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ON A CLASS OF LACUNARY ALMOST NEWMAN POLYNOMIALS MODULO p AND DENSITY THEOREMS

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ABSTRACT. The reduction modulo p of a family of lacunary integer polynomials, associated with the dynamical zeta function $\zeta_\beta(z)$ of the β-shift, for $\beta > 1$ close to one, is investigated. We briefly recall how this family is correlated to the problem of Lehmer. A variety of questions is raised about their numbers of zeroes in $\mathbb{F}_p$ and their factorizations, via Kronecker's Average Value Theorem (viewed as an analog of classical Theorems of Uniform Distribution Theory). These questions are partially answered using results of Schinzel, revisited by Sawin, Shusterman and Stoll, and density theorems (Frobenius, Chebotarev, Serre, Rosen). These questions arise from the search for the existence of integer polynomials of Mahler measure > 1 less than the smallest Salem number 1.176280. Explicit connection with modular forms (or modular representations) of the numbers of zeroes of these polynomials in $\mathbb{F}_p$ is obtained in a few cases. In general it is expected since it must exist according to the Langlands program.

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1. Introduction

In this paper we will address a number of arithmetic questions on a class of lacunary integer polynomials which have all their coefficients in $\{0, 1\}$ except the constant term equal to -1. We have called such polynomials *almost Newman polynomials* in Dutykh and Verger-Gaugry [10], by comparison with Newman polynomials which all have their coefficients in $\{0, 1\}$. Recall that, with probability one, Newman polynomials are irreducible (Breuillard and Varju [3]).

This class is the following. For $n \geq 2$, we denote by $\mathcal{B}$ the class of lacunary polynomials

$$f(x) := -1 + x + x^n + x^{m_1} + x^{m_2} + \cdots + x^{m_s},$$

where

$$s \geq 0, \quad m_1 - n \geq n - 1, \quad m_{q+1} - m_q \geq n - 1 \quad \text{for } 1 \leq q < s,$$

and by $\mathcal{B}_n$ those whose third monomial is exactly x^n, so that

$$\mathcal{B} = \cup_{n \geq 2} \mathcal{B}_n.$$

The case "$s = 0$" corresponds to the trinomials $G_n(z) := -1 + z + z^n$. This class admits a very special type of lacunarity.

This class appears naturally in the study of the existence of reciprocal integer polynomials having small Mahler measure (Smyth [46]): in the Appendix we recall how this class of polynomials is arising from the dynamical zeta function $\zeta_\beta(z)$ of the β-shift (equivalently from the Parry Upper function $f_\beta(z)$), when $\beta > 1$ is close to one. Recall that the Mahler measure of a nonzero algebraic number β, of minimal polynomial

$$P_\beta(X) = a_0 X^m + a_1 X^{m-1} + \cdots + a_m = a_0 \prod_i \big(X - \alpha^{(i)}\big) \in \mathbb{Z}[X],$$

is

$$\mathrm{M}(\beta) = |a_0| \prod_i \max\big\{1, \big|\alpha^{(i)}\big|\big\} =: \mathrm{M}(P_\beta).$$

The search for non-trivial Mahler measures of reciprocal integer polynomials smaller than the smallest Salem number known $1.176280\ldots$ (Lehmer's number) is of particular interest in the Problem of Lehmer (Smyth [46], Verger-Gaugry [52]). When $\beta > 1$ is close to one, it is shown in Dutykh and Verger-Gaugry [11] [53] that a subcollection of zeroes (the lenticular zeroes, cf Section 2) of $f_\beta(z)$ is at the origin of a nontrivial (universal) minorant of $\mathrm{M}(\beta)$. By Hurwitz Theorem, the zeroes of $f_\beta(z)$ in the open unit disk of $\mathbb{C}$ are limit points of zeroes of polynomials of the class $\mathcal{B}$, which are its polynomial sections. It is the reason why this class of lacunary almost Newman polynomials is important for the Problem of Lehmer.

By analogy, there is interest in studying the zeroes of $f(z)$ modulo p, for any $f \in \mathcal{B}$ and any prime number p. In the present note, we start the investigation of (i) the number of zeroes $N_p(f)$ of the polynomials $f \in \mathcal{B}$ in $\mathbb{F}_p$, i.e. modulo a prime number p, as a function of p, n, and the type of lacunarity of f characterized by the sequence $(m_j)_{j=1,\ldots,s}$, (ii) the subcollection of prime numbers p for which $N_p(f)$ is equal to zero or is maximal, (iii) the asymptotics of the averages of $N_p(f)$ when p tends to infinity.

For a nonzero integer polynomial $f(X) \in \mathbb{Z}[X]$ we denote by $c(f)$ the greatest common divisor of its coefficients. Let $\mathbb{Z}[X]^c := \{f \in \mathbb{Z}[X] \setminus \{0\} : c(f) = 1\}$. Let $\mathcal{N}$ be the map $\mathbb{Z}[X]^c \to (\mathbb{N})^{\mathbb{P}}, f \to \big(N_p(f)\big)_{p\in\mathbb{P}}$. Obviously, for $f_1, f_2 \in \mathbb{Z}[X]^c$, $\mathcal{N}(f_1 f_2) = \mathcal{N}(f_1) + \mathcal{N}(f_2)$ since, for any $p \in \mathbb{P}$, $N_p(f_1 f_2) = N_p(f_1) + N_p(f_2)$. Now, integrating $\mathcal{N}$ over the set of prime numbers $\mathbb{P}$ and taking the limit average with respect to $\pi(x)$, which is as usual the number of primes $p \le x$, gives the following result (proof in Section 3).

Theorem 1.1. *Let $f \in \mathbb{Z}[X]^c$. If $f = \prod_{i\in J} f_i^{\nu_i}$ is the decomposition of f into irreducible factors, with all $\nu_i \ge 1$, then*

$$\lim_{x\to\infty} \frac{1}{\pi(x)} \sum_{p\le x} N_p(f) = \sum_{i\in J} \nu_i \tag{1}$$

which is the number of irreducible factors in the decomposition of f.

By the Langlands program closed formulas are expected between the values $N_p(f)$ and the coefficients $a(n)$ of Newforms $\sum a(n)q^n$. Key properties of Newforms, geometrical objects attached to the coefficients, can be found, e.g., in Cohen and Strömberg [6], Ono [29].

Let us observe that equation (1) is an analog of classical Theorems of Uniform Distribution (Kuipers and Niederreiter [24], Strauch [48]) (cf. Section 3).

Therefore the present study on the class $\mathcal{B}$ starts a concrete investigation not only of the quantities $N_p(f), f \in \mathcal{B}$, and the associated Newforms, but also of their limit averages by equation (1), which corresponds to the factorization of f.

Remark 1. In the literature, following Lehmer's work, there are two "Conjectures of Lehmer", which a priori are completely independent. The first one is evoked in Smyth [46], Verger-Gaugry [52], and is concerned by a universal minorant of the Mahler measure of reciprocal nonzero algebraic integers which are not roots of unity. It is of concern in this study. The second one is related to the τ-function of Ramanujan. The present work, though trying to correlate the Problem of Lehmer modulo p to modular forms (Newforms) has no objectives to try to link the two Conjectures. The authors have no idea if the two Conjectures are linked.

The factorization of $f \in \mathcal{B}$ as a function of its reductions modulo p is a deep question (Lenstra and Stevenhagen [27], [47]). It is well-known that a monic integer polynomial is irreducible over the rationals if it is irreducible modulo some prime. The converse is not true in general (Brandl [4]). Even if $f \in \mathcal{B}$ is irreducible a first question is about the density of the set of prime numbers p such that $N_p(f) = 0$. Indeed Guralnick, Schacher and Sonn [20], then Gupta [19], have shown the existence of irreducible integer polynomials which are reducible modulo all primes. Whether the factorization of f modulo p contains linear factors is a basic question. If it is so, it is natural to say that the number $N_p(f)$ can take a priori any value between 1 and the maximal value $\deg(f)$.

