-submodule of M and I is a graded ideal of A such that $0 \neq IL \subseteq K$, then either $L \subseteq K$ or $I \subseteq (K :_A M)$. If A is commutative, this definition is equivalent to: a proper graded A-submodule K of a graded A-module M is said to be graded weakly prime if whenever $a \in h(A)$ and $x \in h(M)$ are such that $0 \neq ax \in K$, then either $x \in K$ or $a \in (K :_A M)$. Let K be a graded A-submodule of M and $g \in G$ such that $K_g \neq M_g$. Then K is said to be a g-prime A-submodule of M if whenever L is an A_e-submodule of M_g and I is an ideal of A_e such that $IL \subseteq K$, then either $L \subseteq K$ or $I \subseteq (K :_A M)$.

The concept of graded weakly classical prime submodules over commutative graded rings has been proposed and studied by Abu-Dawwas and Al-Zoubi in [1]. A proper graded A-submodule K of M over a commutative graded ring A is said to be a graded weakly classical prime if whenever $x, y \in h(A)$ and $m \in h(M)$ such that $0 \neq xym \in K$, then either $xm \in K$ or $ym \in K$. In this article, we introduce and examine the concept of graded classical weakly prime submodules over non-commutative graded rings. Indeed, this article is motivated by the concepts and the techniques that have been examined in [11]. We propose the following:

a proper graded A-submodule K of M over a non-commutative graded ring A is said to be a graded classical weakly prime if whenever $x, y \in h(A)$ and L is a graded A-submodule of M such that $0 \neq xAyL \subseteq K$, then either $xL \subseteq K$ or $yL \subseteq K$. Several properties have been examined. Also, we investigate the structure of graded classical weakly prime submodules in various categories of graded modules.

2. Graded Classical Weakly Prime Submodules

In this section, we introduce and examine graded classical weakly prime submodules over non-commutative graded rings.

Definition 2.1. Let A be a graded ring, M be a graded A-module, K be a graded A-submodule of M, and $g \in G$. Then

(1) K is called a graded classical weakly prime A-submodule of M if $K \neq M$ and whenever $x, y \in h(A)$ and L is a graded A-submodule of M such that $0 \neq xAyL \subseteq K$, then either $xL \subseteq K$ or $yL \subseteq K$.

(2) K is called a graded completely classical weakly prime A-submodule of M if $K \neq M$ and whenever $x, y \in h(A)$ and $z \in h(M)$ such that $0 \neq xyz \in K$, then either $xz \in K$ or $yz \in K$.

(3) K is called a g-classical weakly prime A-submodule of M if $K_g \neq M_g$ and whenever $x, y \in A_e$ and L is an A_e-submodule of M_g such that $0 \neq xA_eyL \subseteq K$, then either $xL \subseteq K$ or $yL \subseteq K$.

Clearly, every graded prime A-submodule of M is a graded classical weakly prime A-submodule of M. Additionally, if A is a commutative graded ring with unity, then the concepts of graded classical weakly prime submodules and graded completely classical weakly prime submodules coincide. However, the following two examples demonstrate that this equivalence does not hold for non-commutative graded rings.

Example 2.2. Consider $A = M = M_2(\mathbb{Z})$ (The 2×2 matrices over the ring of integers $\mathbb{Z}$) and $G = \mathbb{Z}_4$ (The additive group of integers modulo 4). The ring A is G-graded as follows:

$$A_0 = \begin{pmatrix} \mathbb{Z} & 0 \\ 0 & \mathbb{Z} \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & \mathbb{Z} \\ \mathbb{Z} & 0 \end{pmatrix} \quad \text{and} \quad A_1 = A_3 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

The module M is also G-graded with the same grading as A. Let $K = M_2(2\mathbb{Z})$. Then K is a graded prime A-submodule of M, and hence K is a graded classical weakly prime A-submodule of M. On the other hand, K is not a graded completely classical weakly prime A-submodule of M since

$$x = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \in h(A), \quad y = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in h(A) \quad \text{and} \quad z = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} \in h(M)$$

are such that $0 \neq xyz \in K$, $xz \notin K$ and $yz \notin K$.

Example 2.3. Let $A = M = M_3(\mathbb{Z})$ and $G = \mathbb{Z}$. The ring A is G-graded as follows:

$$A_0 = \begin{pmatrix} \mathbb{Z} & 0 & 0 \\ 0 & \mathbb{Z} & 0 \\ 0 & 0 & \mathbb{Z} \end{pmatrix}, \qquad A_1 = \begin{pmatrix} 0 & \mathbb{Z} & 0 \\ 0 & 0 & \mathbb{Z} \\ 0 & 0 & 0 \end{pmatrix},$$

$$A_2 = \begin{pmatrix} 0 & 0 & \mathbb{Z} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad A_n \; ! = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

for $|n| > 2$ and $A_{-n} = A_n^t$ (the transpose of A_n), for $n = 1, 2$. Then $K = M_3(2\mathbb{Z})$ is a graded prime A-submodule of M, and thus K is a graded classical weakly prime A-submodule of M. On the other hand, K is not a graded completely classical weakly prime A-submodule of M since

$$x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in h(A), \qquad y = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \in h(A)$$

and

$$z = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 3 & 0 & 0 \end{pmatrix} \in h(M)$$

are such that $0 \neq xyz \in K$, $xz \notin K$ and $yz \notin K$.

Example 2.4. Consider the ring $A = M_3(\mathbb{F})$ (the ring of 3×3 matrices over a field $\mathbb{F}$) and the group $G = \mathbb{Z}$. To construct a grading of A by G where all components are nonzero, we proceed as follows:

(1) A standard basis for $M_3(\mathbb{F})$ is given by the elementary matrices e_{ij}, where e_{ij} has 1 in the (i, j)-th position and 0 elsewhere. The set $\{e_{ij} \mid 1 \leq i, j \leq 3\}$ forms a basis for the vector space $M_3(\mathbb{F})$.

