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T–S FUZZY BIBO STABILISATION OF NON–LINEAR SYSTEMS UNDER PERSISTENT PERTURBATIONS USING FUZZY LYAPUNOV FUNCTIONS AND NON–PDC CONTROL LAWS

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This paper develops an innovative approach for designing non-parallel distributed fuzzy controllers for continuous-time non-linear systems under persistent perturbations. Non-linear systems are represented using Takagi–Sugeno fuzzy models. These non-PDC controllers guarantee bounded input bounded output stabilisation in closed-loop throughout the computation of generalised inescapable ellipsoids. These controllers are computed with linear matrix inequalities using fuzzy Lyapunov functions and integral delayed Lyapunov functions. LMI conditions developed in this paper provide non-PDC controllers with a minimum $\star$-norm (upper bound of the 1-norm) for the T–S fuzzy system under persistent perturbations. The results presented in this paper can be classified into two categories: local methods based on fuzzy Lyapunov functions with guaranteed bounds on the first derivatives of membership functions and global methods based on integral-delayed Lyapunov functions which are independent of the first derivatives of membership functions. The benefits of the proposed results are shown through some illustrative examples.

Keywords: linear matrix inequalities, Takagi–Suegno fuzzy systems, fuzzy Lyapunov functions, integral delayed Lyapunov functions (IDLFs), non-parallel distributed fuzzy controllers (non-PDC), generalised inescapable ellipsoids.

1. Introduction

Takagi–Sugeno fuzzy systems have been an active research topic in control community since the mid-1990s. The keys behind this activity are mainly two (Tanaka *et al.*, 1998; Tanaka and Wang, 2001):

- T–S fuzzy systems can precisely represent non-linear systems in a compact set;
- parallel distributed compensators (PDC) can be efficiently designed for these systems using linear matrix inequalities (LMIs) (Boyd *et al.*, 1994; Tanaka and Wang, 2001; Sala and Ariño, 2007; Guerra *et al.*, 2015).

LMI problems are convex optimization problems which can be solved in polynomial time (Boyd *et al.*, 1994; Gahinet *et al.*, 1995) with highly efficient solvers (Löfberg, 2004).

The design of PDC controllers can incorporate several kinds of conditions (Tanaka and Wang, 2001; Tuan *et al.*, 2001; Scherer and Weiland, 2000): stability, state and input constraints, $\mathcal{H}_\infty$-norm, etc. Consequently, PDC controllers are able to stabilise the non-linear system represented by a T–S fuzzy system. Moreover, Takagi–Sugeno fuzzy systems are a useful tool to design controllers for non-linear systems (Guerra *et al.*, 2015).

In the literature there are recent publications (Sun *et al.*, 2019; Qiu *et al.*, 2019a; 2019b) which deal with the design of fuzzy output-feedback controllers for non-linear systems under full state constraints and with prescribed performance in closed loop. These manuscripts show the importance that researches are paying to the design of controllers for non-linear systems under real-world conditions.

Moreover, recent advanced stability results based on fuzzy Lyapunov functions (FLFs) and non-PDC control laws have been developed (Cherifi, 2017; Hu *et al.*, 2017; 2018; Márquez *et al.*, 2016; 2017; Lam *et al.*, 2016; Lam,

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2018; Vafamand and Shasadeghi, 2017; Yoneyama, 2017; Vafamand, 2020a; 2020b). In addition, some references have dealt with the case of persistent perturbations (Hu *et al.*, 2019; Vafamand *et al.*, 2016; 2017b; 2017a).

In our previous papers (Salcedo and Martinez, 2008; Salcedo *et al.*, 2008; 2018) we presented results for continuous-time T–S fuzzy models under persistent perturbations based on the concept of the $\star$-norm, using a common quadratic Lyapunov function (CQLF) and PDC controllers. These results guarantee bounded input bounded output (BIBO) stabilisation. The main goal of this paper is to extend such results using FLFs and non-PDC controllers, and to draw a comparison with the results presented in the literature when more complex Lyapunov functions than CQLFs are used (Hu *et al.*, 2019; Vafamand *et al.*, 2017b).

Vafamand *et al.* (2017b) presented some results; cf. Theorem 1 and Corollary 1 based on CQLFs, and Theorem 2 and Corollary 2 based on FLFs with bounds in first derivatives of membership functions. Only Corollary 2 guarantees such bounds using ideas from (Guerra *et al.*, 2012; Pan *et al.*, 2012), although it is conservative compared with the results of this paper. This question will be addressed in the examples of Section 4.

Hu *et al.* (2019) develop an alternative way to deal with the bounds of the first derivatives of membership functions trough Lemma 1 therein. Their bounds outperform bounds from previous publications (Guerra *et al.*, 2012; Pan *et al.*, 2012). Comparisons between that new approach and the results of this paper will be performed throughout examples. As a conclusion, when $q = 1$, that is to say, the multi-index of the FLF and non-PDC control law equals 1, the bounds of (Hu *et al.*, 2019) provide worse values for the $\star$-norm than the results of this paper. Note that this paper does not use multi-indexation formulation for FLFs nor for non-PDC controllers, or equivalently, uses multi-index $q = 1$.

The approaches of Vafamand *et al.* (2017b) and Hu *et al.* (2019) are only local because of bounds for derivatives of membership functions. A way to avoid this problem is to use Lyapunov functions whose derivatives do not depend on derivatives of membership functions. Márquez *et al.* (2016) and Yoneyama (2017) proposed a new kind of Lyapunov functions called IDLFs which satisfy this property. In this article such IDLFs are used to provide global non-PDC controllers under persistent perturbations. To the best of our knowledge, this kind of Lyapunov functions has not been previously used when persistent perturbations are present.

In this paper innovative approaches are developed to improve previous publications which deal with persistent perturbations (Salcedo and Martinez, 2008; Salcedo *et al.*, 2018; Vafamand *et al.*, 2017b; Hu *et al.*, 2019). These innovative methodologies can be summarised as follows:

- Extension of the concepts of the $\star$-norm and inescapable ellipsoids corresponding to CQLFs (Salcedo and Martinez, 2008; Salcedo *et al.*, 2018) when using FLFs, IDLFs and non-PDC controllers under persistent perturbations.
- Development of new guaranteed bounds for first derivatives of membership functions following the ideas of Lee *et al.* (2012; 2014) and de Silva Campos *et al.* (2017) based on LMIs under persistent perturbations.
- Design of local BIBO non-PDC controllers based on FLFs and the new bounds. These controllers will be only valid inside a generalised inescapable ellipsoid.
- Design of global BIBO non-PDC controllers based on IDLFs. Derivatives of IDLFs do not depend on derivatives of membership functions, consequently the derived LMIs can be satisfied globally.

The rest of the paper is organised as follows: Section 2 presents theoretical background and preliminary results. Main results of this paper are developed in Section 3. Section 4 is devoted to application examples. Finally, in Section 5 conclusions are discussed and in Section 6 some future research lines are commented.

2. Definitions, notation and preliminary results

This paper considers the following kind of non-linear models:

$$\begin{aligned} \dot{\boldsymbol{x}}(t) &= f(\boldsymbol{z}(t))\boldsymbol{x}(t) + g(\boldsymbol{z}(t))\boldsymbol{u}(t) + e(\boldsymbol{z}(t))\boldsymbol{\phi}(t), \\ \boldsymbol{y}(t) &= c(\boldsymbol{z}(t))\boldsymbol{x}(t) + d(\boldsymbol{z}(t))\boldsymbol{\phi}(t), \end{aligned} \tag{1}$$

where $\boldsymbol{x}(t) \in \mathbb{R}^{n_x}$ is the state vector, $\boldsymbol{z}(t) \in \mathbb{R}^{p}$ is the premise vector, $\boldsymbol{u}(t) \in \mathbb{R}^{n_u}$ is the control input vector, $\boldsymbol{\phi} \in \mathbb{R}^{n_\phi}$ is the disturbance vector and $\boldsymbol{y}(t) \in \mathbb{R}^{n_y}$ is the controlled output. It is assumed that the premise vector is a subset of the state vector and all the states are measurable. Using different approaches (Tanaka and Wang, 2001) a continuous T–S fuzzy model can be obtained:

$$\begin{aligned} \dot{\boldsymbol{x}}(t) &= \sum_{i=1}^{r} h_i(t)\left(\boldsymbol{A}_i\boldsymbol{x}(t) + \boldsymbol{B}_i\boldsymbol{u}(t) + \boldsymbol{E}_i\boldsymbol{\phi}(t)\right), \qquad (2) \\ &= \boldsymbol{A}_h\boldsymbol{x} + \boldsymbol{B}_h\boldsymbol{u} + \boldsymbol{E}_h\boldsymbol{\phi}, \end{aligned}$$

$$\boldsymbol{y}(t) = \sum_{i=1}^{r} h_i(t)\left(\boldsymbol{C}_i\boldsymbol{x}(t) + \boldsymbol{D}_i\boldsymbol{\phi}(t)\right) = \boldsymbol{C}_h\boldsymbol{x} + \boldsymbol{D}_h\boldsymbol{\phi}, \tag{3}$$

where

$$\boldsymbol{Y}_h \triangleq \sum_{i=1}^{r} h_i(t)\boldsymbol{Y}_i,$$

r is the number of fuzzy rules, and the 'h_i' are known as membership functions satisfying the convex sum property:

$$h_i(t) \geq 0 \quad i = 1, \ldots, r \text{ and } \sum_{i=1}^{r} h_i(t) = 1, \quad \forall\, t. \quad (4)$$

If the T–S model is obtained using the non-linearity sector approach, membership functions have a special structure (da Silva Campos *et al.*, 2017; Márquez *et al.*, 2017):

$$h_i(\boldsymbol{z}) = h_{1+i_1\cdot 2^0+i_2\cdot 2^1+\ldots+i_p\cdot 2^{p-1}}(\boldsymbol{z}) = \prod_{j=1}^{p} w^j_{i_j}(z_j),$$
$$i \in \{1, 2, \ldots, r = 2^p\}, \quad i_j \in \{0, 1\}, \quad j = 1, \ldots, p, \quad (5)$$

where $w^j_{i_j}(z_j)$ are the normalised weighting functions satisfying

$$w^j_{i_j}(z_j) \geq 0, \quad w^j_1(z_j) = 1 - w^j_0(z_j). \quad (6)$$

Note that the j-th normalised weighting function depends only on the j-th premise variable. These functions are related to non-linear terms which are in functions f, g, e, c and d.

A fuzzy parallel distributed compensator (PDC) (Tanaka and Wang, 2001) is a fuzzy controller which has the same premises and membership functions as the T–S model and its consequents are linear state feedback laws:

$$\boldsymbol{u}(t) = \sum_{i=1}^{r} h_i(t)\boldsymbol{F}_i.x(t) = \boldsymbol{F}_h\boldsymbol{x}(t). \quad (7)$$

2.1. 1-Norm and the $\star$-norm. The main objective in this work is to design non-PDC fuzzy state-feedback controllers for continuous-time T–S fuzzy systems, which stabilize the closed loop when the disturbance vector is bounded (persistent disturbance):

$$\boldsymbol{\phi}(t)^T\boldsymbol{\phi}(t) \leq \delta^2, \quad \forall t,\ \delta > 0. \quad (8)$$

This stabilization is known as BIBO since the output vector will always be bounded when $\boldsymbol{\phi}(t)$ is a persistent disturbance:

$$\exists \mu > 0\ \forall t:\ \boldsymbol{y}(t)^T\boldsymbol{y}(t) \leq \mu^2. \quad (9)$$

Remark 1. A persistent disturbance does not necessarily tend asymptotically towards 0 as $t \to \infty$.

The 1-norm is defined by (Boyd *et al.*, 1994; Abedor *et al.*, 1996; Salcedo *et al.*, 2018):

$$||\boldsymbol{G}_{\phi\to y}||_1 \triangleq \sup_{||\boldsymbol{\phi}(t)||_\infty \neq 0} \frac{||\boldsymbol{y}(t)||_\infty}{||\boldsymbol{\phi}(t)||_\infty}, \quad (10)$$

where the ∞-norm of a vector signal is defined as:

$$||\boldsymbol{\phi}(t)||^2_\infty \triangleq \sup_{t\geq 0} \boldsymbol{\phi}(t)^T\boldsymbol{\phi}(t) = \delta^2. \quad (11)$$

This work extends the results presented by Salcedo and Martinez (2008) or Salcedo *et al.* (2018) when a non-PDC state-feedback controller is designed when minimising an upper bound for the 1-norm between $\boldsymbol{\phi}(t)$ and $\boldsymbol{y}(t)$. Salcedo and Martinez (2008) or Salcedo *et al.* (2018) considered only PDC controllers.

It is more complex to calculate the 1-norm than the 2-norm or the $\mathcal{H}_\infty$-norm (Abedor *et al.*, 1996; Sanchez Peña and Sznaier, 1998), although an upper bound can be estimated for it, called the star ($\star$) norm, based on LMIs only (Abedor *et al.*, 1996; Salcedo *et al.*, 2018). This bound allows us to tackle the design of fuzzy controllers using existing techniques (Tanaka and Wang, 2001; Liu and Zhang, 2003; Teixeira *et al.*, 2003; Guerra *et al.*, 2006).

Recall the following result of Salcedo *et al.* (2018):

Lemma 1. ($\star$-Norm computation) *The $\star$-norm between the $\boldsymbol{y}$ output and the $\boldsymbol{\phi}$ input for the closed-loop system formed by* (2) *and* (3)*, and the PDC controller* (7) *is obtained solving the problem:*

$$||\boldsymbol{G}^{CL}_{\phi\to y}||_\star = \inf_{\alpha>0} N(\alpha), \quad (12)$$

where $N(\alpha)$ is calculated of each fixed $\alpha > 0$, as follows:

$$N(\alpha) \triangleq \frac{1}{\delta}\min\left\{\mu \geq 0:\ \bar{\boldsymbol{P}} = \bar{\boldsymbol{P}}^T > 0,\ 0 \leq \beta \leq \alpha,\right.$$
$$\left.\textit{subject to } (13) \textit{ and } (14)\right\},$$

$$\begin{pmatrix} \boldsymbol{A}_h^{CL^T}\bar{\boldsymbol{P}} + \bar{\boldsymbol{P}}\boldsymbol{A}_h^{CL} + \alpha\bar{\boldsymbol{P}} & \delta\bar{\boldsymbol{P}}\boldsymbol{B}_h^{CL} \\ \delta\boldsymbol{B}_h^{CL^T}\bar{\boldsymbol{P}} & -\beta\boldsymbol{I} \end{pmatrix} \leq 0, \quad (13)$$

$$\begin{pmatrix} \alpha\bar{\boldsymbol{P}} & \boldsymbol{0} & \boldsymbol{C}_h^{CL^T} \\ \boldsymbol{0} & (\mu-\beta)\boldsymbol{I} & \delta\boldsymbol{D}_h^{CL^T} \\ \boldsymbol{C}_h^{CL} & \delta\boldsymbol{D}_h^{CL} & \mu\boldsymbol{I} \end{pmatrix} \geq 0. \quad (14)$$

Remark 2. Optimization with respect to α in (12) is carried out calculating the values of $N(\alpha)$ for a sufficiently representative finite set of values for α (gridding procedure).

Conditions of Lemma 1 can be recast as an LMI problem (Salcedo and Martinez, 2008; Salcedo *et al.*, 2018).