By Conjecture ARC (Conjecture (3) below) only 75% of the polynomials f in $\mathcal{B}$ are irreducible. General irreducibility criteria for the polynomials f in $\mathcal{B}$ are missing. Theorem 1.1 (Kronecker's Average Value Theorem), applied to each irreducible $f \in \mathcal{B}$, says

$$\lim_{p\to\infty} \frac{1}{\pi(x)} \sum_{p\leq x} N_p(f) = 1,$$

preventing the values $N_p(f) = \deg(f)$ to occur often, since the sum $\sum_{p\leq x} N_p(f)$ behaves like $x/\log(x)$ at infinity. If $f \in \mathcal{B}$ is irreducible and $\deg(f)$ is a prime number, then $\mathcal{N}(f) = \big(N_p(f)\big)_{p\in\mathbb{P}}$ is such that infinitely many $N_p(f)$ are equal to 0 (Stevenhagen and Lenstra [47]). Examples of the distribution of values $N_p(S_j)$ of the (non-reciprocal parts of the) polynomial sections $S_j(x)$ of the Parry Upper function $f_\tau(x) = -1/\zeta_\tau(x)$, where τ is Lehmer's number (cf Section 2 for its definition), are given in Section 5.

The density Theorems of Frobenius and Chebotarev (Chebotarev [5]) play a role in the frequencies of values taken by $N_p(f)$ as p varies in general. For the class $\mathcal{B}$ we summarize this in Theorem 1.2 below. The question of the non-existence of factors of degree 1 (the "$N_p(f) = 0$" case) in the factorization of f modulo p, is partially answered by a general theorem of Serre (Serre's Density Theorem [43]), which gives a positive lower bound on the density of such primes (reported in Theorem 1.2, 1)).

The density is taken in the following sense: a subset S of the set of primes $\mathbb{P}$ has density c if

$$\lim_{x\to\infty} \frac{\text{number of } p \in S \text{ with } p \leq x}{\pi(x)} = c.$$

The limit need not exist. When it exists the natural density of S is said to be defined; then it is denoted by $\delta(S)$.

Theorem 1.2. *Let $n \geq 3$. Let*

$$f(x) := -1 + x + x^n + x^{m_1} + x^{m_2} + \cdots + x^{m_s} \in \mathcal{B},$$

where $s \geq 0$, $m_1 - n \geq n - 1$, $m_{q+1} - m_q \geq n - 1$ for $1 \leq q < s$. The polynomial f is assumed irreducible. Denote by G the Galois group of $f(x)$ and $g := \#G$. Then

1) *the set $\mathcal{P}_0 := \{p \in \mathbb{P} \mid N_p(f) = 0\}$ is infinite, has a density and its density satisfies*
$$\delta(\mathcal{P}_0) \geq \frac{1}{m_s},$$
with strict inequality if m_s is not a power of a prime,

2) *the set $\mathcal{P}_{max} := \{p \in \mathbb{P} \mid N_p(f) = \deg(f)\}$ is infinite and has density*
$$\delta(\mathcal{P}_{max}) = \frac{1}{g}.$$

Theorem 1.2 (1.) is due to Serre; the proof of 1) is given in Serre [43]. The statement of 2) is Corollary 2 in Rosen [33]. This Corollary 2 is a consequence of a Theorem of Frobenius (Theorem 2 in Rosen [33]).

In Section 4 we continue the direct study of the factorization of the polynomials f in the class $\mathcal{B}$, initiated in Dutykh and Verger-Gaugry [10]. Indeed this study has left open the problem of the existence of reciprocal non-cyclotomic components in the factoring of any such f. The main theorem on the factorization of all $f \in \mathcal{B}$ is Theorem 2.1 in [10]. We recall in Section 2 the Asymptotic Reducibility Conjecture ("ARC") which states that the probability of finding a polynomial f in $\mathcal{B}$ which is irreducible is 3/4, and the Conjecture "B", which states the non-existence of a reciprocal non-cyclotomic component in the factorization of a polynomial f in $\mathcal{B}$. We revisit Conjecture B using a Theorem of Schinzel and a recent new lower bound by Sawin, Shusterman and Stoll [35] for large gaps. We prove that Conjecture B is valid on some infinite subclasses of $\mathcal{B}$. For doing this, we consider the new bound for large gaps given in [35] as a new critical value above which Conjecture B is always true. Then, at intermediate lacunarity, for moderate gaps below this critical value, we show numerically that Conjecture B is also true. The subclasses considered are families of pentanomials in $\mathcal{B}$, chosen from the quadrinomials studied by Finch and Jones [16].

The Problem of Lehmer modulo p, adressed to the class $\mathcal{B}$, with its asymptotics when p tends to infinity, also calls for understanding the interplay between the asymptotics of $N_p(f)$ and the peculiar lacunarity of f, for any $f \in \mathcal{B}$.

The importance of the problem of the factorization of lacunary polynomials was outlined in a series of papers by Schinzel [37] [38] [39] [40] [41]. The link between lacunarity, say the geometry of the gappiness, and irreducibility of any $f \in \mathcal{B}$, is emphasized in Corollary 1.4 in [35], as follows.

Theorem 1.3 (Sawin, Shusterman, Stoll [35]). *Let $f \in \mathcal{B}$, and write it as a polynomial $f(x) = g_N(x) = d(x) + x^N c(x^{-1})$ as in equation (5) and equation (7). Under the assumptions of Schinzel's Theorem 4.2, the set of $N > \deg c + \deg d$ such that f is irreducible is the complement of the union of a finite set with a finite union of arithmetic progressions.*

In Section 6 the quantities $N_p(f)$, p tending to infinity, are studied for trinomials, following Serre [43], as functions of the coefficients of q-expansions and correlated to Newforms and modular forms. Some basic questions about the densities of ps such that $N_p(f)$ is congruent to a fixed integer modulo some integers are asked in Serre's general context (Serre [44] [45]).

To outline the novel strategy of the paper, this note initiates the study of the quantities $N_p(f)$, p any prime number, and also p tending to infinity, for any f in the class $\mathcal{B}$, and concomittantly the factorization of such fs. This difficult problem is tackled by the simplest cases of fs in $\mathcal{B}$, which are trinomials, as in Section 6; and by showing on some subfamilies of $\mathcal{B}$, as in Section 4, that factorization occurs with the absence of reciprocal non-cyclotomic components, which is conjectured to be the general rule.

2. Dynamical zeta function of the β-shift and Lehmer's problem

Let us recall standard definitions. A complex number α is an algebraic integer if there exists a monic polynomial $R(X) \in \mathbb{Z}[X]$ such that $R(\alpha) = 0$. If R is the minimal polynomial of α and is reciprocal, i.e. satisfies $X^{\deg R} R(1/X) = R(X)$, then α is called reciprocal. If α is reciprocal, α and $1/\alpha$ are conjugated. If the minimal polynomial of α is not reciprocal, α is called non-reciprocal. If $\alpha = 1$, or if $\alpha > 1$ and the conjugates $\alpha^{(i)} \neq \alpha$ of α satisfy: $|\alpha^{(i)}| < \alpha$, then α is said to be a Perron number.

Let $n \geq 3$ be a fixed integer. Selmer [42] has shown that the trinomials $-1 + x + x^n \in \mathbb{Z}[x]$ are irreducible if $n \not\equiv 5 \pmod 6$, and, for $n \equiv 5 \pmod 6$, are reducible as product of two irreducible factors whose one is the cyclotomic factor $x^2 - x + 1$, the other factor $(-1 + x + x^n)/(x^2 - x + 1)$ being nonreciprocal of degree $n - 2$. We denote by θ_n the unique zero in $(0, 1)$ of the trinomial $-1 + x + x^n$. The inverses $\theta_n^{-1} > 1$ are non-reciprocal algebraic integers which are Perron numbers, and constitute a decreasing sequence $(\theta_n^{-1})_{n\geq 3}$ tending to 1^+. The smallest Mahler measure known is Lehmer's number τ, root > 1 of $x^{10}+x^9-x^7-x^6-x^5-x^4-x^3+x+1$, such that: $\theta_{12}^{-1} < \tau = 1.176280\ldots < \theta_{11}^{-1}$. Here n is equal to 12. The search for reciprocal algebraic integers βs, of Mahler measure $\mathrm{M}(\beta) \leq 1.176280$, in the intervals $(\theta_n^{-1}, \theta_{n-1}^{-1})$, $n \geq 13$, having a minimal polynomial for which there is no $\mathbb{Z}$-minimal integer polynomial $\widehat{P_\beta}(X)$ such that

$$P_\beta(X) = \widehat{P_\beta}(X^r) \tag{2}$$

for some integer $r \geq 2$ is of importance in the problem of the non-trivial minoration of the Mahler measure (Section 2 in Verger-Gaugry [52]). The existence of very small Mahler measures is still a mystery.