(2) We assign degrees from $\mathbb{Z}$ to each matrix entry in such a way that no component is zero. Specifically, we define the grading as:

$$A = \bigoplus_{g \in \mathbb{Z}} A_g,$$

where $A_g = \text{span}\{e_{ij} \mid \deg(e_{ij}) = g\}$. The degree of each elementary matrix e_{ij} is given by a degree map $\deg : \{e_{ij}\} \to \mathbb{Z}$.

(3) Using $deg(e_{ij}) = j - i$, we assign the following degrees to the elementary matrices:

$$\begin{array}{lll} \deg(e_{11}) = \ \ 0, & \deg(e_{12}) = \ \ 1, & \deg(e_{13}) = 2, \\ \deg(e_{21}) = -1, & \deg(e_{22}) = \ \ 0, & \deg(e_{23}) = 1, \\ \deg(e_{31}) = -2, & \deg(e_{32}) = -1, & \deg(e_{33}) = 0. \end{array}$$

(4) With the degree assignment above, the graded components A_g for $g \in \mathbb{Z}$ are

$$A_0 = \text{span}\{e_{11}, e_{22}, e_{33}\}, \quad A_1 = \text{span}\{e_{12}, e_{23}\}, \quad A_2 = \text{span}\{e_{13}\},$$

$$A_{-1} = \text{span}\{e_{21}, e_{32}\}, \quad A_{-2} = \text{span}\{e_{31}\},$$

and other components are determined by the property

$$A_g A_h = A_{g+h}.$$

All graded components are nonzero, which satisfies the requirement.

Definition 2.5. [2] Let A be a graded ring, M be a graded A-module, K be a graded A-submodule of M and $g \in G$. Then K is said to be a g-prime A-submodule of M if $K_g \neq M_g$ and whenever I is an ideal of A_e and L is an A_e-submodule of M_g such that $IL \subseteq K$, then either

$$L \subseteq K \quad \text{or} \quad I \subseteq (K :_A M) = \{a \in A : aM \subseteq K\}.$$

Clearly, the zero submodule is always graded classical weakly prime and g-classical weakly prime for all $g \in G$ by definition. However, the following example demonstrates that one can find $g \in G$ such that the zero submodule is not a g-prime.

Example 2.6. Consider $A = M = M_2(\mathbb{Z})$ and $G = \mathbb{Z}_4$. Then A is G-graded by

$$A_0 = \begin{pmatrix} \mathbb{Z} & 0 \\ 0 & \mathbb{Z} \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & \mathbb{Z} \\ \mathbb{Z} & 0 \end{pmatrix} \quad \text{and} \quad A_1 = A_3 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

M also is a G-graded left A-module by the same graduation of A. Choose

$$x = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in A_0.$$

Then

$$I = A_0 x A_0 = \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{Z} \end{pmatrix} \text{ is an ideal of } A_0.$$

Choose

$$y = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in M_2.$$

Then

$$L = A_0 y = \begin{pmatrix} 0 & \mathbb{Z} \\ 0 & 0 \end{pmatrix} \text{is an } A_0\text{-submodule of } M_2.$$

Consider the graded A-submodule

$$K = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \text{ of } M.$$

Then

$$K_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \neq M_2 \text{ and } IL \subseteq K,$$

but $L \nsubseteq K$ and $I \nsubseteq (K :_A M)$. Hence, K is not a 2-prime A-submodule of M.

Theorem 2.7. *Let M be a graded A-module, K be a graded A-submodule of M and $g \in G$ such that $K_g \neq M_g$ and every A_e-submodule of M_g is faithful. Then K is a g-classical weakly prime A-submodule of M if and only if, whenever I, J are ideals of A_e and L is an A_e-submodule of M_g with $0 \neq IJL \subseteq K$, we have either $IL \subseteq K$ or $JL \subseteq K$.*

Proof. Suppose K is a g-classical weakly prime A-submodule of M. Let I, J be ideals of A_e and L an A_e-submodule of M_g with $0 \neq IJL \subseteq K$. Suppose that $IL \nsubseteq K$ and $JL \nsubseteq K$. Then there exist $r \in I$ and $s \in J$ such that $rL \nsubseteq K$ and $sL \nsubseteq K$. Since $rA_esL \subseteq IJL \subseteq K$, and K is g-weakly classical prime, this implies $rA_esL = 0$, so $rsL = 0$. Thus, $rs = 0$. To show $IJ = 0$, let $a \in I$ and $b \in J$. If $aL \nsubseteq K$ and $bL \nsubseteq K$, then $ab = 0$ as shown earlier. If $aL \nsubseteq K$ and $bL \subseteq K$, then $(s + b)L \nsubseteq K$, and then $a(s + b) = 0$. Since $as = 0$, $ab = 0$. Similarly, if $aL \subseteq K$ and $bL \nsubseteq K$, then $ab = 0$. If $aL \subseteq K$ and $bL \subseteq K$, then $(r + a)L \nsubseteq K$ and $(s + b)L \nsubseteq K$, and then $(r + a)s = r(s + b) = (r + a)(s + b) = rs = 0$, which gives that $ab = 0$. Thus $IJ = 0$, and then $IJL = 0$, which is a contradiction. Hence, either $IL \subseteq K$ or $JL \subseteq K$. Conversely, let $x, y \in A_e$ and L be an A_e-submodule M_g with $0 \neq xA_eyL \subseteq K$. Then $I = A_ex$ and $J = A_ey$ are ideals of A_e with $0 \neq IJL \subseteq K$, and then either $IL \subseteq K$ or $JL \subseteq K$ by assumption, and hence either $xL \subseteq K$ or $yL \subseteq K$. Thus K is a g-classical weakly prime A-submodule of M. □

Remark 2.8. Upon careful examination, we could not identify an example where the faithfulness condition in Theorem 2.7 does not hold and, consequently, where the theorem fails. This suggests that the faithfulness condition may be essential for the theorem's validity. Faithfulness ensures that the submodules behave consistently, avoiding trivial or degenerate cases. While relaxing this assumption would be valuable, such an example remains elusive, supporting the conjecture that the faithfulness condition is integral to the problem's structure.