Lemma 2. ($\star$-Norm computation with LMIs for PDC controllers) *The $\star$-norm between the output $\boldsymbol{y}$ and the input $\boldsymbol{\phi}$ for the closed-loop system formed by* (2) *and* (3)*,* *and the PDC controller* (7) *can be obtained by solving the LMI problem*

$$||\boldsymbol{G}^{CL}_{\phi\to y}||_\star = \inf_{\alpha>0} N(\alpha), \tag{15}$$

where $N(\alpha)$ *is calculated of each fixed* $\alpha > 0$*, as follows:*

$$N(\alpha) \triangleq \frac{1}{\delta}\min\{\mu \ge 0:\ \boldsymbol{P} > 0,\ 0 \le \beta \le \alpha,$$
$$\text{subject to (16) and (17)}\},$$

$$\Upsilon_{ij} = \begin{pmatrix} \boldsymbol{P}\boldsymbol{A}_i^T + \boldsymbol{A}_i\boldsymbol{P} + \boldsymbol{B}_i\boldsymbol{F}_j + \boldsymbol{F}_j^T\boldsymbol{B}_i^T + \alpha\boldsymbol{P} & \delta\boldsymbol{E}_i \\ \delta\boldsymbol{E}_i^T & -\beta\boldsymbol{I} \end{pmatrix},$$

$$\Psi_i = \begin{pmatrix} \alpha\boldsymbol{P} & \boldsymbol{0} & \boldsymbol{P}\boldsymbol{C}_i^T \\ \boldsymbol{0} & (\mu-\beta)\boldsymbol{I} & \delta\boldsymbol{D}_i^T \\ \boldsymbol{C}_i\boldsymbol{P} & \delta\boldsymbol{D}_i & \mu\boldsymbol{I} \end{pmatrix},$$

$$\Upsilon_{ii} \le 0, \quad \Psi_i \ge 0, \quad i = 1,\dots,r, \tag{16}$$

$$\frac{2}{r-1}\Upsilon_{ii} + \Upsilon_{ij} + \Upsilon_{ji} \le 0, \quad i \ne j, \quad i,j = 1,\dots,r. \tag{17}$$

Remark 3. $V(\boldsymbol{x}) = \boldsymbol{x}^T\boldsymbol{P}^{-1}\boldsymbol{x}$ is a quadratic Lyapunov function for the closed-loop, and the positive definite matrix $\boldsymbol{P}^{-1}$ defines an inescapable ellipsoid (16) (Abedor *et al.*, 1996; Salcedo *et al.*, 2018):

$$\mathcal{E}(\boldsymbol{P}^{-1}) \triangleq \{\boldsymbol{x} : \boldsymbol{x}^T\boldsymbol{P}^{-1}\boldsymbol{x} \le 1\}, \tag{18}$$

which is a robust control positively invariant set for the closed loop.

Remark 4. If an LMI solver based on interior point methods (Boyd *et al.*, 1994) is used, the computational cost of the LMI optimization problem can be estimated as being proportional to $N_{var}^3 \cdot N_{row}$, where N_{var} is the total number of scalar decision variables and N_{row} the total row size of the LMIs (Gahinet *et al.*, 1995). For Lemma 2 we have

$$N_{var}^{Lem2} = 2 + \frac{1}{2}n_x(n_x+1) + r\,n_u\,n_x,$$
$$N_{row}^{Lem2} = 1 + n_x + r^2(n_x+n_\phi) + r(n_x+n_y+n_\phi).$$

2.2. Auxiliary lemmas.

Some useful results are presented here for further developments:

Lemma 3. (Tuan *et al., 2001*) *Given symmetric matrices* Υ_{ij} *of appropriate dimensions, the inequality*

$$\Upsilon_{hh} = \sum_{i=1}^{r}\sum_{j=1}^{r} h_i(z(t))h_j(z(t))\Upsilon_{ij} < 0, \tag{19}$$

is satisfied if

$$\Upsilon_{ii} < 0, \quad i = 1,\dots,r,$$
$$\frac{2}{r-1}\Upsilon_{ii} + \Upsilon_{ij} + \Upsilon_{ji} < 0, \quad i,j = 1,\dots,r, \quad j \ne i. \tag{20}$$

Lemma 4. (Delmotte *et al.*, 2007) *Given matrices* $\boldsymbol{A}$*,* $\boldsymbol{T}_1$*,* $\boldsymbol{T}_2$ *and* $\boldsymbol{T}_3$ *of appropriate dimensions, the next two problems are equivalent:*

1. *Find symmetric* $\boldsymbol{P} > 0$ *with appropriate dimensions such that*

$$\begin{bmatrix} \boldsymbol{T}_1 + \boldsymbol{A}^T\boldsymbol{P} + \boldsymbol{P}\boldsymbol{A} & (*) \\ \boldsymbol{T}_2 & \boldsymbol{T}_3 \end{bmatrix} < 0.$$

2. *Find symmetric* $\boldsymbol{P} > 0$ *and full* $\boldsymbol{L}_1$*,* $\boldsymbol{L}_2$ *and* $\boldsymbol{G}$ *with appropriate dimensions such that*

$$\begin{bmatrix} \boldsymbol{T}_1 + \boldsymbol{A}^T\boldsymbol{L}_1^T + \boldsymbol{L}_1\boldsymbol{A} & (*) & (*) \\ \boldsymbol{T}_2 + \boldsymbol{L}_2\boldsymbol{A} & \boldsymbol{T}_3 & (*) \\ \boldsymbol{P} - \boldsymbol{L}_1^T + \boldsymbol{G}^T\boldsymbol{A} & -\boldsymbol{L}_2^T & -\boldsymbol{G} - \boldsymbol{G}^T \end{bmatrix} < 0.$$

Applying Lemma 4 to Lemma 2, we get the following result.

Corollary 1. *The* $\star$*-norm between the output* $\mathbf{y}$ *and the input* ϕ *for the closed-loop system formed by* (2) *and* (3) *and the PDC controller* (7) *can be obtained solving the LMI problem*

$$||\boldsymbol{G}^{CL}_{\phi\to y}||_\star = \inf_{\alpha>0} N(\alpha), \tag{21}$$

where $N(\alpha)$ *is calculated of each fixed* $\alpha > 0$*, as follows:*

$$N(\alpha) \triangleq \frac{1}{\delta}\min\{\mu \ge 0:\ \boldsymbol{P} > 0,\ 0 \le \beta \le \alpha,$$
$$\text{subject to (22) and (23)}\},$$

$$\Upsilon_{ij} = \begin{pmatrix} \boldsymbol{L}_{1j}\boldsymbol{A}_i^T + \boldsymbol{A}_i\boldsymbol{L}_{1j}^T + \boldsymbol{B}_i\boldsymbol{F}_j + \boldsymbol{F}_j^T\boldsymbol{B}_i^T + \alpha\boldsymbol{P} & (*) & (*) \\ \delta\boldsymbol{E}_i^T + \boldsymbol{L}_{2j}\boldsymbol{A}_i^T & -\beta\boldsymbol{I} & (*) \\ \boldsymbol{P} - \boldsymbol{L}_{1i}^T + \boldsymbol{G}_j^T\boldsymbol{A}_i^T & -\boldsymbol{L}_{2i}^T & -\boldsymbol{G}_i - \boldsymbol{G}_i^T \end{pmatrix},$$

$$\Psi_i = \begin{pmatrix} \alpha\boldsymbol{P} & \boldsymbol{0} & \boldsymbol{P}\boldsymbol{C}_i^T \\ \boldsymbol{0} & (\mu-\beta)\boldsymbol{I} & \delta\boldsymbol{D}_i^T \\ \boldsymbol{C}_i\boldsymbol{P} & \delta\boldsymbol{D}_i & \mu\boldsymbol{I} \end{pmatrix},$$

$$\Upsilon_{ii} \le 0, \quad \Psi_i \ge 0, \quad i = 1,\dots,r, \tag{22}$$

$$\frac{2}{r-1}\Upsilon_{ii} + \Upsilon_{ij} + \Upsilon_{ji} \le 0, \quad i \ne j, \quad i,j = 1,\dots,r, \tag{23}$$

where $\boldsymbol{L}_{1i}$*,* $\boldsymbol{L}_{2i}$ *and* $\boldsymbol{G}_i$ $i = 1,\dots,r$ *are matrices of appropriate dimensions.*

Remark 5. Corollary 1 is more relaxed than Lemma 2. This conclusion is derived from the fact that if LMIs (22) and (23) have a solution then (16) and (17) have a solution. The converse is not necessarily true. This result was analysed by Delmotte *et al.* (2007) and Márquez *et al.* (2016).

Remark 6. For Corollary 1 we have

$$\begin{aligned} N_{var}^{Cor1} &= 2 + \frac{1}{2} n_x(n_x+1) + r\, n_u\, n_x \\ &\quad + r\left(2\, n_x^2 + n_x\, n_\phi\right), \\ N_{row}^{Cor1} &= 1 + n_x + r^2(2\, n_x + n_\phi) \\ &\quad + r(n_x + n_y + n_\phi). \end{aligned}$$

Consequently, Lemma 2 has a lower number of variables and a lower number of rows than Corollary 1, but it is more conservative.

3. Main results

In this article we propose an extension of our previous results (Salcedo and Martinez, 2008; Salcedo *et al.*, 2018; Vafamand *et al.*, 2017b) using non-PDC fuzzy controllers following two strategies based on fuzzy Lyapunov functions:

(i) a global approach based on integral-delayed Lyapunov functions (Márquez *et al.*, 2016; Yoneyama, 2017; Vafamand, 2020a; 2020b);

(ii) a local approach based on fuzzy Lyapunov functions with guaranteed bounded first derivatives of the membership functions (Lee *et al.*, 2012; 2014; Pan *et al.*, 2012; Lee and Kim, 2014; Wang *et al.*, 2015; da Silva Campos *et al.*, 2017; Vafamand *et al.*, 2017b; Vafamand and Shasadeghi, 2017; Hu *et al.*, 2018; 2019).

Integral-delayed Lyapunov functions (Márquez *et al.*, 2016; Vafamand, 2020a) provide global results using membership functions which are integrals of the membership functions of the T–S fuzzy model. The benefits behind this idea are that the time derivatives of the Lyapunov function will not depend on the time derivatives of membership functions which removes the need of bounding such time derivatives, and the results based on this kind of Lyapunov functions are global.

Fuzzy Laypunov functions matching the membership functions of the fuzzy model (Tanaka *et al.*, 2001) require bounding the time derivatives of such membership functions. When designing a fuzzy controller these bounds cannot be either known or estimated in advance, so they have to be included as additional constraints in the LMI problem (Lee *et al.*, 2012; Pan *et al.*, 2012; Wang *et al.*, 2015; Vafamand *et al.*, 2017b; Vafamand and Shasadeghi, 2017; Hu *et al.*, 2019). These bounds imply that stabilization can only be guaranteed in a local subset.

3.1. Results based on integral-delayed Lyapunov functions.

An integral-delayed Lyapunov function (IDLF) is defined as (Márquez *et al.*, 2016)

$$V(\boldsymbol{x}) = \boldsymbol{x}^T \boldsymbol{P}_v^{-1} \boldsymbol{x} = \boldsymbol{x}^T \left(\sum_{i=1}^{r} v_i(z(t)) \boldsymbol{P}_i \right)^{-1} \boldsymbol{x}, \tag{24}$$

with $\boldsymbol{P}_i > 0$ and

$$v_i(z(t)) = \frac{1}{\kappa} \int_{t-\kappa}^{t} h_i(z(\tau))\, \mathrm{d}\tau, \quad \kappa > 0, \tag{25}$$

where κ is taken as a delay.

Lemma 5. (Marquez *et al.*, 2016) *We have that*

1. $0 \le v_i(t) \le 1, \quad \sum_{i=1}^{r} v_i(t) = 1,$
2. $\dot{v}_i(t) = \frac{1}{\kappa}\left(h_i(t) - h_i(t-\kappa)\right),$
3. $\lim_{\kappa \to 0} \dot{v}_i(t) = \dot{h}_i(t),$
4. $\dot{\boldsymbol{P}}_v = \frac{1}{\kappa}\left(\boldsymbol{P}_h - \boldsymbol{P}_{h^-}\right)$, *where* $h_i^- \triangleq h_i(t-\kappa)$.

Together with an IDLF a non-PDC control law (Márquez *et al.*, 2016) is used,

$$\boldsymbol{u}(t) = \boldsymbol{F}_{hh^-v} \boldsymbol{P}_v^{-1} \boldsymbol{x}(t) \tag{26}$$

with $\boldsymbol{F}_{hh^-v} = \sum_{i=1}^{r} \sum_{j=1}^{r} \sum_{k=1}^{r} h_i h_j^- v_k \boldsymbol{F}_{ijk}$, $\boldsymbol{F}_{ijk} \in \mathbb{R}^{n_u \times n_x}$. Non-PDC controllers (Guerra and Vermeiren, 2004) are a generalisation of PDC controllers when the defuzzification of their consequents includes more than one fuzzy summation and/or the inversion of a fuzzy summation. With this non-PDC controller the dynamics of closed-loop are

$$\dot{\boldsymbol{x}} = \overbrace{\left(\boldsymbol{A}_h + \boldsymbol{B}_h \boldsymbol{F}_{hh^-v} \boldsymbol{P}_v^{-1}\right)}^{\boldsymbol{A}^{CL}} \boldsymbol{x} + \boldsymbol{E}_h \boldsymbol{\phi}. \tag{27}$$

Expanding these ideas Yoneyama (2017) proposed the double integral-delayed Lyapunov function (DIDLF)

$$\begin{aligned} V(\boldsymbol{x}) &= \boldsymbol{x}^T \boldsymbol{P}_{v\lambda}^{-1} \boldsymbol{x} \\ &= \boldsymbol{x}^T \left(\sum_{i=1}^{r} \sum_{j=1}^{r} v_i(z(t)) \lambda_j(z(t)) \boldsymbol{P}_{ij} \right)^{-1} \boldsymbol{x}, \end{aligned} \tag{28}$$

with $\boldsymbol{P}_{ij} > 0$ and

$$\lambda_j(z(t)) = \frac{2}{\kappa^2} \int_{-\kappa}^{0} \int_{t+\theta}^{\theta} h_i(z(\tau))\, \mathrm{d}\tau \mathrm{d}\theta. \tag{29}$$

Lemma 6. (Yoneyama, 2017) *We have that*

1. $0 \le \lambda_i(t) \le 1, \quad \sum_{i=1}^{r} \lambda_i(t) = 1,$
2. $\dot{\lambda}_i(t) = \frac{2}{\kappa}\left(h_i(t) - v_i(t)\right),$
3. $\dot{\boldsymbol{P}}_{v\lambda} = \frac{1}{\kappa}\left(\boldsymbol{P}_{h\lambda} - \boldsymbol{P}_{h^-\lambda} + 2\boldsymbol{P}_{vh} - 2\boldsymbol{P}_{vv}\right).$

Remark 7. If $\boldsymbol{P}_{ij} = \boldsymbol{P}_i, \forall j,$ then a DIDLF becomes an IDLF. Consequently, IDLFs are a subset of DIDLFs.

Together with a DIDLF a non-PDC control law (Yoneyama, 2017) is used,

$$\boldsymbol{u}(t) = \boldsymbol{F}_{hh^-vv\lambda}\boldsymbol{P}_{v\lambda}^{-1}\boldsymbol{x}(t), \tag{30}$$

where

$$\boldsymbol{F}_{hh^-vv\lambda} = \sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{k=1}^{r}\sum_{l=1}^{r}\sum_{m=1}^{r} h_i h_j^- v_k v_l \lambda_m \boldsymbol{F}_{ijklm},$$
$$\boldsymbol{F}_{ijklm} \in \mathbb{R}^{n_u \times n_x}.$$

With this non-PDC controller the dynamics of closed-loop are

$$\dot{\boldsymbol{x}} = \overbrace{\left(\boldsymbol{A}_h + \boldsymbol{B}_h\boldsymbol{F}_{hh^-vv\lambda}\boldsymbol{P}_{v\lambda}^{-1}\right)}^{\boldsymbol{A}^{CL}}\boldsymbol{x} + \boldsymbol{E}_h\boldsymbol{\phi}. \tag{31}$$

Remark 8. If $\boldsymbol{F}_{ijklm} = \boldsymbol{F}_{ijk}, \forall l, m$ and $\boldsymbol{P}_{ij} = \boldsymbol{P}_i$, $\forall j$ then the non-PDC control law of (Yoneyama, 2017) (30), becomes the non-PDC control law of Márquez *et al.* (2016); cf. (27).