Let us assume the existence of such a reciprocal algebraic integer $\beta > 1$. It is canonically associated with, and characterized by, two analytic functions:

1) its minimal polynomial function, say $z \to P_\beta(z)$, which is monic and reciprocal; denote $d := \deg P_\beta$, $H :=$ the (naïve) height of P_β,
2) the Parry Upper function $f_\beta(x)$ at β^{-1}, which is the generalized Fredholm determinant of the β-transformation T_β (Section 3 in Verger-Gaugry [53]). It is a (infinite) power series with coefficients in the alphabet $\{0, 1\}$ except the constant term equal to -1, with distanciation between the exponents of the monomials

   $$f_\beta(x) := -1 + x + x^n + x^{m_1} + x^{m_2} + \cdots + x^{m_s} + \cdots,$$

   where $m_1 - n \geq n-1$, $m_{q+1} - m_q \geq n-1$ for $q \geq 1$. β^{-1} is the unique zero of $f_\beta(x)$ in the unit interval $(0, 1)$. The analytic function $f_\beta(z)$ is related to the dynamical zeta function $\zeta_\beta(z)$ of the β-shift (Section 3 in Verger-Gaugry [53]; Flatto, Lagarias and Poonen [17]) by: $f_\beta(z) = -1/\zeta_\beta(z)$. Since β is reciprocal, with the two real roots β and $1/\beta$, the series $f_\beta(x)$ is never a polynomial, by Descartes's rule on sign changes on the coefficient vector. The algebraic integer β is associated with the infinite sequence of exponents (m_j).

Let us observe that all the polynomial sections

$$S_s(x) := -1 + x + x^n + x^{m_1} + x^{m_2} + \cdots + x^{m_s}, \qquad s \geq 1,$$

of $f_\beta(x)$ are polynomials of the class $\mathcal{B}_n$.

The principal motivation to study the class $\mathcal{B}$ for itself in the present note comes from the peculiar form of these polynomial sections.

The polynomials f of the class $\mathcal{B}$ are often irreducible by the following conjecture, formulated in Dutykh and Verger-Gaugry [10].

Asymptotic Reducibility Conjecture (ARC). *Let $n \geq 2$ and $N \geq n$. Let $\mathcal{B}_n^{(N)}$ denote the set of the polynomials $f \in \mathcal{B}_n$ such that* $\deg(f) \leq N$. *Let $\mathcal{B}^{(N)} := \bigcup_{2 \leq n \leq N} \mathcal{B}_n^{(N)}$. The proportion of polynomials in $\mathcal{B} = \cup_{N \geq 2} \mathcal{B}^{(N)}$ which are irreducible is given by the limit, assumed to exist,*

$$\lim_{N\to\infty} \frac{\#\{f \in \mathcal{B}^{(N)} \mid f \text{ irreducible}\}}{\#\{f \in \mathcal{B}^{(N)}\}} \qquad \text{and its value is expected to be } \frac{3}{4}. \tag{3}$$

Let us recall the generic factorization of the polynomials $f \in \mathcal{B}$ which generalizes that of the trinomials $-1 + x + x^n$ by Selmer [42].

Theorem 2.1 (Dutykh, Verger-Gaugry [10]). *For any $f \in \mathcal{B}_n$, $n \geq 3$, denote by*

$$f(x) = A(x)B(x)C(x) = -1 + x + x^n + x^{m_1} + x^{m_2} + \cdots + x^{m_s},$$

where $s \geq 1$, $m_1 - n \geq n - 1$, $m_{j+1} - m_j \geq n - 1$ for $1 \leq j < s$, the factorization of f where A is the cyclotomic part, B the reciprocal noncyclotomic part, C the nonreciprocal part. Then (i) *the nonreciprocal part C is nontrivial, irreducible, and never vanishes on the unit circle,* (ii) *if $\beta > 1$ denotes the real algebraic integer uniquely determined by the sequence $(n, m_1, m_2, \ldots, m_s)$ such that $1/\beta$ is the unique real root of f in (θ_{n-1}, θ_n), the nonreciprocal polynomial $-C^*(X)$ of $C(X)$ is the minimal polynomial of β, and β is a nonreciprocal algebraic integer.*

From numerous experiments on the class $\mathcal{B}$ by Monte-Carlo in Dutykh and Verger-Gaugry [10], the components "B" were never observed. The following conjecture is reasonable to formulate. It will be partially proved in Section 4.

Conjecture B. *The reciprocal non-cyclotomic part B of any $f \in \mathcal{B}_n$, $n \geq 3$, is always trivial.*

Let us mention the link with Lehmer's problem. Lehmer's number $1.1762\ldots$ is the smallest Mahler measure known. By a theorem of Smythé [46] the Mahler measure of a nonzero algebraic integer which is not reciprocal, not a root of unity, is $\geq 1.3247\ldots$, dominant root of $X^3 - X - 1$ and the smallest Pisot number. Then the Mahler measures of algebraic integers which are in the range $(1, 1.3247)$ arise from reciprocal algebraic integers which are not roots of unity. This is the main reason why Conjecture B is important to investigate, about the possible existence of reciprocal parts in the factors of the polynomials of the class $\mathcal{B}$.

Our attention is focused on the search for hypothetical reciprocal algebraic integers $\beta > 1$ for which $\mathrm{M}(\beta) \in (1, 1.176280)$ and equation (2) is satisfied, that is when n is large in Conjecture B. Using intermediate alphabets and periodic representations of $\mathbb{Q}(\beta)$ in the algebraic basis β, it was shown in Dutykh and Verger-Gaugry [11] that the relation between f_β and P_β is a relation of identification on the subcollection of lenticular zeroes of f_β. The definition of a lenticular zero is given in Dutykh and Verger-Gaugry [10], where many examples are proposed.

The lenticular zeroes of f_β are peculiar zeroes, off the unit circle. Let us briefly recall what is a lenticular zero of f_β. Many examples of lenticular zeroes are given in Dutykh and Verger-Gaugry [10]. The following theorem is Theorem 4 in [10].

Theorem 2.2. *Assume $n \geq 260$. There exist two positive constants c_n and $c_{A,n}$, $c_{A,n} < c_n$, such that the roots of any $f \in \mathcal{B}_n$,*

$$f(x) = -1 + x + x^n + x^{m_1} + x^{m_2} + \cdots + x^{m_s},$$

where

$$s \geq 1, \quad m_1 - n \geq n - 1, \quad m_{j+1} - m_j \geq n - 1 \quad \text{for} \quad 1 \leq j < s,$$

lying in

$$-\pi/18 < \arg z < +\pi/18$$

either belong to

$$\left\{ z \in \mathbb{C} : ||z| - 1| < \frac{c_{A,n}}{n} \right\}, \quad \text{or to} \quad \left\{ z \in \mathbb{C} : ||z| - 1| \geq \frac{c_n}{n} \right\}.$$

The *lenticulus of zeroes ω of f* is then defined as

$$\mathcal{L}_\beta := \left\{ \omega \in \mathbb{C} : f(\omega) = 0, |\omega| < 1, -\frac{\pi}{18} < \arg \omega < +\frac{\pi}{18}, \ ||\omega| - 1| \geq \frac{c_n}{n} \right\},$$

where $1/\beta \in \mathcal{L}_\beta$ is the positive real zero of f. If a zero of f belongs to $\mathcal{L}_\beta$ we say that it is a *lenticular zero of f*.