Corollary 2.9. *Let M be a graded A-module, $g \in G$ such that every A_e-submodule of M_g is faithful, and K be a g-classical weakly prime A-submodule of M. Suppose that L is an A_e-submodule of M_g, $x \in A_e$ and I is an ideal of A_e.*

(1) *If $0 \neq xIL \subseteq K$, then either $xL \subseteq K$ or $IL \subseteq K$.*

(2) *If $0 \neq IxL \subseteq K$, then either $xL \subseteq K$ or $IL \subseteq K$.*

Theorem 2.10. *Let M be a graded A-module and K be a g-classical weakly prime A-submodule of M. If L is a faithful A_e-submodule of M_g with $L \nsubseteq K_g$, then $(K_g :_{A_e} L)$ is a weakly prime left ideal of A_e.*

Proof. Clearly, $(K_g :_{A_e} L)$ is a left ideal of A_e, and since $L \nsubseteq K_g$, $(K_g :_{A_e} L)$ is a proper ideal of A_e. Let I, J be two ideals of A_e such that $0 \neq IJ \subseteq (K_g :_{A_e} L)$. Then $0 \neq IJL \subseteq K_g \subseteq K$, and then by Theorem 2.7, either $IL \subseteq K$ or $JL \subseteq K$. On the other hand, $IL \subseteq A_e M_g \subseteq M_g$, and similarly, $JL \subseteq M_g$. So, either $IL \subseteq K \bigcap M_g = K_g$ or $JL \subseteq K \bigcap M_g = K_g$, and thus either $I \subseteq (K_g :_{A_e} L)$ or $J \subseteq (K_g :_{A_e} L)$. Hence, $(K_g :_{A_e} L)$ is a weakly prime left ideal of A_e. □

Similarly, one can prove the following:

Theorem 2.11. *Let M be a graded A-module and K be a g-classical weakly prime A-submodule of M. If L is a faithful A_e-submodule of M_g with $L \nsubseteq K$, then $(K :_{A_e} L)$ is a weakly prime left ideal of A_e.*

Theorem 2.12. *Let M be a graded A-module and K be a g-classical weakly prime A-submodule of M. If $\mathrm{Ann}_{A_e}(M_g) = \{0\}$, then $(K_g :_{A_e} M_g)$ is a weakly prime left ideal of A_e.*

Proof. Since K is a g-classical weakly prime A-submodule of M, $K_g \neq M_g$, and then $(K_g :_{A_e} M_g)$ is a proper left ideal of A_e. Let I, J be two ideals of A_e such that $0 \neq IJ \subseteq (K_g :_{A_e} M_g)$. Then $IJM_g \subseteq K_g$. If $IJM_g \neq 0$, then $0 \neq IJM_g \subseteq K_g \subseteq K$, and then by Theorem 2.7, either $IM_g \subseteq K$ or $JM_g \subseteq K$. On the other hand, $IM_g \subseteq A_e M_g \subseteq M_g$, and similarly, $JM_g \subseteq M_g$. So, either $IM_g \subseteq K \bigcap M_g = K_g$ or $JM_g \subseteq K \bigcap M_g = K_g$, and hence either $I \subseteq (K_g :_{A_e} M_g)$ or $J \subseteq (K_g :_{A_e} M_g)$. If $IJM_g = 0$, then $IJ \subseteq \mathrm{Ann}_{A_e}(M_g) = \{0\}$, a contradiction. Thus, $(K_g :_{A_e} M_g)$ is a weakly prime left ideal of A_e. □

Let M, S be two G-graded A-modules. An A-homomorphism $f : M \to S$ is called a graded A-homomorphism if $f(M_g) \subseteq S_g$ for all $g \in G$ [14].

Theorem 2.13. *Let M, S be two G-graded A-modules and $f : M \to S$ be a graded A-homomorphism.*

(1) *If f is injective and K is a graded classical weakly prime A-submodule of S with $f^{-1}(K) \neq M$, then $f^{-1}(K)$ is a graded classical weakly prime A-submodule of M.*

(2) *If f is surjective and K is a graded classical weakly prime A-submodule of M with $\mathrm{Ker}(f) \subseteq K$, then $f(K)$ is a graded classical weakly prime A-submodule of S.*

P r o o f.

(1) By ([15], Lemma 3.11 (1)), $f^{-1}(K)$ is a graded A-submodule of M. Let $x, y \in h(A)$ and L be a graded A-submodule of M such that $0 \neq xAyL \subseteq f^{-1}(K)$. Then $f(L)$ is a graded A-submodule of S by ([15], Lemma 3.11 (2)) such that $0 \neq xAyf(L) = f(xAyL) \subseteq K$, and then either $f(xL) = xf(L) \subseteq K$ or $f(yL) = yf(L) \subseteq K$, which implies that either $xL \subseteq f^{-1}(K)$ or $yL \subseteq f^{-1}(K)$. Hence, $f^{-1}(K)$ is a graded classical weakly prime A-submodule of M.

(2) By ([15], Lemma 3.11 (2)), $f(K)$ is a graded A-submodule of S. Let $x, y \in h(A)$ and L be a graded A-submodule of S such that $0 \neq xAyL \subseteq f(K)$. Then by ([15], Lemma 3.11 (1)), $T = f^{-1}(L)$ is a graded A-submodule of M such that $f(xAyT) = xAyf(T) = xAyL \subseteq f(K)$, and then $0 \neq xAyT \subseteq K$ as $\mathrm{Ker}(f) \subseteq K$. So, either $xT \subseteq K$ or $yT \subseteq K$, and then either $xL = xf(T) = f(xT) \subseteq f(K)$ or $yL = yf(T) = f(yT) \subseteq f(K)$. Hence, $f(K)$ is a graded classical weakly prime A-submodule of S. □

Let M be a G-graded A-module and T be a graded A-submodule of M. Then M/T is a G-graded A-module with $(M/T)_g = (M_g + T)/T$ for all $g \in G$. By ([18], Lemma 2.11), if K is an A-submodule of M with $T \subseteq K$, then K is a graded A-submodule of M if and only if K/T is a graded A-submodule of M/T.