3.1.1. Theorems for IDLFs.

Now, it is possible to apply Lemma 1 to the IDLF (24) with the non-PDC control law (26).

Theorem 1. ($\star$-Norm computation with IDLF). *The $\star$-Norm between the output $\boldsymbol{y}$ and the input $\boldsymbol{\phi}$ for the closed-loop system* (27) *is obtained the solving the problem*

$$||\boldsymbol{G}_{\phi\to y}^{CL}||_\star = \inf_{\alpha>0} N(\alpha), \tag{32}$$

where $N(\alpha)$ is calculated of each fixed $\alpha > 0$, as follows:

$$N(\alpha) \triangleq \frac{1}{\delta}\min\{\mu \geq 0: \boldsymbol{P}_v > 0,\ 0 \leq \beta \leq \alpha,$$
$$\text{subject to (33) and (34)}\},$$

$$\begin{pmatrix} \boldsymbol{P}_v{\boldsymbol{A}^{CL}}^T + \boldsymbol{A}^{CL}\boldsymbol{P}_v + \alpha\boldsymbol{P}_v - \dot{\boldsymbol{P}}_v & \delta\boldsymbol{B}_h^{CL} \\ \delta{\boldsymbol{B}_h^{CL}}^T & -\beta\boldsymbol{I} \end{pmatrix} \leq 0, \tag{33}$$

$$\begin{pmatrix} \alpha\boldsymbol{P}_v & \boldsymbol{0} & \boldsymbol{P}_v{\boldsymbol{C}_h^{CL}}^T \\ \boldsymbol{0} & (\mu-\beta)\boldsymbol{I} & \delta{\boldsymbol{D}_h^{CL}}^T \\ \boldsymbol{C}_h^{CL}\boldsymbol{P}_v & \delta\boldsymbol{D}_h^{CL} & \mu\boldsymbol{I} \end{pmatrix} \geq 0, \tag{34}$$

with

$$\boldsymbol{A}^{CL} = \left(\boldsymbol{A}_h + \boldsymbol{B}_h\boldsymbol{F}_{hh^-v}\boldsymbol{P}_v^{-1}\right), \quad \boldsymbol{B}_h^{CL} = \boldsymbol{E}_h,$$
$$\boldsymbol{C}_h^{CL} = \boldsymbol{C}_h, \quad \boldsymbol{D}_h^{CL} = \boldsymbol{D}_h, \tag{35}$$

$$\dot{\boldsymbol{P}}_v = \frac{1}{\kappa}\left(\boldsymbol{P}_h - \boldsymbol{P}_{h^-}\right). \tag{36}$$

Proof. From conditions (13) and (14) of Lemma 1, the IDLF (24) and the non-PDC control law (26), the following conditions are obtained:

$$\begin{pmatrix} {\boldsymbol{A}^{CL}}^T\boldsymbol{P}_v^{-1} + \boldsymbol{P}_v^{-1}\boldsymbol{A}^{CL} + \alpha\boldsymbol{P}_v^{-1} + \dfrac{\mathrm{d}\boldsymbol{P}_v^{-1}}{\mathrm{d}t} & \delta P_v^{-1}B_h^{CL} \\ \delta{\boldsymbol{B}_h^{CL}}^T\boldsymbol{P}_v^{-1} & -\beta\boldsymbol{I} \end{pmatrix} \leq 0 \tag{37}$$

$$\begin{pmatrix} \alpha\boldsymbol{P}_v^{-1} & \boldsymbol{0} & {\boldsymbol{C}_h^{CL}}^T \\ \boldsymbol{0} & (\mu-\beta)\boldsymbol{I} & \delta{\boldsymbol{D}_h^{CL}}^T \\ \boldsymbol{C}_h^{CL} & \delta\boldsymbol{D}_h^{CL} & \mu\boldsymbol{I} \end{pmatrix} \geq 0. \tag{38}$$

The term $\mathrm{d}\boldsymbol{P}_v^{-1}/\mathrm{d}t$ appears in block $(1,1)$ of condition (37) because $\boldsymbol{P}_v$ is time dependent. It did not appear in Lemma 1 since it was obtained for quadratic time-independent Lyapunov functions. Conditions (33) and (34) are recovered from (37) and (38) applying a congruence transformation with

$$\mathrm{diag}\left(\boldsymbol{P}_v, \boldsymbol{I}\right) \tag{39}$$

and using the property

$$\frac{\mathrm{d}\boldsymbol{P}_v^{-1}}{\mathrm{d}t} = -\boldsymbol{P}_v^{-1}\dot{\boldsymbol{P}}_v\boldsymbol{P}_v^{-1} \tag{40}$$

■

Remark 9. $V(\boldsymbol{x}) = \boldsymbol{x}^T\boldsymbol{P}_v^{-1}\boldsymbol{x}$ is a non-quadratic Lyapunov function for the closed loop. Moreover, the positive definite fuzzy matrix $\boldsymbol{P}_v^{-1}$ defines an inescapable set (33) (Abedor *et al.*, 1996; Salcedo *et al.*, 2018):

$$\mathcal{E}(\boldsymbol{P}_v^{-1}) \triangleq \left\{\boldsymbol{x} : \boldsymbol{x}^T\boldsymbol{P}_v^{-1}\boldsymbol{x} \leq 1\right\}, \tag{41}$$

which is a robust control positively invariant set for the closed loop. This set is called the generalised inescapable ellipsoid.

Remark 10. Note that (33) and (34) are not LMI conditions. The next two theorems provide a way to recast them as LMIs.

Theorem 2. ($\star$-Norm computation with LMIs for IDLF) *The $\star$-norm between the output $\boldsymbol{y}$ and the input $\boldsymbol{\phi}$ for the closed-loop system* (27) *can be obtained solving the following LMI problem:*

$$||\boldsymbol{G}_{\phi\to y}^{CL}||_\star = \inf_{\alpha>0} N(\alpha). \tag{42}$$

Given $\kappa > 0$, $N(\alpha)$ is calculated of each fixed $\alpha > 0$, as follows:

$$N(\alpha) \triangleq \frac{1}{\delta}\min\{\mu \geq 0: \boldsymbol{P}_i > 0,\ 0 \leq \beta \leq \alpha,$$
$$\text{subject to LMIs (43)–(45)}\},$$

$$\Upsilon_{ijkl} = \begin{pmatrix} \boldsymbol{A}_i\boldsymbol{P}_l + \boldsymbol{P}_l\boldsymbol{A}_i^T + \boldsymbol{B}_i\boldsymbol{F}_{jkl} \\ +\boldsymbol{F}_{jkl}^T\boldsymbol{B}_i^T + \alpha\boldsymbol{P}_l - \dfrac{1}{\kappa}\left(\boldsymbol{P}_i - \boldsymbol{P}_k\right) & \delta\boldsymbol{E}_i \\ \delta\boldsymbol{E}_i^T & -\beta\boldsymbol{I} \end{pmatrix},$$

$$\Psi_{ij} = \begin{pmatrix} \alpha\boldsymbol{P}_j & \boldsymbol{0} & \boldsymbol{P}_j\boldsymbol{C}_i^T \\ \boldsymbol{0} & (\mu-\beta)\boldsymbol{I} & \delta\boldsymbol{D}_i^T \\ \boldsymbol{C}_i\boldsymbol{P}_j & \delta\boldsymbol{D}_i & \mu\boldsymbol{I} \end{pmatrix},$$

$$\Upsilon_{iikl} \leq 0, \quad i, k, l = 1, \ldots, r, \tag{43}$$

$$\frac{2}{r-1}\Upsilon_{iikl}+\Upsilon_{ijkl}+\Upsilon_{jikl}\le 0,\quad i\ne j,k,l=1,\dots,r. \tag{44}$$

$$\Psi_{ij}\ \ge 0,\quad i,j=1,\dots,r \tag{45}$$

Proof. The left member of condition (33) is a fuzzy summation with four indexes: $hhh^{-}v$. Substituting (35) and (36) in it and using Lemma 3, conditions (43), (44) are obtained.

Following a similar procedure, conditions (45) are recovered from condition (34). Note that, if $\boldsymbol{P}_i > 0$, $i=1,\dots,r$, then $\boldsymbol{P}_v > 0$. ■

Remark 11. For Theorem 2 we have

$$N_{var}^{\text{Th2}} = 2+\frac{r}{2}n_x(n_x+1)+r^3\,n_u\,n_x,$$
$$N_{row}^{\text{Th2}} = 1+r\,n_x+r^4(n_x+n_\phi)+r^2(n_x+n_y+n_\phi).$$

LMI conditions of Theorem 2 can be improved using Lemma 4.

Theorem 3. ($\star$-Norm computation with LMIs based on Lemma 4 for IDLF) *The $\star$-norm between the output* **y** *and the input* $\boldsymbol{\phi}$ *for the closed-loop system* (27) *can be obtained by solving the following LMI problem:*

$$||\boldsymbol{G}_{\phi\to y}^{CL}||_\star = \inf_{\alpha>0} N(\alpha), \tag{46}$$

given $\kappa > 0$ $N(\alpha)$ is calculated of each fixed $\alpha > 0$, as follows:

$$N(\alpha)\triangleq\frac{1}{\delta}\min\{\mu\ge 0:\ \boldsymbol{P}_i>0,\ 0\le\beta\le\alpha,\ \textit{subject to LMIs (47)–(49)}\},$$

$$\Upsilon_{ijkl}=\begin{pmatrix} \boldsymbol{A}_i\boldsymbol{L}_{1jkl}^T+\boldsymbol{L}_{1jkl}\boldsymbol{A}_i^T+\boldsymbol{B}_i\boldsymbol{F}_{jkl} +\boldsymbol{F}_{jkl}^T\boldsymbol{B}_i^T+\alpha\boldsymbol{P}_l-\frac{1}{\kappa}(\boldsymbol{P}_i-\boldsymbol{P}_k) & (*) & (*)\\ \delta\boldsymbol{E}_i^T+\boldsymbol{L}_{2jkl}\boldsymbol{A}_i^T & -\beta\boldsymbol{I} & (*)\\ \boldsymbol{P}_l-\boldsymbol{L}_{1jkl}+\boldsymbol{G}_{jkl}^T\boldsymbol{A}_i^T & -\boldsymbol{L}_{2jkl}^T & -\boldsymbol{G}_{jkl}-\boldsymbol{G}_{jkl}^T\end{pmatrix},$$

$$\Psi_{ij}=\begin{pmatrix}\alpha\boldsymbol{P}_j & \boldsymbol{0} & \boldsymbol{P}_j\boldsymbol{C}_i^T\\ \boldsymbol{0} & (\mu-\beta)\boldsymbol{I} & \delta\boldsymbol{D}_i^T\\ \boldsymbol{C}_i\boldsymbol{P}_j & \delta\boldsymbol{D}_i & \mu\boldsymbol{I}\end{pmatrix},$$

$$\Upsilon_{iikl}\le 0,\quad i,k,l=1,\dots,r \tag{47}$$

$$\frac{2}{r-1}\Upsilon_{iikl}+\Upsilon_{ijkl}+\Upsilon_{jikl}\le 0,\quad i\ne j,k,l=1,\dots,r, \tag{48}$$

$$\Psi_{ij}\ \ge 0,\quad i,j=1,\dots,r, \tag{49}$$

where $\boldsymbol{L}_{1jkl}$, $\boldsymbol{L}_{2jkl}$ and $\boldsymbol{G}_{jkl}$ $j,k,l=1,\dots,r$ matrices of appropriate dimensions.

Proof. In condition (33) take $\boldsymbol{A}$ as $\boldsymbol{A}_h^T$, $\boldsymbol{P}$ as $\boldsymbol{P}_v$, $\boldsymbol{T}_1$ as $\boldsymbol{B}_h\boldsymbol{F}_{hh^-v}+\boldsymbol{F}_{hh^-v}^T\boldsymbol{B}_h^T+\alpha\boldsymbol{P}_v-1/\kappa\,(\boldsymbol{P}_h-\boldsymbol{P}_{h^-})$, $\boldsymbol{T}_2$ as $\boldsymbol{E}_h^T$ and $\boldsymbol{T}_3$ as $-\beta\boldsymbol{I}$. Applying Lemma 4 to condition (33) is guaranteed by

$$\begin{pmatrix} \boldsymbol{A}_h\boldsymbol{L}_{1hh^-v}^T+\boldsymbol{L}_{1hh^-v}\boldsymbol{A}_h^T+\boldsymbol{B}_h\boldsymbol{F}_{hh^-v} +\boldsymbol{F}_{hh^-v}^T\boldsymbol{B}_h^T+\alpha\boldsymbol{P}_v-\frac{1}{\kappa}(\boldsymbol{P}_h-\boldsymbol{P}_{h^-}) & (*) & (*)\\ \delta\boldsymbol{E}_h^T+\boldsymbol{L}_{2hh^-v}\boldsymbol{A}_h^T & -\beta\boldsymbol{I} & (*)\\ \boldsymbol{P}_h-\boldsymbol{L}_{1hh^-v}+\boldsymbol{G}_{hh^-v}^T\boldsymbol{A}_h^T & -\boldsymbol{L}_{2hh^-v}^T & -\boldsymbol{G}_{hh^-v}-(*)^T\end{pmatrix}\le 0.$$

This expression can be recast as LMIs (47) and (48) by applying Lemma 3. LMIs (49) are obtained following the proof of Theorem 2. ■

Remark 12. Fot Theorem 3 we have

$$N_{var}^{\text{Th3}} = 2+\frac{r}{2}n_x(n_x+1) + r^3\left(n_u\,n_x+2\,n_x^2+n_x\,n_\phi\right),$$
$$N_{row}^{\text{Th3}} = 1+r\,n_x+r^4(2\,n_x+n_\phi) + r^2(n_x+n_y+n_\phi).$$

Consequently, Theorem 2 has a lower number of variables and a lower number of rows than Theorem 3, but it is more conservative.

Remark 13. Theorem 3 is more relaxed than Theorem 2. This conclusion is derived following the same lines of Remark 5.

It is possible to reduce the number of variables replacing control gains $\boldsymbol{F}_{jkl}$ by $\boldsymbol{F}_j$. This result is expressed as follows.

Corollary 2. ($\star$-Norm computation with LMIs based on Lemma 4 for the IDLF using reduced gains) *The $\star$-norm between the output* **y** *and the input* $\boldsymbol{\phi}$ *for the closed-loop system* (27) *can be obtained by solving the following LMI problem:*

$$||\boldsymbol{G}_{\phi\to y}^{CL}||_\star = \inf_{\alpha>0} N(\alpha). \tag{50}$$

Given $\kappa > 0$, $N(\alpha)$ is calculated of each fixed $\alpha > 0$, as follows:

$$N(\alpha)\triangleq\frac{1}{\delta}\min\{\mu\ge 0:\ \boldsymbol{P}_i>0,\ 0\le\beta\le\alpha,\ \textit{subject to LMIs (47)–(49)}\},$$

$$\Upsilon_{ijkl}=\begin{pmatrix} \boldsymbol{A}_i\boldsymbol{L}_{1jkl}^T+\boldsymbol{L}_{1jkl}\boldsymbol{A}_i^T+\boldsymbol{B}_i\boldsymbol{F}_j +\boldsymbol{F}_j^T\boldsymbol{B}_i^T+\alpha\boldsymbol{P}_l-\frac{1}{\kappa}(\boldsymbol{P}_i-\boldsymbol{P}_k)\\ \delta\boldsymbol{E}_i^T+\boldsymbol{L}_{2jkl}\boldsymbol{A}_i^T\\ \boldsymbol{P}_l-\boldsymbol{L}_{1jkl}+\boldsymbol{G}_{jkl}^T\boldsymbol{A}_i^T \end{pmatrix}$$

$$\left.\begin{matrix} (*) & (*) \\ -\beta \boldsymbol{I} & (*) \\ -\boldsymbol{L}_{2jkl}^T & -\boldsymbol{G}_{jkl} - \boldsymbol{G}_{jkl}^T \end{matrix}\right),$$

Ψ_{ij} being the same as in Theorem 3.