More precisely if Ω is a lenticular zero, then

$$f_\beta(\Omega) = 0 \Longrightarrow P_\beta(\Omega) = 0.$$

3. An analog of uniform distribution theorems – – Proof of Theorem 1.1

Recall (Kuipers and Niederreiter [24], Strauch [48]) that we say a sequence $(x_n)_{n\geq 1} \subseteq [0,1)^s$ is *uniformly distributed on* $[0,1)^s$ if for each box $\mathcal{B} \subseteq [0,1)^s$ which is a cartesian product of intervals contained in $[0,1)$, of volume $|\mathcal{B}|$ we have

$$\lim_{N\to\infty} \frac{1}{N} \#\{1 \leq n \leq N : x_n \in \mathcal{B}\} = |\mathcal{B}|\,.$$

Equivalently, for continuous $f : [0,1)^s \to \mathbb{C}$ we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(x_n) = \int_{[0,1)^s} f(t)\,\mathrm{d}t. \tag{4}$$

In the present context, the left-hand side of equation (1) is the analog of the left-hand side of equation (4). The right-hand side of equation (1) is deduced from Kronecker's Average Value Theorem. This Theorem was announced by Kronecker [23] at the Academy of Sciences, Berlin, in 1880, without proof. It has been given a proof by Rosen in [33].

4. Factorization, large gaps and conjecture "B"

The factorization of integer lacunary polynomials has received a lot of attention, e.g., Prasolov's book [32], Schinzel [36] [37] [39] [41], or Filaseta and his collaborators [9] [12] [13] [15]. Finding irreducibilty criteria is an important topic, and many problems remain open.

Consider polynomials of the form

$$g_N(x) = x^N c\big(x^{-1}\big) + d(x), \tag{5}$$

where c and d are fixed polynomials in $\mathbb{Z}[x]$ with $c(0), d(0) \neq 0$. We are interested in the irreductibility of g_N for large N. Such polynomials have already appeared in different contexts, e.g., in Dobrowolski, Filaseta and Vincent [9], Harrington, Vincent and White [21], Filaseta and Matthews [15], Filaseta, Ford and Konyagin [14], Schinzel [36]. From Schinzel [41] [37] and following Sawin, Shusterman and Stoll [35] we first recall the general statements of Theorem 4.2 and Corollary 4.3, concerned by the factorization of g_N for N large.

The step after, which is important to understand the factorization of any $f \in \mathcal{B}$ containing large gaps, is to recognize such f as a g_N, and apply these general theorems to f. Of course only the $f \in \mathcal{B}$ having large gaps are concerned by these statements. It is expected that the polynomials $f \in \mathcal{B}$ which possess a small gappiness, though not concerned by these statements, have the same factorization properties.

What is the main conclusion at large gaps? Corollary 4.3 implies that any $f \in \mathcal{B}$ would have an irreducible non-cyclotomic component. But, in view of Theorem 2.1, since the non-cyclotomic part of $f(x)$ is $B(x)C(x)$ and that $C(x)$ always exist and is irreducible, it means that the reciprocal non-cyclotomic component $B(x)$ is trivial. So to say, Conjecture B is true at large gaps on $\mathcal{B}$.

Proving Conjecture B on all $f \in \mathcal{B}$ amounts to checking Conjecture B when the gaps in f do not obey the conditions of Theorem 4.2 and Corollary 4.3, that is at small gappiness. Examples of pentanomials with moderate gappiness are given in Table 1.

Let us now state Theorem 4.2 and Corollary 4.3, and make precise the critical bounds N_1, N_2, N_3, N_4.

For a polynomial $u = \sum_{i=0}^{r} a_i x^i \in \mathbb{Z}[x]$, we define $\|u\|$ as the squared Euclidean length of its coefficient vector:

$$\|u\| := \sum_{i=0}^{r} |a_i|^2,$$

and u^* denotes its reciprocal polynomial, as

$$u^*(x) = x^{\deg u} u(1/x) = \sum_{i=0}^{r} a_{r-i} x^i.$$

We say that u is reciprocal when $u^* = u$. It is said to be non-reciprocal if $u^* \neq u$.

From equation (5) the two integers N_1 and N_2 can be defined: let $T := \max\{\deg c, \deg d\}$, and τ be the smallest Salem number known, 1.176280, unique real positive root > 1 of $x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1$ (Lehmer's polynomial). Denote

$$N_1 := \deg c + \deg d + \begin{cases} \frac{2T}{\log \tau} \log(\|c\| + \|d\|) & \text{if } T \leq 27, \\ T\big(\log(6T)\big)^3 \log(\|c\| + \|d\|), & \text{otherwise}; \end{cases} \tag{6}$$

$$N_2 := \deg c + \exp\left(\frac{5}{16} \cdot 2^{(\|c\|+\|d\|)^2}\right)\big(2 + \max\{2, (\deg c)^2, (\deg d)^2\}\big)^{\|c\|+\|d\|}.$$

Definition 4.1. A pair (c, d) of polynomials $c, d \in \mathbb{Z}[x]$ with $c(0), d(0) \neq 0$ is Capellian when $-d(x)/c(x^{-1})$ is a pth power in $\mathbb{Q}(x)$ for some prime p or $d(x)/c(x^{-1})$ is 4 times a fourth power in $\mathbb{Q}(x)$.

The following Theorem 4.2 is Theorem 1.2 in [35], a revisited formulation of a theorem of Schinzel (Theorem 74 in [41]); the upper bound N_2 in item 2) can be found in Schinzel [37].

Theorem 4.2 (Schinzel). *Let $c, d \in \mathbb{Z}[x]$ with $c(0), d(0) \neq 0$. Assume that $c \neq \pm d$, that $\gcd_{\mathbb{Z}[x]}(c^*, d) = 1$ and (c, d) is not Capellian. Then*

1) *there is a bound N_0 depending only on c and d such that for $N > N_0$, the non-reciprocal part of g_N is irreducible,*
2) *the bound N_0 satisfies $N_0 \leq N_2$.*

Remark 2. Theorem 4.2 is probably the best theorem on the subject in general. However, in the case of the class $\mathcal{B}$, a direct application of Ljunggren's tricks provides a better solution, which is Theorem 2.1, obtained recently by the authors in [10].

Filaseta, Ford and Konyagin [14] have shown that the upper bound N_2 in 2) can be replaced by the better upper bound

$$N_3 := \deg c + 2\max\left\{5^{4(\|c\|+\|d\|+t)-15},\, T\big(5^{2(\|c\|+\|d\|+t)-8} + 1/4\big)\right\},$$

where $T = \max\{\deg c, \deg d\}$ and t is the number of terms in c plus the number of terms in d.

Under some assumptions, Sawin, Shusterman and Stoll [35] show that the upper bound N_2 in 2) can still be improved by replacing N_2 by

$$N_4 := (1 + \deg c + \deg d)2^{\|c\|+\|d\|}.$$

The bound N_4 is considerably smaller than N_3, and the bound N_2 given by Schinzel is extremely large. For instance, for the pentanomial $f(x) = -1 + x + x^5 + x^{14} + x^{100}$, the bounds are: $N_1 = 292$ (cf Corollary 4.3 for its use), $N_2 = 1.54 \cdot 10^{4553919}$, $N_3 = 6 \cdot 10^{17}$ whereas

$$N_4 = 480.$$

The authors in [35] also suggest an algorithm to improve further the value N_4, by replacing the exponential $2^{\|c\|+\|d\|}$ by a polynomial function of $\|c\| + \|d\|$. This algorithm is useful in some cases.

The integer N_1 defined in equation (6) is introduced in [35] and both N_1 and N_2 are used when applying numerically the following statement (which can be found in [35]).

Corollary 4.3. *Under the assumptions of Theorem 4.2, if* $N > \max\{N_0, N_1\}$, *then the non-cyclotomic part of* g_N *is irreducible.*

Now let

$$f(x) = -1 + x + x^n + x^{m_1} + \cdots + x^{m_{j-1}} + x^{m_j} + \cdots + x^{m_s} \quad \in \mathcal{B},$$

with $n \geq 3, s \geq 1$. It is easy to write it under the general form g_N as above.