Theorem 2.14. *Let M be a graded A-module and T, K be proper graded A-submodules of M with $T \subsetneq K$. If K is a graded classical weakly prime A-submodule of M, then K/T is a graded classical weakly prime A-submodule of M/T.*

P r o o f. Let $x, y \in h(A)$ and L/T be a graded A-submodule of M/T such that $(0+T)/T \neq (xAyL+T)/T = xAy\big((L+T)/T\big) \subseteq K/T$. Then L is a graded A-submodule of M such that $0 \neq xAyL \subseteq K$, and then either $xL \subseteq K$ or $yL \subseteq K$, which implies that either $x\big((L+T)/T\big) = (xL+T)/T \subseteq (K+T)/T \subseteq K/T$ or $y\big((L+T)/T\big) = (yL+T)/T \subseteq (K+T)/T \subseteq K/T$. Hence, K/T is a graded classical weakly prime A-submodule of M/T. □

Theorem 2.15. *Let M be a graded A-module and T, K be proper graded A-submodules of M with $T \subsetneq K$. If T is a graded classical weakly prime A-submodule of M and K/T is a graded classical weakly prime A-submodule of M/T, then K is a graded classical weakly prime A-submodule of M.*

P r o o f. Let $x, y \in h(A)$ and L be a graded A-submodule of M such that $xAyL \subseteq K$. If $xAyL \subseteq T$, then either $xL \subseteq T \subsetneq K$ or $yL \subseteq T \subsetneq K$ as required. Suppose that $xAyL \nsubseteq T$. Then

$$(0+T)/T \neq xAy\big((L+T)/T\big) = (xAyL+T)/T \subseteq (K+T)/T \subseteq K/T,$$

and then either

$$(xL+T)/T = x\big((L+T)/T\big) \subseteq K/T \quad \text{or} \quad (yL+T)/T = y\big((L+T)/T\big) \subseteq K/T,$$

which implies that either $xL \subseteq K$ or $yL \subseteq K$. Hence, K is a graded classical weakly prime A-submodule of M. □

Graded weakly 2-absorbing ideals over non-commutative graded rings were introduced and examined in [4]. A proper graded ideal P of A is said to be raded weakly 2-absorbing if, whenever $x, y, z \in h(A)$ such that $0 \neq xAyAz \subseteq P$, then $xy \in P$ or $yz \in P$ or $xz \in P$. In this article, we introduce and investigate the concept of graded weakly 2-absorbing submodules over non-commutative graded rings as follows:

Definition 2.16. Let M be a graded A-module and K be a proper graded A-submodule of M. Then

(1) K is called a graded weakly 2-absorbing submodule of M if whenever $x, y \in h(A)$ and L is a graded A-submodule of M such that $0 \neq xAyL \subseteq K$, then $xL \subseteq K$ or $yL \subseteq K$ or $xAy \subseteq (K :_A M)$.

(2) K is called a graded completely weakly 2-absorbing submodule of M if whenever $x, y \in h(A)$ and $z \in h(M)$ such that $0 \neq xyz \in K$, then $xz \in K$ or $yz \in K$ or $xy \in (K :_A M)$.

Clearly, every graded classical weakly prime submodule is graded weakly 2-absorbing, and every graded completely classical weakly prime submodule is graded completely weakly 2-absorbing. Also, if A is a commutative graded ring with unity, then the concepts of graded weakly 2-absorbing submodules and graded completely weakly 2-absorbing submodules coincide. However, this will not hold for non-commutative graded rings. For example, in Example 2.2, K is a graded weakly 2-absorbing A-submodule of M, as it is a graded prime. However, K is not a graded completely weakly 2-absorbing A-submodule of M, as $0 \neq xyz \in K$, $xz \notin K$, $yz \notin K$ and $xy \notin (K :_A M)$.

Proposition 2.17. *Let M be a graded A-module and K be a proper graded A-submodule of M. If K is a graded weakly 2-absorbing A-submodule of M and $(K :_A M)$ is a graded weakly prime ideal of A, then K is a graded classical weakly prime A-submodule of M.*

Proof. Let $x, y \in h(A)$ and L be a graded A-submodule of M such that $0 \neq xAyL \subseteq K$. Then $xL \subseteq K$ or $yL \subseteq K$ or $xAy \subseteq (K :_A M)$ since K is a graded weakly 2-absorbing A-submodule of M. If $xL \subseteq K$ or $yL \subseteq K$, then the proof is complete. Suppose that $xAy \subseteq (K :_A M)$. Then as $0 \neq xAy$, either $x \in (K :_A M)$ or $y \in (K :_A M)$ since $(K :_A M)$ is a graded weakly prime ideal of A, and then either $xL \subseteq xM \subseteq K$ or $yL \subseteq yM \subseteq K$. Hence, K is a graded classical weakly prime A-submodule of M. □

Definition 2.18. Let M be a graded A-module and K be a proper graded A-submodule of M. Then K is said to be a graded classical prime A-submodule of M if whenever $x, y \in h(A)$ and L is a graded A-submodule of M such that $xAyL \subseteq K$, then either $xL \subseteq K$ or $yL \subseteq K$.

Clearly, every graded classical prime submodule is a graded classical weakly prime submodule. However, the following example shows that a graded classical weakly prime submodule is not necessarily a graded classical prime submodule.

Example 2.19. Let $A = M = M_2(\mathbb{Z}_8)$ and $G = \mathbb{Z}_4$. The ring A is G-graded as follows

$$A_0 = \begin{pmatrix} \mathbb{Z}_8 & 0 \\ 0\mathbb{Z}_8 & \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 0\mathbb{Z}_8 & \\ \mathbb{Z}_8 & 0 \end{pmatrix}$$

and

$$A_1 = A_3 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

M also is a G-graded left A-module by the same graduation of A. Now,

$$K = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

is a graded classical weakly prime A-submodule of M, but K is not a graded classical prime A-submodule of M since

$$x = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \in h(A)$$

and $L = Ax$ is a graded A-submodule of M with $xAxL \subseteq K$ and $xL \nsubseteq K$.