Proof. Repeat the proof of Theorem 3 replacing $\boldsymbol{F}_{hh^-v}$ gains by $\boldsymbol{F}_h$ gains. ■

Remark 14. For Corollary 2 we have

$$\begin{aligned} N_{var}^{\text{Cor2}} &= 2 + \frac{r}{2} n_x(n_x+1) + r\, n_u\, n_x \\ &\quad + r^3\left(2\, n_x^2 + n_x\, n_\phi\right), \\ N_{row}^{\text{Cor2}} &= 1 + r\, n_x + r^4(2\, n_x + n_\phi) \\ &\quad + r^2(n_x + n_y + n_\phi). \end{aligned}$$

Consequently, Corollary 2 has a lower number of variables than Theorem 3 but the same number of rows. This could be helpful when solving problems with a high number of rules (r), since it can reduce the complexity of LMI problem to be solved.

3.1.2. Theorems for DIDLFs.

Now, it is possible to extend previous results to DIDLFs applying Lemma 1 to the DIDLF (28) with non-PDC control law (30):

Theorem 4. ($\star$-Norm computation with DIDLF) *The $\star$-norm between the output **y** and the input $\boldsymbol{\phi}$ for the closed-loop system* (31) *is obtained by solving the problem*

$$||\boldsymbol{G}_{\phi\to y}^{CL}||_\star = \inf_{\alpha>0} N(\alpha) \tag{51}$$

where $N(\alpha)$ is calculated of each fixed $\alpha > 0$, as follows:

$$N(\alpha) \triangleq \frac{1}{\delta} \min \{\mu \ge 0 :\ \boldsymbol{P}_{v\lambda} > 0,\ 0 \le \beta \le \alpha,$$
$$\textit{subject to } (52)\text{–}(53)\},$$

$$\begin{pmatrix} \boldsymbol{P}_{v\lambda}\boldsymbol{A}^{CL^T} + \boldsymbol{A}^{CL}\boldsymbol{P}_{v\lambda} + \alpha\boldsymbol{P}_{v\lambda} - \dot{\boldsymbol{P}}_{v\lambda} & \delta\boldsymbol{B}_h^{CL} \\ \delta\boldsymbol{B}_h^{CL^T} & -\beta\boldsymbol{I} \end{pmatrix} \le 0, \tag{52}$$

$$\begin{pmatrix} \alpha\boldsymbol{P}_{v\lambda} & \boldsymbol{0} & \boldsymbol{P}_{v\lambda}\boldsymbol{C}_h^{CL^T} \\ \boldsymbol{0} & (\mu-\beta)\boldsymbol{I} & \delta\boldsymbol{D}_h^{CL^T} \\ \boldsymbol{C}_h^{CL}\boldsymbol{P}_{v\lambda} & \delta\boldsymbol{D}_h^{CL} & \mu\boldsymbol{I} \end{pmatrix} \ge 0, \tag{53}$$

where

$$\boldsymbol{A}^{CL} = \left(\boldsymbol{A}_h + \boldsymbol{B}_h\boldsymbol{F}_{hh^-vv\lambda}\boldsymbol{P}_{v\lambda}^{-1}\right), \quad \boldsymbol{B}_h^{CL} = \boldsymbol{E}_h,$$
$$\boldsymbol{C}_h^{CL} = \boldsymbol{C}_h, \quad \boldsymbol{D}_h^{CL} = \boldsymbol{D}_h, \tag{54}$$

$$\dot{\boldsymbol{P}}_{v\lambda} = \frac{1}{\kappa}\left(\boldsymbol{P}_{h\lambda} - \boldsymbol{P}_{h^-\lambda} + 2\boldsymbol{P}_{vh} - 2\boldsymbol{P}_{vv}\right). \tag{55}$$

Proof. Proceed in the same way as in the proof of Theorem 1 with the DIDLF (28) and the non-PDC control law (30). ■

Remark 15. $V(\boldsymbol{x}) = \boldsymbol{x}^T\boldsymbol{P}_{v\lambda}^{-1}\boldsymbol{x}$ is a non-quadratic Lyapunov function for the closed loop. Moreover, the positive definite fuzzy matrix $\boldsymbol{P}_{v\lambda}^{-1}$ defines an inescapable set (33) (Abedor *et al.*, 1996; Salcedo *et al.*, 2018):

$$\mathcal{E}(\boldsymbol{P}_v^{-1}) \triangleq \left\{\boldsymbol{x} : \boldsymbol{x}^T\boldsymbol{P}_{v\lambda}^{-1}\boldsymbol{x} \le 1\right\}, \tag{56}$$

which is a robust control positively invariant set for the closed loop.

Remark 16. Note that (52) and (53) are not LMI conditions. The next two theorems provide a way to recast them as LMIs.

Theorem 5. ($\star$-Norm computation with LMIs for DIDLF) *The $\star$-norm between the output **y** and the input $\boldsymbol{\phi}$ for the closed-loop system* (31) *can be obtained by solving the following LMI problem:*

$$||\boldsymbol{G}_{\phi\to y}^{CL}||_\star = \inf_{\alpha>0} N(\alpha). \tag{57}$$

Given $\kappa > 0$, $N(\alpha)$ is calculated of each fixed $\alpha > 0$, as follows:

$$N(\alpha) \triangleq \frac{1}{\delta} \min \{\mu \ge 0 :\ \boldsymbol{P}_{ij} > 0,\ 0 \le \beta \le \alpha,$$
$$\textit{subject to LMIs } (58)\text{–}(62)\},$$

$$\Upsilon_{ijklmn} = \begin{pmatrix} \begin{matrix} \boldsymbol{A}_i\boldsymbol{P}_{ln} + \boldsymbol{P}_{ln}\boldsymbol{A}_i^T + \boldsymbol{B}_i\boldsymbol{F}_{jklmn} \\ +\boldsymbol{F}_{jklmn}^T\boldsymbol{B}_i^T + \alpha\boldsymbol{P}_{ln} \\ -\dfrac{1}{\kappa}\left(\boldsymbol{P}_{in} - \boldsymbol{P}_{kn} + 2\boldsymbol{P}_{li} - 2\boldsymbol{P}_{lm}\right) \end{matrix} & \delta\boldsymbol{E}_i \\ \delta\boldsymbol{E}_i^T & -\beta\boldsymbol{I} \end{pmatrix},$$

$$\Psi_{ijk} = \begin{pmatrix} \alpha\boldsymbol{P}_{jk} & \boldsymbol{0} & \boldsymbol{P}_{jk}\boldsymbol{C}_i^T \\ \boldsymbol{0} & (\mu-\beta)\boldsymbol{I} & \delta\boldsymbol{D}_i^T \\ \boldsymbol{C}_i\boldsymbol{P}_{jk} & \delta\boldsymbol{D}_i & \mu\boldsymbol{I} \end{pmatrix},$$

$$\Upsilon_{iiklln} \le 0, \quad i,k,l,n = 1,\dots,r, \tag{58}$$

$$\frac{2}{r-1}\Upsilon_{iiklln} + \Upsilon_{iiklmn} + \Upsilon_{iikmln} \le 0,$$
$$i,k,l \ne m,n = 1,\dots,r, \tag{59}$$

$$\frac{2}{r-1}\Upsilon_{iiklln} + \Upsilon_{ijklln} + \Upsilon_{jiklln} \le 0,$$
$$i \ne j,k,l,n = 1,\dots,r, \tag{60}$$

$$\begin{aligned} &\left(\frac{2}{r-1}\right)^2 \Upsilon_{iiklln} + \frac{2}{r-1}\Upsilon_{ijklln} + \frac{2}{r-1}\Upsilon_{jiklln} \\ &\quad + \frac{2}{r-1}\Upsilon_{iiklmn} + \Upsilon_{ijklmn} + \Upsilon_{jiklmn} \\ &\quad + \frac{2}{r-1}\Upsilon_{iikmln} + \Upsilon_{ijkmln} + \Upsilon_{jikmln} \le 0, \end{aligned}$$

$$i \ne j,k,l \ne m,n = 1,\dots,r, \tag{61}$$

$$\Psi_{ijk} \ge 0 \quad i,j,k = 1,\dots,r. \tag{62}$$

Proof. Follow the proof of Theorem 2 applied to conditions (52) and (53), using closed-loop dynamics (54) and Eqn. (55). Lemma 3 has to be applied several times. ■

Remark 17. For Theorem 5 we have

$$N_{var}^{\text{Th5}} = 2 + \frac{r^2}{2} n_x(n_x+1) + r^5\, n_u\, n_x,$$
$$N_{row}^{\text{Th5}} = 1 + r^2\, n_x + r^6(n_x+n_\phi) + r^3(n_x+n_y+n_\phi).$$

The LMI conditions of Theorem 5 can be improved using Lemma 4.

Theorem 6. ($\star$-Norm computation with LMIs based on Lemma 4 for the DIDLF) *The $\star$-norm between the output $\boldsymbol{y}$ and the input $\boldsymbol{\phi}$ for the closed-loop system* (31) *can be obtained solving the following LMI problem:*

$$||\boldsymbol{G}_{\phi\to y}^{CL}||_\star = \inf_{\alpha>0} N(\alpha). \tag{63}$$

Given $\kappa > 0$, $N(\alpha)$ is calculated of each fixed $\alpha > 0$, as follows:

$$N(\alpha) \triangleq \frac{1}{\delta}\min\{\mu \ge 0:\ \boldsymbol{P}_i > 0,\ 0\le\beta\le\alpha,\ \textit{subject to LMIs (64)–(68)}\},$$

$$\Upsilon_{ijklmn} = \begin{pmatrix} \boldsymbol{A}_i\boldsymbol{L}_{1jklmn}^T + \boldsymbol{L}_{1jklmn}\boldsymbol{A}_i^T + \boldsymbol{B}_i\boldsymbol{F}_{jklmn} + \boldsymbol{F}_{jklmn}^T\boldsymbol{B}_i^T + \alpha\boldsymbol{P}_{ln} - \frac{1}{\kappa}(\boldsymbol{P}_{in} - \boldsymbol{P}_{kn} + 2\boldsymbol{P}_{li} - 2\boldsymbol{P}_{lm}) & (*) & (*) \\ \delta\boldsymbol{E}_i^T + \boldsymbol{L}_{2jklmn}\boldsymbol{A}_i^T & -\beta\boldsymbol{I} & (*) \\ \boldsymbol{P}_{ln} - \boldsymbol{L}_{1jklmn} + \boldsymbol{G}_{jklmn}^T\boldsymbol{A}_i^T & -\boldsymbol{L}_{2jklmn}^T & -\boldsymbol{G}_{jklmn} - (*)^T \end{pmatrix},$$

$$\Psi_{ijk} = \begin{pmatrix} \alpha\boldsymbol{P}_{jk} & \boldsymbol{0} & \boldsymbol{P}_{jk}\boldsymbol{C}_i^T \\ \boldsymbol{0} & (\mu-\beta)\boldsymbol{I} & \delta\boldsymbol{D}_i^T \\ \boldsymbol{C}_i\boldsymbol{P}_{jk} & \delta\boldsymbol{D}_i & \mu\boldsymbol{I} \end{pmatrix},$$

$$\Upsilon_{iiklln} \le 0, \quad i,k,l,n = 1,\dots,r, \tag{64}$$

$$\frac{2}{r-1}\Upsilon_{iiklln} + \Upsilon_{iiklmn} + \Upsilon_{iikmln} \le 0, \quad i,k,l\ne m,n = 1,\dots,r, \tag{65}$$

$$\frac{2}{r-1}\Upsilon_{iiklln} + \Upsilon_{ijklln} + \Upsilon_{jiklln} \le 0, \quad i\ne j,k,l,n = 1,\dots,r, \tag{66}$$

$$\left(\frac{2}{r-1}\right)^2\Upsilon_{iiklln} + \frac{2}{r-1}\Upsilon_{ijklln} + \frac{2}{r-1}\Upsilon_{jiklln} + \frac{2}{r-1}\Upsilon_{iiklmn} + \Upsilon_{ijklmn} + \Upsilon_{jiklmn} + \frac{2}{r-1}\Upsilon_{iikmln} + \Upsilon_{ijkmln} + \Upsilon_{jikmln} \le 0,$$
$$i\ne j,k,l\ne m,n = 1,\dots,r, \tag{67}$$

$$\Psi_{ijk} \ge 0, \quad i,j,k = 1,\dots,r, \tag{68}$$

with $\boldsymbol{L}_{1jklmn}$, $\boldsymbol{L}_{2jklmn}$ and $\boldsymbol{G}_{jklmn}$ $j,k,l,m,n = 1,\dots,r$, as matrices of appropriate dimensions.

Proof. Proceed in the same way as in the proof of Theorem 3, this time applying it to condition (52). ■

Remark 18. For Theorem 6 we have

$$N_{var}^{\text{Th6}} = 2 + \frac{r^2}{2} n_x(n_x+1) + r^5\left(n_u\, n_x + 2\, n_x^2 + n_x\, n_\phi\right),$$
$$N_{row}^{\text{Th6}} = 1 + r^2\, n_x + r^6(2\, n_x + n_\phi) + r^3(n_x+n_y+n_\phi).$$

Consequently, Theorem 5 has a lower number of variables and a lower number of rows than Theorem 6, but it is more conservative.

Remark 19. Theorem 6 is more relaxed than Theorem 5. This conclusion is derived following the lines of Remark 5.

It is possible to reduce the number of variables replacing control gains $\boldsymbol{F}_{jklmn}$ by $\boldsymbol{F}_j$. This result is expressed as follows.

Corollary 3. ($\star$-Norm computation with LMIs based on Lemma 4 for the DIDLF using reduced gains) *The $\star$-norm between the output $\boldsymbol{y}$ and the input $\boldsymbol{\phi}$ for the closed-loop system* (31) *can be obtained solving the following LMI problem:*

$$||\boldsymbol{G}_{\phi\to y}^{CL}||_\star = \inf_{\alpha>0} N(\alpha). \tag{69}$$

Given $\kappa > 0$, $N(\alpha)$ is calculated of each fixed $\alpha > 0$, as follows:

$$N(\alpha) \triangleq \frac{1}{\delta}\min\{\mu \ge 0:\ \boldsymbol{P}_i > 0,\ 0\le\beta\le\alpha,\ \textit{subject to LMIs (64)–(68)}\},$$

$$\Upsilon_{ijklmn} = \begin{pmatrix} \boldsymbol{A}_i\boldsymbol{L}_{1jklmn}^T + \boldsymbol{L}_{1jklmn}\boldsymbol{A}_i^T + \boldsymbol{B}_i\boldsymbol{F}_j + \boldsymbol{F}_j^T\boldsymbol{B}_i^T + \alpha\boldsymbol{P}_{ln} - \frac{1}{\kappa}(\boldsymbol{P}_{in} - \boldsymbol{P}_{kn} + 2\boldsymbol{P}_{li} - 2\boldsymbol{P}_{lm}) \\ \delta\boldsymbol{E}_i^T + \boldsymbol{L}_{2jklmn}\boldsymbol{A}_i^T \\ \boldsymbol{P}_{ln} - \boldsymbol{L}_{1jklmn} + \boldsymbol{G}_{jklmn}^T\boldsymbol{A}_i^T \end{pmatrix}$$

$$\begin{pmatrix} \dots & (*) & (*) \\ \dots & -\beta \boldsymbol{I} & (*) \\ \dots & -\boldsymbol{L}_{2jklmn}^T & -\boldsymbol{G}_{jklmn} - (*)^T \end{pmatrix},$$

Ψ_{ijk} being the same as in Theorem 6.