Define $m_0 = n$ for coherency. With the distanciation rules we have missing monomials, those between $x^n = x^{m_0}$ and x^{m_1}, between x^{m_1} and x^{m_2}, ..., and between $x^{m_{s-1}}$ and x^{m_s}. Let us fix an integer $j \in \{1, 2, \dots, s\}$ and write f as

$$f(x) = d(x) + x^{m_j} x^{m_s - m_j} c\big(x^{-1}\big) \tag{7}$$

with

$$d(x) = -1 + x + x^n + x^{m_1} + \cdots + x^{m_{j-1}}$$

and

$$c(x) = x^{m_s - m_j} + x^{m_s - m_{j+1}} + x^{m_s - m_{j+2}} + \cdots + x^{m_s - m_{s-1}} + 1.$$

The two polynomials c and d are fixed. Let us make the link with equation (5). We consider the family of polynomials deduced from f by taking arbitrarily sizes of the "hole" left between the monomials $x^{m_{j-1}}$ and x^{m_j}, as follows. We consider the subcollection

$$\widetilde{f^{(j)}} = \left\{ f_N(x) := d(x) + x^N c\left(x^{-1}\right) \mid N \geq m_s \right\} \subset \mathcal{B}_n,$$

and are interested in the irreducibility of f_N for large N. Let us note that $f_{m_s} = f$. The superscript $^{(j)}$ means that this family is "associated" with f and its jth "hole". Let us observe that the integer $\|c\| + \|d\|$ is the number of monomials of any f_N in $\widetilde{f^{(j)}}$. It is an invariant of the family: for any f_N in $\widetilde{f^{(j)}}$, the integer N_3 is the same, and is equal to

$$N_3 := \deg c + 2 \max \left\{ 5^{8(\|c\|+\|d\|)-15},\ T\left(5^{4(\|c\|+\|d\|)-8} + 1/4\right) \right\}.$$

Example. We consider infinite families of pentanomials which present a variable gappiness at the last monomial. The polynomial c is taken equal to 1. When $c = 1$, then assumptions of Theorem 4.2 are satisfied. The pentanomials are defined below. When applying Corollary 4.3 to the polynomials

$$f_N \in \widetilde{f^{(j)}} \quad \text{for } j = m_2 \quad \text{and} \quad N > \max\{N_4, N_1\},$$

then Conjecture B is valid for all the $f_N \in \widetilde{f^{(j)}}$. In Table 1 we check the validity of Conjecture B for the intermediate values of N in the range

$$m_1 + (n-1) \leq N \leq N_4.$$

Theorem 4.4 (Finch, Jones)**.** *Let $d \in \mathcal{B}$,*

$$d(x) = -1 + x + x^n + x^{m_1}.$$

Let

$$e_1 = \gcd(m_1, n-1), \qquad e_2 = \gcd(n, m_1 - 1).$$

The quadrinomial $d(x)$ is irreducible over $\mathbb{Q}$ if and only if

$$m_1 \not\equiv 0 \pmod{2\,e_1}, \qquad n \not\equiv 0 \pmod{2\,e_2}.$$

In Table 1 the Conjecture B is tested on all the pentanomials $f(x) = d(x) + x^N c(x^{-1})$ with $d(x) = -1 + x + x^n + x^{m_1}$, $c(x^{-1}) = 1$, and for N in the range $\{m_1 + n - 1, m_1 + n, \ldots, N_4\}$. The quadrinomials $d(x)$ are chosen to be either irreducible (in which case they are labelled "+") or reducible (in which case they are labelled "-") after Finch-Jones's Theorem 4.4. The pentanomials $f(x) = 1 + x + x^n + x^{m_1} + x^N$ obtained are either irreducible, or have cyclotomic factors $\Phi_k(x)$ with $k = 3, 6, 9, 10, 12, 18, 24$ or 30. No reciprocal non-cyclotomic factor appears in the factorizations.

TABLE 1. Numerical verification of the Conjecture B for all low degree pentanomials $f(x) = -1 + x + x^n + x^{m_1} + x^N$ for which $m_1 + (n-1) \leq N \leq N_4$. For each experiment number, the polynomials $f(x)$ are either irreducible or have cyclotomic factors in $\{\Phi_k\}$ (e.g., for Exp. num. equal to 1, only $\Phi_{k=3}$ and $\Phi_{k=6}$ are encountered).

Exp. num.	n	m_1	Quad. irred.	N_4	$\{\Phi_k(x)\}$	$B-$conj.
1	3	5	+	192	{3, 6}	✓
2	3	6	+	224	{3, 6}	✓
3	3	7	+	256	{10}	✓
4	3	8	−	288	{3}	✓
5	3	9	+	320	{3, 10}	✓
6	3	10	+	352	{18}	✓
7	3	11	+	384	{3, 6}	✓
8	3	12	−	416	{3, 6}	✓
9	3	13	+	448	∅	✓
10	3	17	+	576	{3, 6, 10}	✓
11	4	7	−	256	{6, 9}	✓
12	4	8	−	288	∅	✓
13	4	9	+	320	{9}	✓
14	4	10	−	352	∅	✓
15	4	11	−	384	{6}	✓
16	4	12	−	416	∅	✓
17	4	13	+	448	{6, 24}	✓
18	4	17	+	576	{6}	✓
19	5	9	+	320	{3, 6, 12}	✓
20	5	10	+	352	{6}	✓
21	5	11	+	384	{6}	✓
22	5	12	+	416	{3, 6, 12}	✓
23	5	13	+	448	{6}	✓
24	5	14	+	480	{6}	✓
25	5	15	+	512	{3, 6}	✓
26	5	16	−	544	{6, 30}	✓
27	5	17	+	576	{6}	✓

REMARK 3. The assumptions of Theorem 4.2 are not strong and are compatible with Conjecture B. Indeed, if $f(x) = d(x) + x^N c^*(x)$ belongs to $\mathcal{B}$, then $c \neq \pm d$ always, the couple (c, d) is never Capellian. What about the assumption $\gcd_{\mathbb{Z}[x]}(c^*, d) = 1$? The polynomial $c^*(x)$ is a Newman polynomial since all the coefficients are in $\{0, 1\}$. By the Odlyzko-Poonen Conjecture (cf [3]) it is irreducible with probability 1. But it has no zero in the interval $[0, 1]$, which is not the case of the non-reciprocal part of $d(x) \in \mathcal{B}$ by Theorem 2.1.

Therefore, with probability one, c^* cannot be the non-reciprocal part of d. Therefore, with probability one, it is an irreducible cyclotomic polynomial, which is a cyclotomic factor of d. At worst, $\gcd_{\mathbb{Z}[x]}(c^*, d)$ would be a cyclotomic factor of f with probability one.

Applying Corollary 4.3, the numerical investigation reported in Table 2 allows to complement completely the study of Conjecture B at large gaps by the one at intermediate lacunarity, and gives a proof to the following result.

Proposition 4.5. *The family of pentanomials $f(x) = -1 + x + x^5 + x^{14} + x^N$, $N \geq 18$, of $\mathcal{B}_5$, admits the bounds $N_1 = 292$, $N_4 = 480$. All the polynomials of this family satisfy Conjecture B.*

Remark 4. The other infinite families of pentanomials whose first quadrinomial is given in the list of Table 1 present different values of N_1 and N_4, and may be studied in the same way, with respect to Conjecture B

In the continuation of search for the conditions of existence of very small Mahler measures $\mathrm{M}(\beta) > 1$ of reciprocal algebraic integers $\beta > 1$, close to 1, it should be noticed that the gaps of the Parry Upper functions

$$f_\beta(x) := -1 + x + x^n + x^{m_1} + x^{m_2} + \cdots + x^{m_s} + \cdots ,$$

where $m_1 - n \geq n - 1$, $m_{q+1} - m_q \geq n - 1$ for $q \geq 1$, are never large by the following asymptotic upper bound (Verger-Gaugry [50]):

$$\limsup_{j \to \infty} \frac{m_{j+1}}{m_j} \leq \frac{\log \mathrm{M}(\beta)}{\log \beta}.$$

Consequently the polynomial sections of $f_\beta(z)$ have asymptotically a moderate gappiness, which is the intermediate domain of study for the non-existence of reciprocal non-cyclotomic factors. If the domain of very large gaps is covered by Corollary 4.3, the existence of non-zero reciprocal algebraic integers $\beta > 1$ would lead to the difficult domain of intermediate gappinesses.