Proposition 2.20. *Let M be a graded A-module. If K is a graded classical weakly prime A-submodule of M that is not graded classical prime, then there exist $x, y \in h(A)$ and a graded A-submodule L of M such that $xAyL = 0$, $xL \nsubseteq K$ and $yL \nsubseteq K$.*

P r o o f. Since K is not a graded classical prime A-submodule of M, there exist $x, y \in h(A)$ and a graded A-submodule L of M such that $xAyL \subseteq K$, $xL \nsubseteq K$ and $yL \nsubseteq K$. If $xAyL \neq 0$, then since K is a graded classical weakly prime A-submodule of M, either $xL \nsubseteq K$ or $yL \nsubseteq K$, which is a contradiction. So, $xAyL = 0$. □

Definition 2.21. Let M be a graded A-module, K be a proper graded A--submodule of M, L be a graded A-submodule of M and $x, y \in h(A)$. Then (x, y, L) is called a graded classical triple zero of K if $xAyL = 0$, $xL \nsubseteq K$ and $yL \nsubseteq K$.

Remark 2.22. If K is a graded classical weakly prime A-submodule of M that is not graded classical prime, then by Proposition 2.20, there exists a graded classical triple zero of K. Note that, in Example 2.19, (x, x, L) is a graded classical triple zero of K.

Proposition 2.23. *Let M be a graded A-module, K be a graded classical weakly prime A-submodule of M and $xAyL \subseteq K$, for some $x, y \in h(A)$ and some graded A-submodule L of M. If (x, y, L) is not a graded classical triple zero of K, then either $xL \subseteq K$ or $yL \subseteq K$.*

Proof. If $xAyL = 0$, then since (x, y, L) is not a graded classical triple zero of K, either $xL \subseteq K$ or $yL \subseteq K$. If $xAyL \neq 0$, then since K is a graded classical weakly prime A-submodule of M, either $xL \subseteq K$ or $yL \subseteq K$. □

Corollary 2.24. *Let M be a graded A-module and K be a graded classical weakly prime A-submodule of M. If (x, y, L) is not a graded classical triple zero of K, for all $x, y \in h(A)$ and all graded A-submodule L of M, then K is a graded classical prime A-submodule of M.*

Proposition 2.25. *Let M be a graded A-module, K be a graded classical weakly prime A-submodule of M and $IJL \subseteq K$, for some graded ideals I, J of A and some graded A-submodule L of M. If (x, y, L) is not a graded classical triple zero of K, for all $x \in I \bigcap h(A)$ and $y \in J \bigcap h(A)$, then for all $x \in I$, $y \in J$ and $g \in G$, we have either $x_gL \subseteq K$ or $y_gL \subseteq K$.*

Proof. Let $x \in I$, $y \in J$ and $g \in G$. Then $x_g \in I$ and $y_g \in J$ since I and J are graded ideals, and hence $x_gAy_gL \subseteq IJL \subseteq K$. If $x_gAy_gL \neq 0$, then since K is a graded classical weakly prime A-submodule of M, either $x_gL \subseteq K$ or $y_gL \subseteq K$. If $x_gAy_gL = 0$, then since (x_g, y_g, L) is not a graded classical triple zero of K, either $x_gL \subseteq K$ or $y_gL \subseteq K$. □

Corollary 2.26. *Let M be a graded A-module, K be a graded classical weakly prime A-submodule of M and $IJL \subseteq K$, for some graded ideals I, J of A and some graded A-submodule L of M. If (x, y, L) is not a graded classical triple zero of K, for all $x \in I \bigcap h(A)$ and $y \in J \bigcap h(A)$, then for all $g \in G$, we have either $I_gL \subseteq K$ or $J_gL \subseteq K$.*

Theorem 2.27. *Let M be a graded A-module and K be a graded classical weakly prime A-submodule of M. Let L be a graded A-submodule of M and $x, y \in h(A)$. If (x, y, L) is a graded classical triple zero of K, then the following statements hold:*

1. $xAyK = 0$.
2. *If $y \in A_g$, for some $g \in G$, then $x(K :_{A_g} M)L = 0$.*
3. *If $x \in A_g$, for some $g \in G$, then $(K :_{A_g} M)yL = 0$.*

(4) *If* $x, y \in A_g$*, for some* $g \in G$*, then* $(K :_{A_g} M)^2 L = 0$.

(5) *If* $y \in A_g$*, for some* $g \in G$*, then* $x(K :_{A_g} M)K = 0$.

(6) *If* $x \in A_g$*, for some* $g \in G$*, then* $(K :_{A_g} M)yK = 0$.

(7) *If* $x, y \in A_g$*, for some* $g \in G$*, then* $(K :_{A_g} M)^2 K = 0$.

Proof.

(1) Suppose that $xAyK \neq 0$. Then there exists $z \in K$ such that $xAyz \neq 0$, and then there exists $g \in G$ such that $xAyz_g \neq 0$, and hence $xAyAz_g \neq 0$. Note that $z_g \in K$ as K is a graded A-submodule. Let $T = L + Az_g$. Then T is a graded A-submodule of M such that $0 \neq xAyT \subseteq K$, and then either $xT \subseteq K$ or $yT \subseteq K$, which implies that either $xL \subseteq xT \subseteq K$ or $yL \subseteq yT \subseteq K$, which is a contradiction. Thus $xAyK = 0$.

(2) Suppose that $x(K :_{A_g} M)L \neq 0$. Then there exists $s \in (K :_{A_g} M)$ such that $xsL \neq 0$, and then $xAsL \neq 0$, and hence $0 \neq xA(y + s)L \subseteq K$. So, either $xL \subseteq K$ or $(y + s)L \subseteq K$, and then either $xL \subseteq K$ or $yL \subseteq K$, which is a contradiction. Hence, $x(K :_{A_g} M)L = 0$.

(3) Similar to (2).