Proof. Repeat the proof of Theorem 6 replacing $\boldsymbol{F}_{hh-vv\lambda}$ gains by $\boldsymbol{F}_h$ gains. ■

Remark 20. For Corollary 3 we have

$$\begin{aligned} N_{var}^{\text{Cor3}} &= 2 + \frac{r^2}{2} n_x(n_x+1) + r\, n_u\, n_x \\ &\quad + r^5 \left(2\, n_x^2 + n_x\, n_\phi\right), \\ N_{row}^{\text{Cor3}} &= 1 + r^2\, n_x + r^6 (2\, n_x + n_\phi) \\ &\quad + r^3 (n_x + n_y + n_\phi). \end{aligned}$$

Consequently, Corollary 3 has a lower number of variables than Theorem 6 but the same number of rows. This could be helpful when solving problems with a high number of rules (r), since it can reduce the complexity of LMI problem to be solved.

Remark 21. Because of Remarks 7 (IDLFs are a subset of DIDLFs) and 8 (the non-PDC control law used with IDLFs is a particular case of the non-PDC control law used with DIDLFs) it is possible to conclude that:

- Theorem 2 is a special case of Theorem 5,
- Theorem 3 is a special case of Theorem 6,
- Corollary 2 is a special case of Corollary 3.

Remark 22. Using Remarks 11, 12, 14, 17, 18 and 20 it is possible to establish the following comparisons between the numbers of variables and rows that each result requires:

$$\begin{gathered} N_{var}^{\text{Th2}} < N_{var}^{\text{Th5}}, \\ N_{var}^{\text{Th3}} < N_{var}^{\text{Th6}}, \\ N_{var}^{\text{Cor2}} < N_{var}^{\text{Cor3}}, \\ N_{var}^{\text{Th2}} \underset{\text{If } n_u \le 2n_x + n_\phi}{<} N_{var}^{\text{Cor2}} < N_{var}^{\text{Th3}}, \\ N_{var}^{\text{Th5}} \underset{\text{If } n_u \le 2n_x + n_\phi}{<} N_{var}^{\text{Cor3}} < N_{var}^{\text{Th6}}, \\ N_{row}^{\text{Th2}} < N_{row}^{\text{Cor2}} = N_{row}^{\text{Th3}} < N_{row}^{\text{Th5}} < N_{row}^{\text{Cor3}} = N_{row}^{\text{Th6}}. \end{gathered}$$

3.2. Results based on fuzzy Lyapunov functions with guaranteed bounds for first derivatives of membership functions.

A quadratic fuzzy Lyapunov function matching the membership functions of the fuzzy model (Tanaka *et al.*, 2001) is an interesting way of generalising a quadratic Lyapunov function:

$$V(\boldsymbol{x}) = \boldsymbol{x}^T \boldsymbol{P}_h \boldsymbol{x}. \tag{70}$$

Nevertheless, to guarantee stability, first derivatives of membership functions must be bounded (Tanaka *et al.*, 2001; Lee *et al.*, 2012):

$$|\dot{h}_i| \le \varphi_i, \quad i = 1, \dots, r. \tag{71}$$

The main problem with this approach is that bounds φ_i have to be known in advance, and these bounds cannot estimated because they usually depend on the control law. In order to cope with this problem, recent research has been conducted to guarantee such upper bound using LMIs (Mozelli *et al.*, 2009; Mozelli, 2011; Lee *et al.*, 2012; 2014; Guedes *et al.*, 2013; da Silva Campos *et al.*, 2017; Vafamand and Shasadeghi, 2017; Márquez *et al.*, 2017; Hu *et al.*, 2018; 2019). Da Silva Campos *et al.* (2017) compare different approaches which were used in the literature (Mozelli *et al.*, 2009; Mozelli, 2011; Guedes *et al.*, 2013; Márquez *et al.*, 2017) in order to guarantee upper bounds for first derivatives of membership functions. In this article, results of da Silva Campos *et al.* (2017) are going to be applied for BIBO stabilisation using the $\star$-norm following the lines of Section 3.1. The best two approaches according to da Silva Campos *et al.* (2017) are the following:

- For the general type of the T–S fuzzy model (2), (3) the first approach is based on the assumption that the vector of the first derivatives of membership functions $\dot{\boldsymbol{h}}(\boldsymbol{z})$ belongs to a polytope defined by bounds (71) and the convex sum property (5) $\sum_{i=1}^{r} \dot{h}_i = 0$:

  $$\dot{\boldsymbol{h}}(\boldsymbol{z}) \in \operatorname{co}(\boldsymbol{v}_1, \dots, \boldsymbol{v}_m), \tag{72}$$

  where $\boldsymbol{v}_i$ are the vertices of the polytope defined by the intersection of the hyper-rectangle related to bounds (71) and the hyper-plane associated with $\sum_{i=1}^{r} \dot{h}_i = 0$.

- If the T–S fuzzy model has been obtained using the non-linearity sector approach, instead of using bounds in $\dot{h}_i$, bounds on the first derivatives of normalised weighting functions can be used:

  $$|\dot{w}_{i_k}^k| \le \theta_k, \quad k = 1, \dots, p, \quad i_k \in \{0, 1\}. \tag{73}$$

  The second approach assumes that the vector of the first derivatives of normalised weighting funcitions belongs to the hyper-rectangle defined by bounds (73):

  $$\dot{\boldsymbol{w}}(\boldsymbol{z}) \in \operatorname{co}(\boldsymbol{q}_1, \dots, \boldsymbol{q}_{2^p}), \tag{74}$$

  where $\boldsymbol{q}_j$ are the vertices of such hyper-rectangle.

In both the approaches the LMI conditions of Lee *et al.* (2014) and da Silva Campos *et al.* (2017) have been extended to the $\star$-norm providing Theorems 7 and 10.

The fuzzy Lyapunov function (70) is not suitable for designing fuzzy controllers. Instead, the following non-quadratic fuzzy Lyapunov function (n-Q FLF) is going to be used (Lee *et al.*, 2012; 2014; Wang *et al.*, 2015; da Silva Campos *et al.*, 2017; Vafamand *et al.*, 2017):

$$V(\boldsymbol{x}) = \boldsymbol{x}^T \boldsymbol{P}_h^{-1} \boldsymbol{x}, \tag{75}$$

together with the non-PDC control law

$$\boldsymbol{u}(t) = \boldsymbol{F}_h \boldsymbol{P}_h^{-1} \boldsymbol{x}(t). \tag{76}$$

With this non-PDC controller the dynamics of the closed-loop system are

$$\dot{\boldsymbol{x}} = \overbrace{\left(\boldsymbol{A}_h + \boldsymbol{B}_h \boldsymbol{F}_h \boldsymbol{P}_h^{-1}\right)}^{\boldsymbol{A}^{CL}} \boldsymbol{x} + \boldsymbol{E}_h \boldsymbol{\phi}. \tag{77}$$

3.2.1. Theorems for n-Q FLFs when $\dot{\boldsymbol{h}}(\boldsymbol{z})$ belongs to a polytope. Applying Lemma 1 for the general type of the T–S fuzzy model (2) and (3), to n-Q FLFs (75) with non-PDC control law (76) under conditions (71) and (72), the following result is obtained.

Theorem 7. ($\star$-Norm computation using n-Q FLF with bounded derivatives of h_i) *The $\star$-norm between the output $\boldsymbol{y}$ and the input $\boldsymbol{\phi}$ for the closed-loop system* (77) *is obtained by solving the problem*

$$||\boldsymbol{G}^{CL}_{\phi\to y}||_\star = \inf_{\alpha>0} N(\alpha), \tag{78}$$

where $N(\alpha)$ is calculated of each fixed $\alpha > 0$, as follows:

$$N(\alpha) \triangleq \frac{1}{\delta} \min \{\mu \geq 0 : \boldsymbol{P}_h > 0,\ 0 \leq \beta \leq \alpha,$$
$$\textit{subject to } (79)\text{–}(81)\},$$

$$\begin{pmatrix} \boldsymbol{P}_h {\boldsymbol{A}^{CL}}^T + \boldsymbol{A}^{CL}\boldsymbol{P}_h + \alpha \boldsymbol{P}_h - \sum_{i=1}^{r} \dot{h}_i \boldsymbol{P}_i & \delta \boldsymbol{B}_h^{CL} \\ \delta {\boldsymbol{B}_h^{CL}}^T & -\beta \boldsymbol{I} \end{pmatrix} \leq 0, \tag{79}$$

$$\begin{pmatrix} \alpha \boldsymbol{P}_h & \boldsymbol{0} & \boldsymbol{P}_h {\boldsymbol{C}_h^{CL}}^T \\ \boldsymbol{0} & (\mu-\beta)\boldsymbol{I} & \delta {\boldsymbol{D}_h^{CL}}^T \\ \boldsymbol{C}_h^{CL}\boldsymbol{P}_h & \delta \boldsymbol{D}_h^{CL} & \mu \boldsymbol{I} \end{pmatrix} \geq 0, \tag{80}$$

$$\dot{\boldsymbol{h}}(\boldsymbol{z}) \in \operatorname{co}(\boldsymbol{v}_1, \dots, \boldsymbol{v}_m), \quad |\dot{h}_i| \leq \varphi_i, \quad i = 1, \dots, r, \tag{81}$$

where

$$\boldsymbol{A}^{CL} = \left(\boldsymbol{A}_h + \boldsymbol{B}_h \boldsymbol{F}_h \boldsymbol{P}_h^{-1}\right), \quad \boldsymbol{B}_h^{CL} = \boldsymbol{E}_h,$$
$$\boldsymbol{C}_h^{CL} = \boldsymbol{C}_h, \quad \boldsymbol{D}_h^{CL} = \boldsymbol{D}_h.$$

Proof. Apply the procedure of the proof of Theorem 1 with n-Q FLF (75) and non-PDC control law (76), taking into account conditions (71) and (72). ∎

Remark 23. $V(\boldsymbol{x}) = \boldsymbol{x}^T \boldsymbol{P}_h^{-1} \boldsymbol{x}$ is a non-quadratic Lyapunov function for the closed-loop. Moreover, the positive definite fuzzy matrix $\boldsymbol{P}_h^{-1}$ defines an inescapable set (79) (Abedor *et al.*, 1996; Salcedo *et al.*, 2018):

$$\mathcal{E}(\boldsymbol{P}_h^{-1}) \triangleq \left\{\boldsymbol{x} : \boldsymbol{x}^T \boldsymbol{P}_h^{-1} \boldsymbol{x} \leq 1\right\}, \tag{82}$$

which is a robust control positively invariant set for the closed loop.

Remark 24. Note that (79)–(81) are not LMI conditions. The next two theorems provide a way to recast them as LMIs.

Theorem 8. ($\star$-Norm computation using LMIs for n-Q FLF with bounded derivatives of h_i) *The $\star$-norm between the output $\boldsymbol{y}$ and the input $\boldsymbol{\phi}$ for the closed-loop system* (77) *under conditions* (71) (72) *can be obtained solving the following LMI problem:*

$$||\boldsymbol{G}^{CL}_{\phi\to y}||_\star = \inf_{\alpha>0} N(\alpha), \tag{83}$$

where $N(\alpha)$ is calculated of each fixed $\alpha > 0$, as follows:

$$N(\alpha) \triangleq \frac{1}{\delta} \min \{\mu \geq 0 : \boldsymbol{P}_i > 0,\ 0 \leq \beta \leq \alpha,$$
$$\textit{subject to LMIs } (84)\text{–}(89)\},$$

$$\Upsilon_{ijk} = \begin{pmatrix} \begin{matrix} \boldsymbol{A}_i\boldsymbol{P}_j + \boldsymbol{P}_j\boldsymbol{A}_i^T + \boldsymbol{B}_i\boldsymbol{F}_j \\ +\boldsymbol{F}_j^T\boldsymbol{B}_i^T + \alpha\boldsymbol{P}_i - \sum_{l=1}^{r} \upsilon_{k,l}\boldsymbol{P}_l \end{matrix} & \delta\boldsymbol{E}_i \\ \delta\boldsymbol{E}_i^T & -\beta\boldsymbol{I} \end{pmatrix},$$

$$\Psi_{ij} = \begin{pmatrix} \alpha\boldsymbol{P}_j & \boldsymbol{0} & \boldsymbol{P}_j\boldsymbol{C}_i^T \\ \boldsymbol{0} & (\mu-\beta)\boldsymbol{I} & \delta\boldsymbol{D}_i^T \\ \boldsymbol{C}_i\boldsymbol{P}_j & \delta\boldsymbol{D}_i & \mu\boldsymbol{I} \end{pmatrix},$$

$$\Lambda_{ijln} = \begin{pmatrix} \frac{1}{1+\delta^2}\begin{pmatrix} \boldsymbol{P}_i & 0 \\ 0 & \boldsymbol{I} \end{pmatrix} & (*) \\ \boldsymbol{\xi}_{ln}^T \left[\boldsymbol{A}_i\boldsymbol{P}_j + \boldsymbol{B}_i\boldsymbol{F}_j \quad \boldsymbol{E}_i\right] & \varphi_l^2 \end{pmatrix},$$

$$\Upsilon_{iik} \leq 0, \quad i = 1, \dots, r, \quad k = 1, \dots, m, \tag{84}$$

$$\frac{2}{r-1}\Upsilon_{iik} + \Upsilon_{ijk} + \Upsilon_{jik} \leq 0,$$
$$i \neq j = 1, \dots, r, \quad k = 1, \dots, m, \tag{85}$$

$$\Psi_{ii} \geq 0, \quad i = 1, \dots, r, \tag{86}$$

$$\frac{2}{r-1}\Psi_{ii} + \Psi_{ij} + \Psi_{ji} \geq 0, \quad i \neq j, = 1, \dots, r, \tag{87}$$

$$\Lambda_{iiln} \geq 0, \quad i, l = 1, \dots, r, \quad n = 1, \dots, s, \tag{88}$$

$$\frac{2}{r-1}\Lambda_{iiln} + \Lambda_{ijln} + \Lambda_{jiln} \geq 0,$$
$$i \neq j, l = 1, \dots, r, \quad n = 1, \dots, s, \tag{89}$$

where $\upsilon_{k,l}$ is the l-th element of vertex $\boldsymbol{v}_k$, and vectors $\boldsymbol{\xi}_{ln}$ are a fuzzy approximation of partial derivatives $\partial h_l / \partial \boldsymbol{x}$ with membership functions υ_{ln}, $n = 1, \dots, s$,

$$\frac{\partial h_l}{\partial \boldsymbol{x}} = \sum_{n=1}^{s} \upsilon_{ln}\boldsymbol{\xi}_{ln}, \quad \sum_{n=1}^{s} \upsilon_{ln} = 1, \quad \upsilon_{ln} \geq 0. \tag{90}$$

The designed non-PDC controller (75) *provides BIBO stability inside $\mathcal{E}(\boldsymbol{P}_h^{-1})$.*

Proof. LMI conditions (84) and (85) are obtained from condition (79) applying the procedure of the proof Theorem 2 and taking into account condition (72). To do so, it is enough to check condition (79) at vertices of (72). LMI conditions (86) and (87) come from condition (80) using Lemma 3.