5. On the lower bound in Serre's density theorem

In this paragraph we show on examples that the lower bound given by Serre in Theorem 1.2 1) of the density of the set $\mathcal{P}_0$ is far from being sharp for the polynomials of the class $\mathcal{B}$.

We consider the set of the polynomial sections arising from $\zeta_\tau(z)$ with $\tau = 1.176280$ Lehmer's number.

Let $f_\tau(x) = -1 + x + x^{12} + x^{31} + x^{44} + x^{63} + x^{86} + x^{105} + x^{118} + \cdots = -1/\zeta_\tau(x)$ be the Parry Upper function at Lehmer's number $\tau = 1.176280\ldots$ All the polynomial sections belong to the subclass $\mathcal{B}_{12}$.

Denote:

$$\begin{aligned}
S_0 &= -1+x+x^{12}, && \textit{irreducible},\\
S_1 &= -1+x+x^{12}+x^{31}, && \textit{reducible},\\
&= (x^2+1)(x^4-x^2+1)C_1(x), &&\\
S_2 &= -1+x+x^{12}+x^{31}+x^{44}, && \textit{irreducible},\\
S_3 &= -1+x+x^{12}+x^{31}+x^{44}+x^{63}, && \textit{irreducible},\\
S_4 &= -1+x+x^{12}+x^{31}+x^{44}+x^{63}+x^{86}, && \textit{irreducible},\\
S_5 &= -1+x+x^{12}+x^{31}+x^{44}+x^{63}+x^{86}+x^{105}, && \textit{reducible},\\
&= \Phi_3(x)\Phi_4(x)\Phi_{12}(x)C_5(x), &&\\
S_6 &= -1+x+x^{12}+x^{31}+x^{44}+x^{63}+x^{86}+x^{105}+x^{118}, && \textit{irreducible}.
\end{aligned}$$

The numbers of solutions of the non-reciprocal parts of S_j, $C_j(x) \equiv 0 \bmod p$ with $p \leq 43$, are given in Table 2, for $0 \leq j \leq 6$. In this table, on each line, the frequencies of zeroes are substantially higher than the ones deduced from the general bounds $1/\deg(C_j)$ of Theorem 1.2, 1).

TABLE 2. Values of the quantities $N_p(S_j)$ for all primes p in the range $\{2, 3, \ldots, 43\}$, where S_j is the jth polynomial section of the Parry Upper function $f_\tau(x) = -1/\zeta_\tau(x)$, and τ Lehmer's number, if S_j is irreducible. When S_j is not irreducible the quantity $N_p(S_j)$ represented is replaced by $N_p(C_j)$ where C_j is the non-reciprocal part of S_j. The contributions of the cyclotomic parts modulo p, removed from the lines $j = 1$ and $j = 5$ are indicated underneath (calculated with PARI/GP).

j	2	3	5	7	11	13	17	19	23	27	31	37	41	43
0	0	0	0	0	2	0	2	1	1	1	0	1	0	1
1	0	0	0	0	0	0	1	1	1	1	0	1	2	1
2	0	1	0	1	0	0	1	1	0	0	2	2	2	0
3	1	0	0	1	0	0	1	1	1	1	1	0	1	1
4	0	0	1	0	0	1	2	2	0	1	0	3	0	1
5	0	0	0	0	0	1	1	0	0	2	1	0	0	0
6	0	0	0	1	0	1	1	1	1	0	1	0	3	1
$N_p(X^2+X+1)$	0	1	0	2	0	2	0	2	0	0	2	2	0	2
$N_p(X^2+1)$	1	0	2	0	0	2	2	0	0	2	0	2	2	0
$N_p(X^4-X^2+1)$	0	0	0	0	0	4	0	0	0	0	0	4	0	0

6. Counting solutions mod p and letting p tend to infinity

For understanding the quantities $N_p(f)$, $f \in \mathcal{B}$, and possibly relate them to the coefficients of some power series, by the global correspondence Langlands program, we follow the general strategy of Serre in [43] [44] [45]. The starting point is the study of the roots of the trinomials $-1+x+x^n$ mod p, then of the quadrinomials of $\mathcal{B}$ mod p, whose first three terms are $-1+x+x^n$, etc, then ideally all the elements of the class $\mathcal{B}$.

On one side the numbers $N_p(f)$, $f \in \mathcal{B}$, are correlated to the factorization of the polynomials f via Kronecker's Average Value Theorem 1.1. On the other side the numbers $N_p(f)$ are related to questions of modularity and geometry.

6.1. Trinomials $-1+z+z^n$ mod p and Newforms

The case $n=2$: $f(x) = -1+x+x^2$. The discriminant of f is 5. The polynomial f has a double root mod 5; then $N_5(f)=1$. If $p \neq 2,5$, the roots of f in $\overline{\mathbb{F}_p}$ are $(1 \pm \sqrt{5})/2$. If 5 is a square mod p then there are two roots, $N_p(f)=2$. If not $N_p(f)=0$.

For p and q two distinct odd prime numbers, define the Legendre symbol as

$$\left(\frac{q}{p}\right) = \begin{cases} +1 & \text{if } r^2 \equiv q \bmod p \text{ for some integer } r, \\ -1, & \text{otherwise.} \end{cases}$$

The law of quadratic reciprocity says

$$\left(\frac{q}{p}\right)\left(\frac{p}{q}\right) = (-1)^{\frac{p-1}{2}\frac{q-1}{2}}.$$

Then 5 is a square mod p if and only if $p \equiv \pm 1$ mod 5. Therefore

$$N_p(-1+x+x^2) = \begin{cases} 0 & \text{if } p \equiv \pm 2 \mod 5, \\ 2 & \text{if } p \equiv \pm 1 \mod 5. \end{cases}$$

We deduce the distribution of values of $N_p(-1+x+x^2)$ in the first column of Table 3. This distribution seems fairly regular. The probability limit for each value 0 and 2 is 1/2, but there is a Chebyshev bias (Rubinstein and Sarnak [34]), mentioned in Serre [43], which slightly shifts the probability distribution to 0 preferentially. It is observed in Table 4. This bias occurs for polynomials $f \in \mathcal{B}$ and will be studied elsewhere.

Using the change of variable x to $-x$, the results of Serre [43] section 5.2, can be directly applied to the trinomials $-1+x+x^2$, as follows.

Let us consider the q-series $F = \sum_{m=0}^{\infty} a_m q^m$, defined by

$$F = \frac{q-q^2-q^3+q^4}{1-q^5} = q-q^2-q^3+q^4+q^6-q^7-q^8+q^9+\cdots$$

Then
$$N_p(-1+x+x^2) = a_p + 1 \qquad \text{for all prime numbers } p.$$
Note that the coefficients (a_m) of F have the property to be strongly multiplicative, in the sense:
$$a_{rq} = a_r a_q \qquad \text{for all integers } r, q \geq 1.$$
The corresponding Dirichlet series (L M F D B [25]) is the L-series
$$\sum_{m=1}^{\infty} \frac{a_m}{m^s} = \prod_p \left(1 - \left(\frac{p}{5}\right) p^{-s}\right)^{-1}.$$

The case $n = 3$: the discriminant of $f(x) = -1+x+x^3$ is -31. Modulo 31, the polynomial f has a double root and a simple root. Hence $N_{31}(f) = 2$. For $p \neq 31$, one has
$$N_p(f) = \begin{cases} 0 \text{ or } 3 & \text{if } \left(\frac{p}{31}\right) = +1, \\ 1 & \text{if } \left(\frac{p}{31}\right) = -1. \end{cases} \tag{8}$$
This explains the values reported in the corresponding column in Table 3.

The case $n = 4$: using the change of variable x to $-x$, the case of the trinomial $-1+x+x^4$ may be deduced from the case of $-1-x-x^4$. In Section 5.4 of [43] S e r r e gives the values $N_p(-1-x+x^4)$ from coefficients of Newforms. Let us summarize the expressions he obtains, for the trinomials $-1+x+x^4$.