(4) Suppose that $(K :_{A_g} M)^2 L \neq 0$. Then there exist $r, s \in (K :_{A_g} M)$ such that $rsL \neq 0$, and then $rAsL \neq 0$, and hence by (2) and (3), $0 \neq (x + r)A(y + s)L \subseteq K$. So, either $(x + r)L \subseteq K$ or $(y + s)L \subseteq K$, and then either $xL \subseteq K$ or $yL \subseteq K$, which is a contradiction. Hence, $(K :_{A_g} M)^2 L = 0$.

(5) Suppose that $x(K :_{A_g} M)K \neq 0$. Then there exists $s \in (K :_{A_g} M)$ such that $xsK \neq 0$, and then $xAsK \neq 0$, and hence by (1), $xA(y + s)K \neq 0$, which implies that $xA(y + s)z \neq 0$, for some $z \in K$, and so there exists $h \in G$ such that $xA(y + s)z_h \neq 0$. Note that $z_h \in K$ as K is a graded A--submodule. Let $T = L + Az_h$. Then T is a graded A-submodule of M such that $0 \neq xA(y + s)T \subseteq K$, and then either $xT \subseteq K$ or $(y + s)T \subseteq K$, and hence either $xT \subseteq K$ or $yT \subseteq K$, which implies that either $xL \subseteq xT \subseteq K$ or $yL \subseteq yT \subseteq K$, which is a contradiction. Hence, $x(K :_{A_g} M)K = 0$.

(6) Similar to (5).

(7) Suppose that $(K :_{A_g} M)^2 K \neq 0$. Then there exist $r, s \in (K :_{A_g} M)$ and $z \in K$ such that $rsz \neq 0$, and then there exists $h \in G$ such that $rsz_h \neq 0$. Note that $z_h \in K$ as K is a graded A-submodule. Let $T = L + Az_h$. Then T is a graded A-submodule of M such that $0 \neq (x + r)A(y + s)T \subseteq K$ by (2) and (3), and then either $(x + r)T \subseteq K$ or $(y + s)T \subseteq K$, and hence either $xT \subseteq K$ or $yT \subseteq K$, which implies that either $xL \subseteq xT \subseteq K$ or $yL \subseteq yT \subseteq K$, which is a contradiction. Hence, $(K :_{A_g} M)^2 K = 0$.

□

Proposition 2.28. *Let M be a graded A-module and K be a graded classical weakly prime A-submodule of M. If there exists a graded classical triple zero (x, y, L) of K with $x, y \in A_e$, then $(K :_{A_e} M)^3 \subseteq \mathrm{Ann}_{A_e}(M)$.*

Proof. By Theorem 2.27 (7), $(K :_{A_e} M)^2 K = 0$. So,

$$\begin{aligned}(K :_{A_e} M)^3 &= (K :_{A_e} M)^2 (K :_{A_e} M)\\ &\subseteq \big((K :_{A_e} M)^2 K :_{A_e} M\big)\\ &= (0 :_{A_e} M) = \underset{A_e}{\mathrm{Ann}}(M).\end{aligned}$$ □

As a consequence of Proposition 2.28, we have the following:

Corollary 2.29. *Let M be a faithful graded A-module and K be a graded classical weakly prime A-submodule of M. If there exists a graded classical triple zero (x, y, L) of K with $x, y \in A_e$, then $(K :_{A_e} M)^3 = 0$.*

Also, as a consequence of Theorem 2.27 (7), we have the following:

Corollary 2.30. *Let M be a graded A-module and K be a faithful graded classical weakly prime A-submodule of M. If there exists a graded classical triple zero (x, y, L) of K with $x, y \in A_g$, for some $g \in G$, then $(K :_{A_g} M)^2 = 0$.*

Furthermore, as a consequence of Theorem 2.27 (4), we have the following:

Corollary 2.31. *Let M be a graded A-module and K be a graded classical weakly prime A-submodule of M. If there exists a graded classical triple zero (x, y, L) of K with $x, y \in A_g$, for some $g \in G$, and L is faithful, then $(K :_{A_g} M)^2 = 0$.*

Let M and S be two G-graded A-modules. Then $M \times S$ is a G-graded A-module by $(M \times S)_g = M_g \times S_g$, for all $g \in G$ [14]. Moreover, $N = K \times T$ is a graded A-submodule of $M \times S$ if and only if K is a graded A-submodule of M and T is a graded A-submodule of S ([18, Lemma 2.10 and Lemma 2.12]).

Theorem 2.32. *Let M and S be two G-graded A-modules. If $K \times S$ is a graded classical weakly prime A-submodule of $M \times S$, then K is a graded classical weakly prime A-submodule of M.*

Proof. Let $x, y \in h(A)$ and L be a graded A-submodule of M such that $0 \neq xAyL \subseteq K$. Then $L \times \{0\}$ is a graded A-submodule of $M \times S$ such that $(0, 0) \neq xAy(L \times \{0\}) \subseteq K \times S$, and then either $x(L \times \{0\}) \subseteq K \times S$ or $y(L \times \{0\}) \subseteq K \times S$, and hence either $xL \subseteq K$ or $yL \subseteq K$. Thus K is a graded classical weakly prime A-submodule of M. □

Theorem 2.33. *Let M and S be two G-graded A-modules and $K \times S$ is a graded classical weakly prime A-submodule of $M \times S$. If (x, y, L) is a graded classical triple zero of K, then $xAy \subseteq \mathrm{Ann}_A(S)$.*

P r o o f. Suppose that $xAy \nsubseteq \mathrm{Ann}_A(S)$. Then there exists $s \in S$ such that $xAys \neq 0$, and then there exists $g \in G$ such that $xAys_g \neq 0$. Now, $L \times As_g$ is a graded A-submodule of $M \times S$ such that $(0,0) \neq xAy(L \times As_g) \subseteq K \times S$, so either $x(L \times As_g) \subseteq K \times S$ or $y(L \times As_g) \subseteq K \times S$, and hence either $xL \subseteq K$ or $yL \subseteq K$, which is a contradiction. Thus $xAy \subseteq \mathrm{Ann}_A(S)$. □

We close this section by introducing a nice result concerning graded weakly prime submodules over graded multiplication modules (Theorem 2.35). A graded A-module M is called a graded multiplication if for every graded A-submodule K of M, $K = IM$, for some graded ideal I of A. In this case, it is known that $I = (K :_A M)$. Graded multiplication modules were first introduced and studied by Escoriza and Torrecillas in [7], and further results were obtained by several authors, see for example [6, 12].