Finally, LMI conditions (88) and (89) guarantee that $|\dot{h}_i| \leq \varphi_i, i = 1,\dots,r$ inside $\mathcal{E}(\boldsymbol{P}_h^{-1})$. They are an extension of conditions obtained in Theorem 4 of (da Silva Campos *et al.*, 2017) when persistent perturbations are present. They are validated in Lemma A1 of Appendix. ■

Remark 25. Bounds on the first derivatives of membership functions (71) are guaranteed by LMI conditions (88) and (89).

Remark 26. For Theorem 8 we have

$$\begin{aligned} N_{var}^{\mathrm{Th8}} &= 2 + \frac{r}{2} n_x(n_x+1) + r\, n_u\, n_x, \\ N_{row}^{\mathrm{Th8}} &= 1 + r\, n_x + r^2 \cdot m(n_x + n_\phi) \\ &\quad + r^3 \cdot s(n_x + n_\phi + 1) \\ &\quad + r^2(n_x + n_y + n_\phi). \end{aligned}$$

The LMI conditions of Theorem 8 can be improved using Lemma 4.

Theorem 9. ($\star$-Norm computation using LMIs based on Lemma 4 for the n-Q FLF with bounded derivatives of h_i) *The $\star$-norm between the output* $\boldsymbol{y}$ *and the input* $\boldsymbol{\phi}$ *for the closed-loop system* (77) *under conditions* (71) (72) *can be obtained by solving the following LMI problem:*

$$||\boldsymbol{G}_{\phi\to y}^{CL}||_\star = \inf_{\alpha>0} N(\alpha), \tag{91}$$

where $N(\alpha)$ is calculated of each fixed $\alpha > 0$, as follows:

$$N(\alpha) \triangleq \frac{1}{\delta} \min \{\mu \geq 0 :\ \boldsymbol{P}_i > 0,\ 0 \leq \beta \leq \alpha, \text{ subject to LMIs (92)–(89)}\},$$

$$\Upsilon_{ijk} = \begin{pmatrix} \boldsymbol{A}_i\boldsymbol{L}_{1j}^T + \boldsymbol{L}_{1j}\boldsymbol{A}_i^T + \boldsymbol{B}_i\boldsymbol{F}_j + \\ +\boldsymbol{F}_j^T\boldsymbol{B}_i^T + \alpha\boldsymbol{P}_i - \sum_{l=1}^{r} v_{k,l}\boldsymbol{P}_l & (*) & (*) \\ \delta\boldsymbol{E}_i^T + \boldsymbol{L}_{2j}\boldsymbol{A}_i^T & -\beta\boldsymbol{I} & (*) \\ \boldsymbol{P}_i - \boldsymbol{L}_{1i} + \boldsymbol{G}_j^T\boldsymbol{A}_i^T & -\boldsymbol{L}_{2i}^T & -\boldsymbol{G}_i - \boldsymbol{G}_i^T \end{pmatrix},$$

$$\Upsilon_{iik} \leq 0, \quad i = 1,\dots,r, \quad k = 1,\dots,m, \tag{92}$$

$$\frac{2}{r-1}\Upsilon_{iik} + \Upsilon_{ijk} + \Upsilon_{jik} \leq 0, \quad i \neq j = 1,\dots,r, \quad k = 1,\dots,m, \tag{93}$$

where $v_{k,l}$ are the same as described in Theorem 8, with $\boldsymbol{L}_{1i}$, $\boldsymbol{L}_{2i}$ and $\boldsymbol{G}_i$ $i = 1,\dots,r$ as matrices of appropriate dimensions. The designed non-PDC controller (75) *provides BIBO stability inside $\mathcal{E}(\boldsymbol{P}_h^{-1})$.*

Proof. Follow the proof of Theorem 3 applied to condition (79) which has to be satisfied at vertices of (72). ■

Remark 27. Theorem 9 is more relaxed than Theorem 8. This conclusion is derived following the lines of Remark 5.

Remark 28. For Theorem 9 we have

$$\begin{aligned} N_{var}^{\mathrm{Th9}} &= 2 + \frac{r}{2} n_x(n_x+1) \\ &\quad + r(n_u\, n_x + 2n_x^2 + n_x\, n_\phi), \\ N_{row}^{\mathrm{Th9}} &= 1 + r\, n_x + r^2 m(2n_x + n_\phi) \\ &\quad + r^3 s(n_x + n_\phi + 1) \\ &\quad + r^2(n_x + n_y + n_\phi). \end{aligned}$$

Consequently, Theorem 8 has a lower number of variables and a lower number of rows than Theorem 9, but it is more conservative.

Remark 29. Using Remarks 11, 12, 17, 18, 26, and 28 the following relationships can be established:

$$\begin{aligned} N_{var}^{\mathrm{Th8}} &< N_{var}^{\mathrm{Th2}} < N_{var}^{\mathrm{Th5}}, \\ N_{var}^{\mathrm{Th9}} &< N_{var}^{\mathrm{Th3}} < N_{var}^{\mathrm{Th6}}. \end{aligned}$$

However, it is difficult to compare theoretically the expressions corresponding to number of rows obtained in Sections 3.1 and 3.2.1. Instead, in Section 4 a numerical comparison is performed through some examples.

3.2.2. Theorems for n-Q FLFs when $\dot{\boldsymbol{w}}(\boldsymbol{z})$ belongs to a hyper-rectangle.

If the T–S fuzzy model has been obtained using the non-linearity sector approach (5) and (6), instead of using bounds in $\dot{h}_i$, bounds on the first derivatives of normalised weighting functions, $\dot{w}_{i_k}^k$, are used. In such a case, applying Lemma 1 to n-Q FLF (75) with non-PDC control law (76) under conditions (73) and (74), we get the following result.

Theorem 10. ($\star$-Norm computation using n-Q FLF with bounded derivatives of $w_{i_k}^k$) *The $\star$-norm between the output* $\boldsymbol{y}$ *and the input* $\boldsymbol{\phi}$ *for the closed-loop system* (77) *is obtained solving the problem:*

$$||\boldsymbol{G}_{\phi\to y}^{CL}||_\star = \inf_{\alpha>0} N(\alpha), \tag{94}$$

where $N(\alpha)$ is calculated of each fixed $\alpha > 0$, as follows:

$$N(\alpha) \triangleq \frac{1}{\delta} \min \{\mu \geq 0 :\ \boldsymbol{P}_h > 0,\ 0 \leq \beta \leq \alpha, \text{ subject to (95)–(97)}\},$$

$$\begin{pmatrix} \boldsymbol{P}_h\boldsymbol{A}^{CL^T} + \boldsymbol{A}^{CL}\boldsymbol{P}_h + \alpha\boldsymbol{P}_h - \\ -\sum_{i=1}^{r} h_i \sum_{k=1}^{p} \dot{w}_{i_k}^k \left(\boldsymbol{P}_i - \boldsymbol{P}_{\bar{s}(i,k)}\right) & \delta\boldsymbol{B}_h^{CL} \\ \delta\boldsymbol{B}_h^{CL^T} & -\beta\boldsymbol{I} \end{pmatrix} \leq 0, \tag{95}$$

$$\begin{pmatrix} \alpha \boldsymbol{P}_h & \boldsymbol{0} & \boldsymbol{P}_h \boldsymbol{C}_h^{CL^T} \\ \boldsymbol{0} & (\mu-\beta)\boldsymbol{I} & \delta \boldsymbol{D}_h^{CL^T} \\ \boldsymbol{C}_h^{CL}\boldsymbol{P}_h & \delta \boldsymbol{D}_h^{CL} & \mu \boldsymbol{I} \end{pmatrix} \geq 0, \quad (96)$$

$$\dot{\boldsymbol{w}}(\boldsymbol{z}) \in \operatorname{co}(\boldsymbol{q}_1,\dots,\boldsymbol{q}_{2^p}), \quad |w_{i_k}^k| \leq \theta_k, \quad k=1,\dots,p, \quad (97)$$

where

$$\boldsymbol{A}^{CL} = \left(\boldsymbol{A}_h + \boldsymbol{B}_h \boldsymbol{F}_h \boldsymbol{P}_h^{-1}\right), \quad \boldsymbol{B}_h^{CL} = \boldsymbol{E}_h,$$
$$\boldsymbol{C}_h^{CL} = \boldsymbol{C}_h, \quad \boldsymbol{D}_h^{CL} = \boldsymbol{D}_h.$$

Here i_k are computed from i using

$$i = 1 + \sum_{k=1}^{p} i_k 2^{p-1}, \quad i_k \in \{0,1\}, \quad (98)$$

and $\bar{s}(i,k)$ is an integer such that

$$h_{\bar{s}(i,k)} = (1-w_{i_k}^k)\prod_{\substack{l=1\\ l\neq k}}^{p} w_{i_l}^l. \quad (99)$$

Proof. Follow the procedure of Theorem 7 taking into account condition (74) and the fact that (da Silva Campos *et al.*, 2017)

$$\dot{\boldsymbol{P}}_h = \sum_{i=1}^{r} h_i \sum_{k=1}^{p} \dot{w}_{i_k}^k \left(\boldsymbol{P}_i - \boldsymbol{P}_{\bar{s}(i,k)}\right),$$

with i_k and $\bar{s}(i,k)$ satisfying conditions (98) and (99). ■

Remark 30. $V(\boldsymbol{x}) = \boldsymbol{x}^T \boldsymbol{P}_h^{-1} \boldsymbol{x}$ is a non-quadratic Lyapunov function for the closed-loop system. Moreover, the positive definite fuzzy matrix $\boldsymbol{P}_h^{-1}$ defines an inescapable set (79) (Abedor *et al.*, 1996; Salcedo *et al.*, 2018):

$$\mathcal{E}(\boldsymbol{P}_h^{-1}) \triangleq \left\{\boldsymbol{x} : \boldsymbol{x}^T \boldsymbol{P}_h^{-1} \boldsymbol{x} \leq 1\right\} \quad (100)$$

which is a robust control positively invariant set for the closed loop.

Remark 31. Note that (95)–(97) are not LMI conditions. The next two theorems provide a way to recast them as LMIs.

Theorem 11. ($\star$-Norm computation using LMIs for n-Q FLF with bounded derivatives of $w_{i_j}^j$) *The $\star$-norm between the output* $\boldsymbol{y}$ *and the input* $\boldsymbol{\phi}$ *for the closed-loop system* (77) *under conditions* (73) *and* (74) *can be obtained solving the following LMI problem:*

$$||\boldsymbol{G}_{\phi\to y}^{CL}||_\star = \inf_{\alpha>0} N(\alpha), \quad (101)$$

where $N(\alpha)$ is calculated of each fixed $\alpha > 0$, as follows:

$$N(\alpha) \triangleq \frac{1}{\delta}\min\{\mu \geq 0 : \boldsymbol{P}_i > 0,\ 0 \leq \beta \leq \alpha,$$
subject to LMIs (102)–(109)},

$$\Upsilon_{ijk} = \begin{pmatrix} \begin{matrix} \boldsymbol{A}_i\boldsymbol{P}_j + \boldsymbol{P}_j\boldsymbol{A}_i^T + \boldsymbol{B}_i\boldsymbol{F}_j \\ +\boldsymbol{F}_j^T\boldsymbol{B}_i^T + \alpha\boldsymbol{P}_i - \sum_{l=1}^{p} \bar{q}_{k,l}\left(\boldsymbol{P}_i - \boldsymbol{P}_{\bar{s}(i,k)}\right)\end{matrix} & \delta\boldsymbol{E}_i \\ \delta\boldsymbol{E}_i^T & -\beta\boldsymbol{I} \end{pmatrix},$$

$$\Psi_{ij} = \begin{pmatrix} \alpha\boldsymbol{P}_j & \boldsymbol{0} & \boldsymbol{P}_j\boldsymbol{C}_i^T \\ \boldsymbol{0} & (\mu-\beta)\boldsymbol{I} & \delta\boldsymbol{D}_i^T \\ \boldsymbol{C}_i\boldsymbol{P}_j & \delta\boldsymbol{D}_i & \mu\boldsymbol{I} \end{pmatrix},$$

$$\Lambda_{ijk,min} = \begin{pmatrix} \frac{1}{1+\delta^2}\begin{pmatrix} \boldsymbol{P}_i & 0 \\ 0 & \boldsymbol{I}\end{pmatrix} & (*) \\ \tau_{min}^k \boldsymbol{L}_k \left[\boldsymbol{A}_i\boldsymbol{P}_j + \boldsymbol{B}_i\boldsymbol{F}_j \quad \boldsymbol{E}_i\right] & \theta_k^2 \end{pmatrix},$$

$$\Lambda_{ijk,\max} = \begin{pmatrix} \frac{1}{1+\delta^2}\begin{pmatrix} \boldsymbol{P}_i & 0 \\ 0 & \boldsymbol{I}\end{pmatrix} & (*) \\ \tau_{\max}^k \boldsymbol{L}_k \left[\boldsymbol{A}_i\boldsymbol{P}_j + \boldsymbol{B}_i\boldsymbol{F}_j \quad \boldsymbol{E}_i\right] & \theta_k^2 \end{pmatrix},$$

$$\Upsilon_{iik} \leq 0, \quad i=1,\dots,r, \quad k=1,\dots,2^p, \quad (102)$$

$$\frac{2}{r-1}\Upsilon_{iik} + \Upsilon_{ijk} + \Upsilon_{jik} \leq 0,$$
$$i \neq j = 1,\dots,r, \quad k=1,\dots,2^p, \quad (103)$$

$$\Psi_{ii} \geq 0, \quad i=1,\dots,r, \quad (104)$$

$$\frac{2}{r-1}\Psi_{ii} + \Psi_{ij} + \Psi_{ji} \geq 0, \quad i \neq j, = 1,\dots,r, \quad (105)$$

$$\Lambda_{iik,\min} \geq 0, \quad i=1,\dots,r, \quad k=1,\dots,p, \quad (106)$$

$$\frac{2}{r-1}\Lambda_{iik,\min} + \Lambda_{ijk,\min} + \Lambda_{ijk,\min} \geq 0,$$
$$i \neq j = 1,\dots,r, \quad k=1,\dots,p, \quad (107)$$

$$\Lambda_{iik,\max} \geq 0, \quad i=1,\dots,r, \quad k=1,\dots,p, \quad (108)$$

$$\frac{2}{r-1}\Lambda_{iik,\max} + \Lambda_{ijk,\max} + \Lambda_{ijk,\max} \geq 0,$$
$$i \neq j = 1,\dots,r, \quad k=1,\dots,p, \quad (109)$$

$$\bar{q}_{k,l} = \begin{cases} q_{k,l} & \textit{if } i_k = 0 \\ -q_{k,l} & \textit{if } i_k = 1 \end{cases}$$

where $q_{k,l}$ is the l-th element of vertex $\boldsymbol{q}_k$ and equals θ_k or $-\theta_k$ (see (73)), $\bar{s}(i,k)$ are the same as described in Theorem 10, $\partial w_0^k/\partial z_k \in [\tau_{\min}^k, \tau_{\max}^k]$ and $z_k = \boldsymbol{L}_k\boldsymbol{x}$, with $\boldsymbol{L}_k$ a constant row vector. The designed non-PDC controller (75) *provides BIBO stability inside $\mathcal{E}(\boldsymbol{P}_h^{-1})$.*

Proof. Apply the procedure of the proof of Theorem 8 subject to (74). In this case, condition (95) has to be checked in the vertices of hyper-rectangle defined by (74).