The discriminant of $f(x) = -1+x+x^4$ is -283. Modulo 283, f has one double root and two simple roots. Then $N_{283}(f) = 3$. If $p \neq 283$, one has
$$N_p(f) = \begin{cases} 0 \text{ or } 4 & \text{if } \; p \text{ can be written as } \; x^2+xy+71y^2, \\ 1 & \text{if } \; p \text{ can be written as } \; 7x^2+5xy+11y^2, \\ 0 \text{ or } 2 & \text{if } \; \left(\frac{p}{283}\right) = -1. \end{cases}$$

A complete determination of $N_p(f)$ can be obtained via a Newform $F = \sum_{m=0}^{\infty} a_m q^m$ of weight 1 and level 283 (L M F D B [25]) whose first hundred terms are given in C r e s p o [8]:
$$\begin{aligned} F = q &+ i\sqrt{2}q^2 - i\sqrt{2}q^3 - q^4 - i\sqrt{2}q^5 + 2q^6 - q^7 - q^9 + 2q^{10} + q^{11} i\sqrt{2}q^{12} \\ &+ q^{13} - i\sqrt{2}q^{14} - 2q^{15} - q^{16} - i\sqrt{2}q^{18} + i\sqrt{2}q^{19} + i\sqrt{2}q^{20} + i\sqrt{2}q^{21} \\ &+ i\sqrt{2}q^{22} - q^{23} - q^{25} + i\sqrt{2}q^{26} + q^{28} - q^{29} - 2i\sqrt{2}q^{30} + i\sqrt{2}q^{31} \\ &- i\sqrt{2}q^{32} - i\sqrt{2}q^{33} + i\sqrt{2}q^{35} + q^{36} - 2q^{38} - i\sqrt{2}q^{39} + q^{41} - 2q^{42} + \cdots \end{aligned}$$

Then one has
$$N_p(f) = 1 + (a_p)^2 - \left(\frac{p}{283}\right) \qquad \text{for all primes} \quad p \neq 283.$$
This explains the values reported in the corresponding column in Table 3.

TABLE 3. Values of the numbers $N_p(-1+x+x^n)$ for all primes p in the range $\{2,3,\dots,101\}$ and n in the range $\{2,3,\dots,15\}$ (calculated with PARI/GP).

p	$n=2$	3	4	5	6	7	8	9	10	11	12	13	14	15
2	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	0	1	0	1	0	1	0	1	0	1	0	1	0	1
5	1	0	0	1	1	0	0	1	1	0	0	1	1	0
7	0	0	1	2	0	1	0	0	1	2	0	1	0	0
11	2	1	1	1	0	1	1	0	0	1	2	1	1	1
13	0	1	1	2	1	0	0	0	0	2	0	1	0	1
17	0	1	2	1	1	0	0	1	0	1	2	1	1	0
19	2	0	0	3	1	0	1	1	0	3	1	0	1	1
23	0	1	1	0	0	2	2	2	0	0	1	0	1	1
29	2	1	1	1	2	0	0	2	0	1	1	0	1	0
31	2	2	0	0	1	1	1	1	1	3	0	1	1	0
37	0	1	2	0	1	0	0	0	0	2	1	0	1	1
41	2	0	1	1	0	0	0	0	1	0	0	0	0	1
43	0	1	0	1	1	2	0	1	0	2	1	0	0	1
47	0	3	0	0	2	2	1	0	2	2	0	2	0	2
53	0	1	2	0	1	0	0	1	0	2	0	0	0	0
59	2	0	1	1	1	2	1	0	2	1	0	0	2	1
61	2	1	1	1	1	1	1	1	0	2	0	1	0	1
67	0	3	2	0	1	0	0	0	0	3	0	1	1	1
71	2	0	0	0	0	0	1	1	1	1	0	1	2	1
73	0	1	0	1	0	0	0	0	2	3	0	0	1	0
79	2	1	2	0	2	1	1	0	2	2	1	1	0	0
83	0	1	4	0	3	3	2	1	0	1	0	0	2	1
89	2	1	1	1	0	0	0	1	0	1	1	1	2	0
97	0	0	1	0	1	1	2	0	0	3	1	1	1	0
101	2	0	0	1	0	0	0	0	0	1	1	1	0	0

TABLE 4. Chebyshev bias for $-1+x+x^2 \bmod p$.

	$x=101$	1001	10001
$\#\{p \le x \mid N_p(-1+x+x^2)=0\}$	14	89	619
$\#\{p \le x \mid N_p(-1+x+x^2)=2\}$	11	78	609
$\#\{p \le x\}$	26	168	1229

The case $n = 7$: the Galois group G of the trinomial $-1 + x + x^7$ is S_7 (Cohen, Movahhedi, Salinier [7]). Applying Theorem 1.2, 2) gives $1.984 \cdot 10^{-4} = (\#G)^{-1}$ for the density of primes p such that $N_p(-1+x+x^7) = 7$. Indeed, the only two prime numbers $\leq 10^5$ realizing the maximality identity $N_p(-1+x+x^7) = 7$ are: $p = 41143$ and $p = 82883$. By Theorem 1.2, 1) the density of prime numbers p such that $N_p(1 + x + x^7) = 0$ exists and is above $1/7$. This is compatible with the corresponding column in Table 3.

The general case n: following Serre [43] there should exist formulas for the numbers $N_p(f)$ coming from the coefficients of q-series, newforms, etc.

6.2. Densities and lacunarity

Let $n \geq 3$ and $f(x) = (-1+x+x^n)+x^{m_1}+x^{m_2}+\cdots+x^{m_s} \in \mathcal{B}$. To align the statements on the presentation of Serre [44] [45], let us adopt the geometric language of schemes. The system of algebraic equations defining the variety is reduced to one equation. Denote by $A = \mathbb{Z}[x]/(f)$ the finitely generated ring over $\mathbb{Z}$. Let $X = \mathrm{Spec}(A)$. The solutions mod p of $f(x) \equiv 0$ correspond to the elements $x \in X(\mathbb{F}_p)$ in the fiber. We denote $N_p(f)$ by $N_X(p)$. The polynomial f is fixed and p varies.

Theorem 6.1. *For any integer $q \geq 1$, any $\gamma \in \mathbb{Z}/q\mathbb{Z}$, the set*

$$\overline{\mathcal{P}_\gamma} := \{p \mid N_X(p) \equiv \gamma \mod q\}$$

has a density. This density is a rational number.

Proof. Serre [44] [45]. □

For $q = 2$ the question of the density of $\overline{\mathcal{P}_\gamma}$ amounts to understand when $N_X(p)$ is even, and when $N_X(p)$ is odd. In general the set $\overline{\mathcal{P}_\gamma}$ is empty, is a finite set or has a density which is > 0. A case when the density is > 0 is of topological origin and comes from the topology of the complex space $X(\mathbb{C})$, resp. of the real space $X(\mathbb{R})$. Denote by $\chi\big(X(\mathbb{C})\big)$, resp $\chi_c\big(X(\mathbb{R})\big)$, in $\mathbb{Z}$, the Euler characteristic of $X(\mathbb{C})$, resp. the Euler characteristic with compact support of $X(\mathbb{R})$.

Theorem 6.2. *For any integer $q \geq 1$,*

$$\text{(i)} \qquad \delta\Big(\overline{\mathcal{P}_\gamma}\Big) > 0 \quad \textit{for } \gamma = \chi\big(X(\mathbb{C})\big),$$

$$\text{(ii)} \qquad \delta\Big(\overline{\mathcal{P}_\gamma}\Big) > 0 \quad \textit{for } \gamma = \chi_c\big(X(\mathbb{R})\big).$$

Proof. Serre [44] [45]. □

The decomposition of f as the sum of the trinomial part

$$-1 + x + x^n$$

and the perturbation term

$$x^{m_1} + x^{m_2} + \cdots + x^{m_s}$$

for $m_1 - n \geq n - 1$, $m_{j+1} - m_j \geq n - 1$, for $1 \leq j \leq s - 1$ suggests to define $Y = \mathrm{Spec}\big(\mathbb{Z}[X]/(-1 + x + x^n)\big)$ and to study $N_Y(p)$ for p tending to infinity. A first question is about the comparison between $N_X(p)$ and $N_Y(p)$, and when p tends to infinity.