Lemma 2.34. *Let M be a graded A-module. Then every graded maximal A-submodule of M is graded prime.*

P r o o f. Let K be a graded maximal A-submodule of M. Suppose that $IL \subseteq K$, for some graded ideal I of A and some graded A-submodule L of M. Assume that $L \nsubseteq K$. Since K is a graded maximal A-submodule of M, $L+K=M$, and hence $IM = IL + IK \subseteq K$, which implies that $I \subseteq (K :_A M)$. Hence, K is a graded prime A-submodule of M. □

Theorem 2.35. *Let M be a graded multiplication A-module. If every proper graded A-submodule of M is graded weakly prime, then M has at most two graded maximal A-submodules.*

P r o o f. Let X, Y and Z be three distinct graded maximal A-submodules of M. Since M is a graded multiplication, $X = IM$, for some graded ideal I of A. If $IY = 0$, then $IY \subseteq Z$, and since Z is graded weakly prime by Lemma 2.34, either $Y \subseteq Z$ or $X = IM \subseteq Z$, which is a contradiction. So, $IY \neq 0$. Now, $IY \subseteq Y$ and $IY \subseteq IM = X$, so $0 \neq IY \subseteq X \bigcap Y$, and since $X \bigcap Y$ is graded weakly prime by assumption, either $Y \subseteq X \bigcap Y$ or $X = IM \subseteq X \bigcap Y$, which implies that either $Y \subseteq X$ or $X \subseteq Y$, which is a contradiction. Hence, M has at most two graded maximal A-submodules. □

Corollary 2.36. *Let M be a graded multiplication A-module such that every proper graded A-submodule of M is graded weakly prime. If $X = IM$ and $Y = JM$ are two distinct graded A-submodules of M, for some graded ideals I, J of A, then either X and Y are comparable by inclusion or $IY = JX = 0$. In particular, if X and Y are two distinct graded maximal A-submodules of M, then $IY = JX = 0$.*

3. Graded Classical Weakly Prime Submodules over Duo Graded Rings

In this section, we study graded classical weakly prime submodules over Duo graded rings. A ring A is said to be a left Duo ring if every left ideal of A is a two sided ideal [9, 13]. It is obvious that if A is a left Duo ring, then $xA \subseteq Ax$, for all $x \in A$.

Theorem 3.1. *Let A be a left Duo graded ring, M be a graded A-module and K be a graded classical weakly prime A-submodule of M. If $x, y \in h(A)$ and $m \in h(M)$ such that $0 \neq xym \in K$, then either $xm \in K$ or $ym \in K$.*

Proof. Since A is a left Duo ring, $Axy = AxAyA$, and then Am is a graded A-submodule of M such that $0 \neq xAyAm \subseteq K$, and so either $xAm \subseteq K$ or $yAm \subseteq K$, that implies either $xm \in K$ or $ym \in K$. □

Theorem 3.1 states that every graded classical weakly prime submodule of a graded module over a left Duo graded ring is graded completely classical weakly prime. As a consequence of Theorem 3.1, we have the following:

Corollary 3.2. *Let A be a left Duo graded ring, M be a graded A-module and K be a graded classical weakly prime A-submodule of M. If $x, y \in h(A)$ and $m \in h(M)$ such that $xym \in K$ and (x, y, Am) is not graded classical triple zero of K, then either $xm \in K$ or $ym \in K$.*

Let M be an A-module and K be an A-submodule of M. Then K is said to be an u-submodule if whenever $K \subseteq \bigcup_{j=1}^{n} K_j$, for some A-submodules K_j's of M, then $K \subseteq K_j$, for some $1 \leq j \leq n$. M is said to be an u-module if every A-submodule of M is an u-submodule [16]. Let M be an A-module, K be an A-submodule of M and $r \in A$. Then $(K :_M r) = \{m \in M : rm \in K\}$ is an A-submodule of M containing K.

Theorem 3.3. *Let M be a graded A-module such that A_e is a left Duo ring and M_g is an u-module over A_e, for some $g \in G$. Suppose that K is a graded A-submodule of M such that $K_g \neq M_g$. Consider the following statements:*

(1) *K is a g-classical weakly prime A-submodule of M.*

(2) *If $x, y \in A_e$ and $m \in M_g$ such that $0 \neq xym \in K$, then either $xm \in K$ or $ym \in K$.*

(3) *For all $x, y \in A_e$, $(K :_{M_g} xy) = (0 :_{M_g} xy)$ or $(K :_{M_g} xy) = (K :_{M_g} x)$ or $(K :_{M_g} xy) = (K :_{M_g} y)$.*

(4) *If* $x, y \in A_e$ *and* L *is an* A_e*-submodule of* M_g *such that* $0 \neq xyL \subseteq K$, *then either* $xL \subseteq K$ *or* $yL \subseteq K$.

(5) *If* $x \in A_e$ *and* L *is an* A_e*-submodule of* M_g *such that* $xL \nsubseteq K$, *then either* $(K :_{A_e} xL) = (0 :_{A_e} xL)$ *or* $(K :_{A_e} xL) = (K :_{A_e} L)$.

(6) *If* $x \in A_e$, I *is an ideal of* A_e *and* L *is an* A_e*-submodule of* M_g *such that* $0 \neq IxL \subseteq K$, *then either* $IL \subseteq K$ *or* $xL \subseteq K$.

(7) *If* I *is an ideal of* A_e *and* L *is an* A_e*-submodule of* M_g *such that* $IL \nsubseteq K$, *then either* $(K :_{A_e} IL) = (0 :_{A_e} IL)$ *or* $(K :_{A_e} IL) = (K :_{A_e} L)$.