Finally, LMI conditions (106)–(109) guarantee that $|w_{i_k}^k| \leq \theta_k, k = 1,\dots,p$ inside $\mathcal{E}(\boldsymbol{P}_h^{-1})$. They are an extension of conditions obtained in Theorem 5 of (da Silva Campos *et al.*, 2017) when persistent perturbations are present. They are justified in Lemma A2 of Appendix. ■

Remark 32. Bounds on the first derivatives of normalised weighting functions (73) are guaranteed by LMI conditions (106)–(109).

Remark 33. Theorem 11 we have

$$\begin{aligned} N_{var}^{\text{Th11}} &= 2 + \frac{r}{2} n_x(n_x+1) + r\, n_u\, n_x, \\ N_{row}^{\text{Th11}} &= 1 + r\, n_x + r^3(n_x+n_\phi) + 2r^3(n_x+n_\phi+1) \\ &\quad + r^2(n_x+n_y+n_\phi). \end{aligned}$$

LMI conditions of Theorem 11 can be improved using Lemma 4.

Theorem 12. ($\star$-Norm computation using LMIs based on Lemma 4 for n-Q FLF with bounded derivatives of $w_{i_j}^j$) *The $\star$-norm between the output* $\boldsymbol{y}$ *and the input* $\boldsymbol{\phi}$ *for the closed-loop system* (77) *under conditions* (73) *and* (74) *can be obtained by solving the following LMI problem:*

$$\|\boldsymbol{G}_{\phi\to y}^{CL}\|_\star = \inf_{\alpha>0} N(\alpha), \tag{110}$$

where $N(\alpha)$ is calculated of each fixed $\alpha > 0$, as follows:

$$N(\alpha) \triangleq \frac{1}{\delta} \min\{\mu \ge 0 : \boldsymbol{P}_i > 0,\ 0 \le \beta \le \alpha,$$
$$\textit{subject to LMIs } (111),\ (112),\ (104)\text{–}(109)\},$$

$$\varUpsilon_{ijk} = \begin{pmatrix} \begin{matrix} \boldsymbol{A}_i \boldsymbol{L}_{1j}^T + \boldsymbol{L}_{1j}\boldsymbol{A}_i^T + \boldsymbol{B}_i\boldsymbol{F}_j \\ +\boldsymbol{F}_j^T\boldsymbol{B}_i^T + \alpha\boldsymbol{P}_i - \sum_{l=1}^{p} \bar{q}_{k,l}\left(\boldsymbol{P}_i - \boldsymbol{P}_{\bar{s}(i,k)}\right) \end{matrix} & (*) & (*) \\ \delta\boldsymbol{E}_i^T + \boldsymbol{L}_{2j}\boldsymbol{A}_i^T & -\beta\boldsymbol{I} & (*) \\ \boldsymbol{P}_i - \boldsymbol{L}_{1i} + \boldsymbol{G}_j^T\boldsymbol{A}_i^T & -\boldsymbol{L}_{2i}^T & -\boldsymbol{G}_i - \boldsymbol{G}_i^T \end{pmatrix},$$

$$\varUpsilon_{iik} \le 0, \quad i = 1,\dots,r, \quad k = 1,\dots,2^p, \tag{111}$$

$$\frac{2}{r-1}\varUpsilon_{iik} + \varUpsilon_{ijk} + \varUpsilon_{jik} \le 0,$$
$$i \ne j = 1,\dots,r, \quad k = 1,\dots,2^p, \tag{112}$$

where $\bar{q}_{k,l}$ and $\bar{s}(i,k)$ are the same as described in Theorem 11, and with $\boldsymbol{L}_{1i}$, $\boldsymbol{L}_{2i}$ and $\boldsymbol{G}_i$ $i = 1,\dots,r$ matrices of appropriate dimensions. The designed non-PDC controller (75) *provides BIBO stability inside $\mathcal{E}(\boldsymbol{P}_h^{-1})$.*

Proof. Follow the proof of Theorem 3 applied to condition (95), which has to be satisfied at vertices of (74). ∎

Remark 34. Theorem 12 is more relaxed than Theorem 11. This conclusion is derived following the same lines of Remark 5.

Remark 35. For Theorem 12 we have

$$\begin{aligned} N_{var}^{\text{Th12}} &= 2 + \frac{r}{2} n_x(n_x+1) + r\, n_u\, n_x + 2r\, n_x^2 \\ &\quad + r\, n_x\, n_\phi, \\ N_{row}^{\text{Th12}} &= 1 + r\, n_x + r^3(2n_x+n_\phi) \\ &\quad + 2r^3(n_x+n_\phi+1) \\ &\quad + r^2(n_x+n_y+n_\phi). \end{aligned}$$

Consequently, Theorem 11 has a lower number of variables and a lower number of rows than Theorem 12, but it is more conservative.

Remark 36. As the number of vertices of polytope (72), m, is greater or equal than the number of rules, $r = 2^p$, and the number of rules of fuzzy approximation of $\partial h_i/\partial \boldsymbol{x}$ (90), s, is greater than or equal to 2, the following relationships are obtained:

$$\begin{aligned} N_{var}^{\text{Th8}} &= N_{var}^{\text{Th11}}, \\ N_{var}^{\text{Th9}} &= N_{var}^{\text{Th12}}, \\ N_{row}^{\text{Th11}} &\le N_{row}^{\text{Th8}}, \\ N_{row}^{\text{Th12}} &\le N_{row}^{\text{Th9}}. \end{aligned}$$

Consequently, Theorems 11 and 12 have the or equal number of rows no exceeding those of Theorems 8 and 9. However, Theorems 11 and 12 are only applicable if the T–S fuzzy model has been obtained using the non-linearity sector approach.

Remark 37. Using Remarks 29, 33, 35 and 36 the following relationships can be established:

$$\begin{aligned} N_{var}^{\text{Th8}} &= N_{var}^{\text{Th11}} < N_{var}^{\text{Th2}} < N_{var}^{\text{Th5}}, \\ N_{var}^{\text{Th9}} &= N_{var}^{\text{Th12}} < N_{var}^{\text{Th3}} < N_{var}^{\text{Th6}}. \end{aligned}$$

However, it is difficult to compare theoretically the expressions corresponding to number of rows obtained in Sections 3.1, 3.2.1 and 3.2.2. Instead in Section 4 a numerical comparison is performed through some examples.

To complete this section, Fig. 1 shows a flow chart which explains the main steps to obtain a non-PDC controller using any result of this section.

4. Examples

Example 1. (*Example 3 of Vafamand* et al. *(2017)*) . Consider a T–S system (2) with $p = 1$, $r = 2$ and

$$\boldsymbol{A}_1 = \begin{pmatrix} 2 & -10 \\ 2 & 0 \end{pmatrix}, \quad \boldsymbol{A}_2 = \begin{pmatrix} a & -5 \\ 1 & 1 \end{pmatrix},$$

$$\boldsymbol{B}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \boldsymbol{B}_2 = \begin{pmatrix} b \\ 2 \end{pmatrix}, \quad \boldsymbol{C}_1 = \boldsymbol{C}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$\boldsymbol{D}_1 = \boldsymbol{D}_2 = \begin{pmatrix} 0 & 0 \end{pmatrix}, \quad \boldsymbol{E}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \boldsymbol{E}_2 = \begin{pmatrix} b \\ 1 \end{pmatrix},$$

$$w_0^1(x_1) = \frac{1-\sin(x_1)}{2}, \quad h_1(x_1) = w_0^1,$$
$$h_2(x_1) = 1 - h_1, \quad |x_1| \le \pi/2.$$

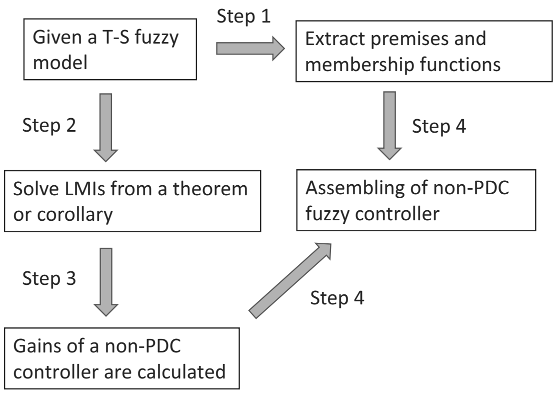

Fig. 1. Steps to obtain a non-PDC controller.

If $a = 4$, $b = 0$, $|\dot{h}_1| \leq 3500$, Corollary 2 of Vafamand *et al.* (2017b) provides an L_1 gain of 1.0231 with $\alpha = 1$. We can compare these values with those of Theorems 2, 3, 5, 6, 8, 9, 11 and 12, and Corollaries 2 and 3. Table 1 compares all the results related to these theorems and Corollary 2 of Vafamand *et al.* (2017b): $\star$-norm, α, number of variables, number of rows and the complexity of the computation (Hu *et al.*, 2018) based on the formula $\log 10\left(N_{var}^3 N_{row}\right)$ for several approaches.

It can be concluded that all the results presented in this paper outperform the L_1 gain provided by Corollary 2 of Vafamand *et al.* (2017b). Even Lemma 2 requires a lower number of variables, number of rows and lower computational complexity than Corollary 2 of Vafamand *et al.* (2017b).

On the other hand, it is possible to compare the conditions developed in Theorems 8 and 9, (88) and (89), for bounding first derivatives of membership functions (71) with the conditions of Lemma 1 of Hu *et al.* (2019) when $q = 1$[1]. From Table 1 it is deduced that when using the conditions of Lemma 1 of Hu *et al.* (2019) together with Theorems 8 and 9 the results are poorer and it is required more computational complexity.

It is possible to compare the results presented in this paper with each other. The best $\star$-norm is provided by Theorems 8, 9, 11 and 12. However, these methods are only valid inside (locally) the generalised inescapable ellipsoid (82). The best global $\star$-norm is guaranteed by Theorems 5 and 6 and Corollary 3. From a computational point of view, global methods based in IDLFs and DIDLFs are the most demanding, specially those using DIDLFs.

Theorems 8 and 9 provide the same results; however, Theorem 8 requires a lower computational cost. The same situation happens with Theorems 11 and 12. Furthermore, Theorems 8 and 11 have a reasonable computational cost compared with Lemma 2 (related to a common Lyapunov function). Moreover, Theorems 8 and 11 have the least computation cost if we discard Lemma 2. ♦

Example 2. (*Example 7 of Lee* et al. *(2012)*) Consider a T–S system (2) with $p = 1$, $r = 2$ and

$$A_1 = \begin{pmatrix} -a & -4 \\ -1 & -2 \end{pmatrix}, \quad A_2 = \begin{pmatrix} -2 & -4 \\ 20 & -2 \end{pmatrix},$$

$$B_1 = \begin{pmatrix} 1 \\ 10 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 1 \\ b \end{pmatrix}, \; C_1 = C_2 = \begin{pmatrix} 1 & 0 \end{pmatrix},$$

$$D_1 = D_2 = 0, \quad E_1 = \begin{pmatrix} 0 \\ 0.1 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0.1 \\ 1.2 \end{pmatrix},$$

$$w_0^1(x_1) = \frac{1 + \sin(x_1)}{2}, \quad h_1(x_1) = w_0^1,$$

$$h_2(x_1) = 1 - h_1, \quad |x_1| \leq \frac{\pi}{2}.$$

A comparison between different results of this article is performed when looking for the minimum value of a when $b = 1$ which provides a finite $\star$-norm with a stable closed-loop. Table 2 shows the minimum value of a, the $\star$-norm when $a = -5$, the number of variables, the number of rows and the complexity of the computation for several approaches. Theorems 1 and 2, and Corollary 2 of Vafamand *et al.* (2017b) cannot be applied because matrices C_i are not equal to the identity matrix. Table 2 also provides values when bounding the first derivatives of membership functions using Lemma 1 of Hu *et al.* (2019) with $q = 1$.

As a conclusion, the results of this paper outperform the results of Vafamand *et al.* (2017b) and Hu *et al.* (2019), and the best $\star$-norm is provided by Theorems 11 and 12. Moreover, the best global $\star$-norm is guaranteed by Theorem 6. From a computational point of view, it is also shown that DIDLFs results are the most demanding.

Theorems 11 and 12 provide a lower $\star$-norm than Theorems 8 and 9. This conclusion is correlated with the fact that bounding first derivatives of normalised weighting functions should be less conservative than bounding first derivatives of membership functions.

Finally, it is concluded that Theorems 8 and 11 have, again, the least computational cost if we discard Lemma 2, and they have a reasonable computational cost compared with Lemma 2. ♦

Example 3. (*Example of da Silva Campos* et al. *(2017)*) Consider a T–S system (2) with $p = 2$, $r = 4$ and

$$A_1 = \begin{pmatrix} 4 & -4 \\ -1 & -2 \end{pmatrix}, \quad A_2 = \begin{pmatrix} -2 & -4 \\ -1 & -2 \end{pmatrix},$$

$$A_3 = \begin{pmatrix} 4 & -4 \\ 20 & -2 \end{pmatrix}, \quad A_4 = \begin{pmatrix} -2 & -4 \\ 20 & -2 \end{pmatrix},$$

[1]If $q = 1$ the n-Q FLF proposed by Hu *et al.* (2019) is the same as in Theorems 8 and 9.

Table 1. Comparison of different results in Example 1.

Result	$\star$-Norm	α	N_{var}	N_{row}	$\log 10\left(N_{var}^3 N_{row}\right)$
Corollary 2 of Vafamand *et al.* (2017b) ($\theta_k = 3500$)	1.0231	1	11	50	4.82
Lemma 2 and Corollary 1	1.0056	1.01	9 / 29	25 / 33	4.26 / 5.91
Theorems 8, 9, 11 and 12 ($\varphi_i, \theta_k = 3500$)	1.0055	1.01	12 / 32	113 / 129	5.29 / 6.63
Theorems 8–12 ($\varphi_i, \theta_k = 500$)	1.0041	1.01	॥	॥	॥
Theorems 8–12 ($\varphi_i, \theta_k = 10$)	0.8714	0.97	॥	॥	॥
Theorems 8 and 9 ($\varphi_i = 3500$) using bounds of Hu *et al.* (2019)	21.1914	0.97	18 / 38	101 / 117	5.77 / 6.81
Theorems 8 and 9 ($\varphi_i = 500$) using bounds of Hu *et al.* (2019)	32.0884	0.64	॥	॥	॥
Theorems 8 and 9 ($\varphi_i = 10$) using bounds of Hu *et al.* (2019)	Infeasible	–	॥	॥	॥
Theorems 2 and 3 and Corollary 2	1.0056	1.01	24 / 104 / 92	73 / 105 / 105	6.00 / 8.07 / 7.91
Theorems 5 and 6 and Corollary 3	1.0051	1.01	78 / 398 / 338	241 / 369 / 369	8.06 / 10.37 / 10.15

Table 2. Comparison of different results in Example 2.