Question 1: For any integer $q \geq 1$, any $\gamma \in \mathbb{Z}/q\mathbb{Z}$ has

$$\overline{\mathcal{P}_\gamma} := \{p \mid N_X(p) - N_Y(p) \equiv \gamma \bmod q\}$$

a density?

Question 2: What is the role of n in the sets

$$\{p \mid N_X(p) - N_Y(p) \equiv \gamma \bmod q\},$$

in particular when n becomes very large?

7. Appendix: The expression of the dynamical zeta function $\zeta_\beta(z)$ of the β-shift when $\beta > 1$ is close to 1

The importance of the class of polynomials $\mathcal{B}$ comes from the formulation of the dynamical zeta function $\zeta_\beta(z)$ of the β-shift when $\beta > 1$ is close to 1. Let us recall the main steps, leaving the details for the reader.

The notion of dynamical zeta function was introduced by M. Artin and B. Mazur [1] in 1965. Let $h : V \mapsto V$ be a diffeomorphism of a compact manifold V, such that its iterates h^k all have isolated fixed points. Then they defined

$$\zeta_\beta(z) := \exp\left(\sum_{n=1}^{\infty} \frac{\#\{x \in V \mid h^n(x) = x\}}{n}\, z^n\right), \tag{9}$$

and showed that for a dense set of diffeomorphisms h of V the power series of such an expression converged in a neighbourhood of $z = 0$. The dynamical zeta function of a dynamical system, when defined, is an analytic function which concentrates a lot of information on the dynamical system, and therefore is a powerful tool (Pollicott [31]).

From Theorem 2 in Baladi and Keller [2], concerning the Rényi-Parry dynamical numeration system ($V = [0,1], h = T_\beta$), where $\beta > 1$ and $T_\beta : x \to \beta x \mod 1$ is the β-transformation (Itô and Takahashi [22], Lagarias [26]), we deduce

Theorem 7.1. *Let $\beta \in (1, \theta_2^{-1})$. Then, the Artin-Mazur dynamical zeta function*

$$\zeta_\beta(z) := \exp\left(\sum_{n=1}^{\infty} \frac{\#\{x \in [0,1] \mid T_\beta^n(x) = x\}}{n} z^n\right), \tag{10}$$

counting the number of periodic points of period dividing n, is nonzero and meromorphic in $\{z \in \mathbb{C} : |z| < 1\}$, and such that $1/\zeta_\beta(z)$ is holomorphic in $\{z \in \mathbb{C} : |z| < 1\}$,

Let $\beta > 1$ be a real number. Denote $\mathcal{A}_\beta := \{0, 1, 2, \ldots, \lceil \beta - 1 \rceil\}$, where $\lceil \beta - 1 \rceil$ denotes the upper integer part of $\beta - 1$. If β is not an integer, then $\lceil \beta - 1 \rceil = \lfloor \beta \rfloor$ which is the usual integer part of β. Using ergodic theory, Takahashi [49] and Itô and Takahashi [22] obtained the reformulation of equation (10) as follows.

Theorem 7.2. *Let $\beta > 1$ be a real number. Then*

$$\zeta_\beta(z) = \frac{1 - z^N}{(1 - \beta z)\left(\sum_{n=0}^{\infty} T_\beta^n(1)\, z^n\right)}, \tag{11}$$

where N, which depends upon β, is the minimal positive integer such that: $T_\beta^N(1) = 0$; in the case where $T_\beta^j(1) \neq 0$ for all $j \geq 1$, "z^N" has to be replaced by "0". Up to the sign, the expansion of the power series of the denominator in the equation (11) is the Parry Upper function $f_\beta(z)$ at β. It satisfies

(i) $$f_\beta(z) = -\frac{1 - z^N}{\zeta_\beta(z)} \qquad \textit{in the first case,} \tag{12}$$

(ii) $$f_\beta(z) = -\frac{1}{\zeta_\beta(z)} \qquad \textit{in the second case,} \tag{13}$$

and, denoting by $t_1, t_2, \ldots \in \mathcal{A}_\beta$ the coefficients in

$$-1 + t_1 z + t_2 z^2 + t_3 z^3 + \cdots = f_\beta(z) = -(1 - \beta z)\left(\sum_{n=0}^{\infty} T_\beta^n(1)\, z^n\right), \tag{14}$$

$f_\beta(z)$ is such that $0.t_1 t_2 t_3 \ldots$ is the Rényi β-expansion of unity $d_\beta(1)$. The Parry Upper function $f_\beta(z)$ has no zero in $\{z \in \mathbb{C} : |z| \leq 1/\beta\}$ except $z = 1/\beta$ which is a simple zero.

The total ordering $<$ on $(1,+\infty)$ is uniquely in correspondence with the lexicographical ordering $<_{lex}$ on Rényi expansions of 1 by the following Proposition, which is Lemma 3 in Parry [30].

Proposition 7.3. *Let $\alpha > 1$ and $\beta > 1$. If the Rényi α-expansion of 1 is*

$$d_\alpha(1) = 0.t'_1 t'_2 t'_3 \dots, \qquad i.e., \quad 1 = \frac{t'_1}{\alpha} + \frac{t'_2}{\alpha^2} + \frac{t'_3}{\alpha^3} + \cdots$$

and the Rényi β-expansion of 1 is

$$d_\beta(1) = 0.t_1 t_2 t_3 \dots, \qquad i.e., \quad 1 = \frac{t_1}{\beta} + \frac{t_2}{\beta^2} + \frac{t_3}{\beta^3} + \cdots,$$

then $\alpha < \beta$ if and only if $(t'_1, t'_2, t'_3, \dots) <_{lex} (t_1, t_2, t_3, \dots)$.

Theorem 7.4. *Let $n \geq 2$. A real number $\beta \in (1, \frac{1+\sqrt{5}}{2}]$ belongs to $[\theta_{n+1}^{-1}, \theta_n^{-1})$ if and only if the Rényi β-expansion of unity $d_\beta(1)$ is of the form*

$$d_\beta(1) = 0.10^{n-1}10^{n_1}10^{n_2}10^{n_3}\dots, \tag{15}$$

with $n_k \geq n-1$ for all $k \geq 1$.

Proof. Since

$$d_{\theta_{n+1}^{-1}}(1) = 0.10^{n-1}1 \quad and \quad d_{\theta_n^{-1}}(1) = 0.10^{n-2}1,$$

Proposition 7.3 implies that the condition is sufficient. It is also necessary: $d_\beta(1)$ begins as $0.10^{n-1}1$ for all β such that $\theta_{n+1}^{-1} \leq \beta < \theta_n^{-1}$. For such βs we write $d_\beta(1) = 0.10^{n-1}1u$ with digits in the alphabet $\mathcal{A}_\beta = \{0,1\}$ common to all βs, that is

$$u = 1^{h_0}0^{n_1}1^{h_1}0^{n_2}1^{h_2}\dots$$

and $h_0, n_1, h_1, n_2, h_2, \dots$ integers ≥ 0. The *Conditions of Parry* (Lothaire [28] Chap. 7) applied to the sequence $(1, 0^{n-1}, 1^{1+h_0}, 0^{n_1}, 1^{h_1}, 0^{n_2}, 1^{h_3}, \dots)$, which characterizes uniquely the base of numeration β, readily implies $h_0 = 0$ and $h_k = 1$ and $n_k \geq n-1$ for all $k \geq 1$. □

The polynomials of the class $\mathcal{B}$ are all the polynomial sections of the power series $f_\beta(z)$ for β in the interval $\big(1, (1+\sqrt{5})/2\big)$. Indeed, from equation (15) and equation (14), the power series in equation (14) takes the form

$$-1 + x + x^n + x^{m_1} + x^{m_2} + \cdots + x^{m_s} + \cdots$$

with the distanciation conditions:

$$m_1 - n \geq n-1, \qquad m_{q+1} - m_q \geq n-1 \quad \text{for } 1 \leq q.$$

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