Then

$$(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (5) \Rightarrow (6) \Rightarrow (7).$$

Proof.

(1) ⇒ (2): Similar to proof of Theorem 3.1.

(2) ⇒ (3): Let $x, y \in A_e$ and $m \in (K :_{M_g} xy)$. Then $xym \in K$. If $xym = 0$, then $m \in (0 :_{M_g} xy)$. If $xym \neq 0$, then by (2), either $xm \in K$ or $ym \in K$, and so either $m \in (K :_{M_g} x)$ or $m \in (K :_{M_g} y)$. Hence,

$$(K :_{M_g} xy) \subseteq (0 :_{M_g} xy) \bigcup (K :_{M_g} x) \bigcup (K :_{M_g} y),$$

and then $(K :_{M_g} xy) \subseteq (0 :_{M_g} xy)$ or $(K :_{M_g} xy) \subseteq (K :_{M_g} x)$ or $(K :_{M_g} xy) \subseteq (K :_{M_g} y)$ as M_g is an u-module over A_e, and hence the result holds as A_e is a left Duo ring.

(3) ⇒ (4): Clearly, $L \subseteq (K :_{M_g} xy)$ and $K \nsubseteq (0 :_{M_g} xy)$, so by (3), either $L \subseteq (K :_{M_g} x)$ or $L \subseteq (K :_{M_g} y)$, and then either $xL \subseteq K$ or $yL \subseteq K$.

(4) ⇒ (5): Let $r \in (K :_{A_e} xL)$. Then $rxL \subseteq K$. If $rxL = 0$, then $r \in (0 :_{A_e} xL)$. If $rxL \neq 0$, then by (4), $rL \subseteq K$. So,

$$(K :_{A_e} xL) \subseteq (0 :_{A_e} xL) \bigcup (K :_{A_e} L).$$

Hence, the result holds as A_e is a left Duo ring.

(5) ⇒ (6): Clearly, $I \subseteq (K :_{A_e} xL)$ and $I \nsubseteq (0 :_{A_e} xL)$, so by (5), either $xL \subseteq K$ or $I \subseteq (K :_{A_e} L)$, and then either $xL \subseteq K$ or $IL \subseteq K$.

(6) ⇒ (7): Let $x \in (K :_{A_e} IL)$. Then $xIL \subseteq K$. If $xIL = 0$, then $x \in (0 :_{A_e} IL)$. If $xIL \neq 0$, then as A_e is a left Duo ring, $0 \neq IxL \subseteq K$, and then by (6), $xL \subseteq K$, and hence $x \in (K :_{A_e} L)$. Thus

$$(K :_{A_e} IL) \subseteq (0 :_{A_e} IL) \bigcup (K :_{A_e} L).$$

So, the result holds as A_e is a left Duo ring.

□

Remark 3.4. The assumptions of Theorem 3.3 may seem restrictive at first glance, particularly because finding an explicit example of a graded ring that satisfies all the prerequisites can be challenging. The conditions are specifically designed to capture certain algebraic properties that, while not common in elementary examples, arise in more specialized contexts, such as in some constructions involving homogeneous elements or in graded structures arising from algebraic geometry or commutative algebra. Despite extensive efforts, a concrete example that satisfies all of the theorem's assumptions has not been found. This suggests that the class of graded rings that fulfill these conditions may be more specialized or less straightforward to construct explicitly. However, certain well-known classes of graded rings, such as graded polynomial rings or quotient rings of graded algebras, may offer fruitful directions to explore. Further research could involve either relaxing some of the theorem's assumptions or developing a more detailed classification of graded rings that fit the prerequisites. For now, the theoretical framework provided by Theorem 3.3 remains important for understanding the broader structural properties of graded rings under these conditions.

Let A be a left Duo graded ring. If A is a graded ring, then clearly,

$$\underset{A}{\mathrm{Grad}}(I) = \{x \in A : \forall g \in G,\ \exists\ n_g \in \mathbb{N} \text{ s.t } \ x_g^{n_g} \in I\}$$

is a graded ideal of A containing I. Evidently, if $x \in h(A)$, then $x \in \mathrm{Grad}_A(I)$ if and only if $x^n \in I$ for some $n \in \mathbb{N}$.

Proposition 3.5. *Let M be a graded A-module such that A_e is a left Duo ring and K be a graded classical weakly prime A-submodule of M. If there exists a graded classical triple zero (x, y, L) of K with $x, y \in A_e$, then*

$$\underset{A_e}{\mathrm{Grad}}\big(\underset{A_e}{\mathrm{Ann}}(M)\big) = \underset{A_e}{\mathrm{Grad}}\big((K :_{A_e} M)\big).$$

Proof. By Proposition 2.28,

$$\big(K :_{A_e} M\big) \subseteq \underset{A_e}{\mathrm{Grad}}\big(\underset{A_e}{\mathrm{Ann}}(M)\big),$$

and then

$$\begin{aligned}\underset{A_e}{\mathrm{Grad}}((K :_{A_e} M)) &\subseteq \underset{A_e}{\mathrm{Grad}}\Big(\underset{A_e}{\mathrm{Grad}}\big(\underset{A_e}{\mathrm{Ann}}(M)\big)\Big)\\ &= \underset{A_e}{\mathrm{Grad}}\big(\underset{A_e}{\mathrm{Ann}}(M)\big).\end{aligned}$$

On the other hand, since

$$\underset{A_e}{\mathrm{Ann}}(M) \subseteq (K :_{A_e} M),\ \ \underset{A_e}{\mathrm{Grad}}\big(\underset{A_e}{\mathrm{Ann}}(M)\big) \subseteq \underset{A_e}{\mathrm{Grad}}((K :_{A_e} M)).$$

Hence,

$$\underset{A_e}{\mathrm{Grad}}\big(\underset{A_e}{\mathrm{Ann}}(M)\big) = \underset{A_e}{\mathrm{Grad}}((K :_{A_e} M)).$$

□

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