Result	Min. of a	$\star$-Norm ($a = -5$)	N_{var}	N_{row}	$\log 10\left(N_{var}^3 N_{row}\right)$
Theorems 1 and 2 and Corollary 2 of Vafamand *et al.* (2017b)	Not applicable	–	–	–	–
Lemma 2	-3.69	Infeasible	9	23	4.23
Corollary 1	-3.69	Infeasible	29	31	5.87
Theorem 2	-3.69	Infeasible	24	69	5.98
Theorem 3	-3.69	Infeasible	104	101	8.06
Corollary 2	-3.69	Infeasible	92	101	7.90
Theorem 5	-5.96	0.5896	78	233	8.04
Theorem 6	-5.96	0.5887	398	361	10.36
Corollary 3	-5.85	0.6652	338	361	10.14
Theorems 8 and 9 ($\varphi_i = 15$)	-5.18	0.5328	12 / 32	109 / 125	5.28 / 6.61
Theorems 8 and 9 ($\varphi_i = 10$)	-6.20	0.3515	॥	॥	॥
Theorems 8 and 9 ($\varphi_i = 5$)	-7.82	0.3185	॥	॥	॥
Theorems 8 and 9 ($\forall\, \varphi_i$) using bounds of Hu *et al.* (2019)	-3.69	Infeasible	18 / 38	97 / 113	5.75 / 6.79
Theorems 11 and 12 ($\theta_k = 15$)	-5.55	0.4261	12 / 32	109 / 125	5.28 / 6.61
Theorems 11 and 12 ($\theta_k = 10$)	-6.70	0.3014	॥	॥	॥
Theorems 11 and 12 ($\theta_k = 5$)	-8.43	0.2786	॥	॥	॥

$$\boldsymbol{B}_1 = \boldsymbol{B}_2 = \begin{pmatrix} 1 \\ 10 \end{pmatrix}, \quad \boldsymbol{B}_3 = \boldsymbol{B}_4 = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$
$$\boldsymbol{C}_1 = \boldsymbol{C}_2 = \boldsymbol{C}_3 = \boldsymbol{C}_4 = \begin{pmatrix} 0 & 1 \end{pmatrix},$$
$$\boldsymbol{D}_1 = \boldsymbol{D}_3 = 0.1, \quad \boldsymbol{D}_2 = \boldsymbol{D}_4 = -0.1,$$
$$\boldsymbol{E}_1 = \boldsymbol{E}_3 = \begin{pmatrix} -5 \cdot 10^{-3} \\ 5 \cdot 10^{-3} \end{pmatrix}, \quad \boldsymbol{E}_2 = \boldsymbol{E}_4 = \begin{pmatrix} -0.075 \\ 0.075 \end{pmatrix},$$
$$w_0^1(x_1) = \frac{1 - \sin(x_1)}{2}, \quad w_0^2(x_2) = \frac{1 - \sin(x_2)}{2},$$
$$h_1(x_1, x_2) = w_0^1(x_1) w_0^2(x_2),$$
$$h_2(x_1, x_2) = w_0^1(x_1) w_1^2(x_2),$$
$$h_3(x_1, x_2) = w_1^1(x_1) w_0^2(x_2),$$
$$h_4(x_1, x_2) = w_1^1(x_1) w_1^2(x_2),$$
$$|x_i| \le 20, \quad i = 1, 2.$$

As commented in Example 2, results of Vafamand *et al.* (2017b) cannot be applied to this example either.

This example is more demanding from a computational point of view than Examples 1 and 2 since it has 4 rules instead of 2. This is also supported by N_{var}, N_{row} and $\log 10\left(N_{var}^3 N_{row}\right)$ values shown in Table 3.

From Table 3 it is possible to get some conclusions:

- Theorems 8 and 9 using bounds of Hu *et al.* (2019) resulted in infeasibility.

- Only global methods based on DIDLFs are feasible.
- Local methods based on Theorems 8, 9, 11 and 12 provide a finite $\star$-norm for several values of bounds. As expected, they outperform the $\star$-norm of global methods.
- Theorem 5 gives the same $\star$-norm as Theorem 6 with less computational effort and fewer variables and rows.
- Corollary 3 is outperformed by Theorem 5.
- Comparing local methods and DIDLFs results, it is realized that the latter require a huge number of variables and a huge number of rows, and the complexity of the computation is much bigger.
- In this example, Theorem 11 has the least computational cost if we discard Lemma 2 followed by Theorem 8. Both of them have a reasonable computational cost compared with Lemma 2 if we consider the remaining results.

♦

Example 4. (*Example 2.3 of Cherifi (2017)*) Consider the non-linear model corresponding to the unstable ball and beam system (Hauser *et al.*, 1992):

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\end{bmatrix}=\begin{pmatrix}0 & 1 & 0 & 0\\ bx_4^2 & 0 & bg\dfrac{\sin(x_3)}{x_3} & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0\end{pmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix}+\begin{pmatrix}0\\0\\0\\1\end{pmatrix}u, \tag{113}$$

where x_1 and x_2 are, respectively, the position and the speed of the ball, x_3 and x_4 are, respectively, the angular position and the angular speed of the beam, and u is the torque applied to beam, $b = 0.9605$ is a mechanical parameter of the system and $g = 9.81\ \mathrm{m/s^{-2}}$ is the gravity constant. Using the non-linearity sector approach non-linear model equation (113) can be exactly represented by this T–S model with $p = 2$, $r = 4$:

$$\boldsymbol{A}_1=\begin{pmatrix}0 & 1 & 0 & 0\\ 0 & 0 & -b\cdot g\cdot\frac{2}{\pi} & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0\end{pmatrix},$$

$$\boldsymbol{A}_2=\begin{pmatrix}0 & 1 & 0 & 0\\ b & 0 & -b\cdot g\cdot\frac{2}{\pi} & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0\end{pmatrix},$$

$$\boldsymbol{A}_3=\begin{pmatrix}0 & 1 & 0 & 0\\ 0 & 0 & -b\cdot g & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0\end{pmatrix},$$

$$\boldsymbol{A}_4=\begin{pmatrix}0 & 1 & 0 & 0\\ b & 0 & -b\cdot g & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0\end{pmatrix},$$

$$\boldsymbol{B}_1=\boldsymbol{B}_2=\boldsymbol{B}_3=\boldsymbol{B}_4=\begin{pmatrix}0\\0\\0\\1\end{pmatrix},$$

$$w_0^1(x_1)=\frac{1-\dfrac{\sin(x_3)}{x_3}}{1-\frac{2}{\pi}},\quad w_0^2(x_2)=1-x_4^2,$$

$$h_1(x_1,x_2)=w_0^1(x_1)w_0^2(x_2),$$
$$h_2(x_1,x_2)=w_0^1(x_1)w_1^2(x_2),$$
$$h_3(x_1,x_2)=w_1^1(x_1)w_0^2(x_2),$$
$$h_4(x_1,x_2)=w_1^1(x_1)w_1^2(x_2),$$
$$|x_3|\le\frac{\pi}{2},\quad |x_4|\le 1.$$

Taking x_3 as the controlled output and adding a persistent perturbation in the system, the rest of matrices of the T–S model are

$$\boldsymbol{C}_1=\boldsymbol{C}_2=\boldsymbol{C}_3=\boldsymbol{C}_4=\begin{pmatrix}0 & 0 & 1 & 0\end{pmatrix},$$
$$\boldsymbol{D}_1=\boldsymbol{D}_3=0.5,\quad \boldsymbol{D}_2=\boldsymbol{D}_4=-0.5,$$
$$\boldsymbol{E}_1=\boldsymbol{E}_3=\begin{pmatrix}-0.1\\0.1\\-0.1\\0.1\end{pmatrix},$$
$$\boldsymbol{E}_2=\boldsymbol{E}_4=\begin{pmatrix}-0.15\\0.15\\-0.15\\0.15\end{pmatrix}.$$

From Table 4 some conclusions are obtained:

- Theorems 8 and 9 using bounds of Hu *et al.* (2019) provide worse results than the original theorems proposed in this work.
- Local methods based on Theorems 8, 9, 11 and 12 provide a finite $\star$-norm for several values of bounds. As expected, they outperform the $\star$-norm of global methods.
- Once more, Theorem 5 gives the same $\star$-norm as Theorem 6 with less computational effort and fewer variables and rows.
- Corollary 3 is outperformed, again, by Theorem 5.
- Comparing local methods and DIDLFs results, it is realized, again, that the latter require a huge number of variables and a huge number of rows, and the complexity of the computation is much bigger.

Table 3. Comparison of different results in Example 3.

Result	$\star$-Norm	α	N_{var}	N_{row}	$\log 10\left(N_{var}^3 N_{row}\right)$
Theorems 1 and 2 and Corollary 2 of Vafamand *et al.* (2017b)	Not applicable	–	–	–	–
Lemma 2 and Corollary 1	Infeasible	–	13/53	67 / 99	5.17 / 7.17
Theorems 8 and 9 ($\varphi_i = 1$)	0.1073	2.25	22 / 62	1289 / 1417	7.14 / 8.53
Theorems 8 and 9 ($\varphi_i = 5$)	0.1083	1.35	〃	〃	〃
Theorems 8 and 9 ($\varphi_i = 10$)	0.1119	0.85	〃	〃	〃
Theorems 8 and 9 ($\varphi_i = 20$)	0.1394	0.20	〃	〃	〃
Theorems 8 and 9 ($\forall\varphi_i$) using bounds of Hu *et al.* (2019)	Infeasible	–	34 / 74	633 / 761	7.40 / 8.49
Theorems 11 and 12 ($\theta_k = 1$)	0.1052	3.80	22 / 62	777 / 905	6.92 / 8.33
Theorems 11 and 12 ($\theta_k = 5$)	0.1068	2.15	〃	〃	〃
Theorems 11 and 12 ($\theta_k = 10$)	0.1101	1.10	〃	〃	〃
Theorems 11 and 12 ($\theta_k = 20$)	0.1281	0.30	〃	〃	〃
Theorems 2 and 3 and Corollary 2	Infeasible	–	142 / 782 / 662	841 / 1353 / 1353	9.31 / 11.81 / 11.59
Corollary 3	0.1322	0.40	10298	20769	16.36
Theorems 5 and 6	0.1260	0.50	2098 / 12338	12577 / 20769	14.07 / 16.59

Table 4. Comparison of different results in Example 4.

Result	$\star$-Norm	α	N_{var}	N_{row}	$\log 10\left(N_{var}^3 N_{row}\right)$
Theorems 1 and 2 and Corollary 2 of Vafamand *et al.* (2017b)	Not applicable	–	–	–	–
Lemma 2 and Corollary 1	0.5323	0.65	28 / 172	109 / 173	6.38 / 8.95
Theorems 8 and 9 ($\varphi_i = 100$)	0.5549	1.35	58 / 202	2129 / 2513	8.62 / 10.32
〃 ($\varphi_i = 1000$)	0.5390	0.90	〃	〃	〃
〃 ($\varphi_i = 10000$)	0.5342	0.70	〃	〃	〃
Theorems 8 and 9 ($\varphi_i = 100$) using bounds of Hu *et al.* (2019)	8.1409	0.35	70 / 214	1240 / 1633	8.63 / 10.20
〃 ($\varphi_i = 1000$)	1.2397	1.4	〃	〃	〃
〃 ($\varphi_i = 10000$)	1.0314	1.55	〃	〃	〃
Theorems 11 and 12 ($\theta_k = 10$)	0.5209	0.7	58 / 202	1201 / 1457	8.37 / 10.08
〃 ($\theta_k = 50$)	0.5213	0.7	〃	〃	〃
〃 ($\theta_k = 100$)	0.5224	0.60	〃	〃	〃
Theorems 2 and 3 and Corollary 2	0.5322	0.6	298 / 2602 / 2362	1393 / 2417 / 2417	10.57 / 13.63 / 13.50
Corollary 3	0.5221	0.60	37042	37313	18.28
Theorems 5 and 6	0.5218	0.65	4258 / 41122	20929 / 37313	15.21 / 18.41

- In this example, Theorem 11 has, again, the least computational cost if we discard Lemma 2 followed by Theorem 8. Once more, both of them have a reasonable computational cost compared with Lemma 2 if we consider the remaining results.

♦

5. Conclusions

In this work several innovative approaches to design BIBO stabilising non-PDC control laws for T–S continuous time fuzzy systems under persistent perturbations based on fuzzy Laypunov functions have been presented. These approaches are based on minimising the $\star$-norm of the closed loop and on two kinds of fuzzy Lyapunov functions:

- integral-delayed Lyapunov functions: Theorems 2, 3, 5 and 6, and Corollaries 2 and 3,
- fuzzy Laypunov functions with guaranteed bounds for the first derivatives of membership functions: Theorems 8, 9, 11 and 12.

IDLFs are global methods and they are the most demanding from a computational point of view. This question has been showed in the presented examples. Within this category two branches have been analysed: single (Theorems 2, 3 and Corollary 2) and double IDLFs (Theorems 5, 6 and Corollary 3). DIDLFs outperforms the results of singular DIDLFs in all the examples. However, DIDLFs require much more computational resources.

FLFs with bounded first derivatives provide a local method to compute the $\star$-norm. These methods have been shown the best ones when compared the values of such a norm in all the examples. Also, they have a lower computation cost compared with global IDLFs. However, their results are only valid in the generalised inescapable ellipsoid $\mathcal{E}(\boldsymbol{P}_h^{-1})$.

FLFs methods are based in two approaches: bounding $|\dot{h}_i|$ (Theorems 8, 9) or bounding $|\dot{\omega}^j_{i_j}|$ (Theorems 11, 12). Both methodologies require the same number of variables but the approach based on bounding first derivatives of normalised weighting functions uses fewer or the same number of rows. This fact has been proofed theoretically (Remark 36) and computed in the examples. On the other hand, the approach based on bounding the first derivatives of normalised weighting functions provides better results if $r > 2$. However, this method is only applicable if the T–S fuzzy model has been obtained using non-linearity sector approach. Consequently, the approach based on bounding the first derivatives of membership functions is more general.

After analysing the results of all the examples, Theorem 11 has the least computational cost if we discard Lemma 2 (related to a common quadratic Lyapunov function) followed by Theorem 8. Moreover, these theorems have a reasonable computational cost compared with Lemma 2 when considering the remaining results.

Considering the reasoning of the previous paragraph, for the case of non-linear systems with a high number of fuzzy rules authors can conclude that the most suitable results are Theorems 11 and 8 because they require the least computational cost, in case Lemma 2 does not provide a good result for BIBO stabilization and/or the $\star$-norm.

Finally, results of this work have been compared with those of Vafamand *et al*. (2017b) and Hu *et al*. (2019). These articles deal also with T–S continuous time fuzzy systems under persistent disturbances. In all the examples it is shown that results from this work outperform results from those previous articles.

6. Future work

The results of this paper could be extended and improved along several ways:

- Design of non-PDC controllers under inputs and state constraints including additional LMIs conditions.
- Use of multi-indexation in fuzzy Lyapunov functions, integral-delayed Lyapunov functions and non-PDC controllers.
- Reduction of the computational burden of methods based on double integral-delayed Lyapunov functions and those based on fuzzy Lyapunov functions with guaranteed bounds for first derivatives of membership functions.
- Inclusion of a fuzzy observer to design output-feedback controllers when not all the states are measurable.
- Extension of these methodologies for designing BIBO stabilising non-PDC controllers for discrete time T–S fuzzy systems.

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Appendix

Lemma A1. (LMI bounds for $|\dot{h}_l|$ inside $\mathcal{E}(\boldsymbol{P}_h^{-1})$) *LMIs* (88) *and* (89) *guarantee* $|\dot{h}_l| \leq \varphi_l$ *inside* $\mathcal{E}(\boldsymbol{P}_h^{-1})$.

Proof. Lemma 3 and LMIs (88) and (89) are sufficient conditions for

$$\begin{pmatrix} \frac{1}{1+\delta^2}\begin{pmatrix} \boldsymbol{P}_h & 0 \\ 0 & \boldsymbol{I} \end{pmatrix} & (*) \\ \frac{\partial h_l}{\partial \boldsymbol{x}}\begin{bmatrix} \boldsymbol{A}_h\boldsymbol{P}_h + \boldsymbol{B}_h\boldsymbol{F}_h & \boldsymbol{E}_h \end{bmatrix} & \varphi_l^2 \end{pmatrix} \geq 0.$$

Applying a congruence transformation with $\operatorname{diag}\left(\boldsymbol{P}_h^{-1}, \boldsymbol{I}, \boldsymbol{I}\right)$ and Schur complement (cf. p. 7 in the work of Boyd *et al.* (1994)),

$$\frac{1}{1+\delta^2}\begin{pmatrix} \boldsymbol{P}_h^{-1} & 0 \\ 0 & \boldsymbol{I} \end{pmatrix} - \frac{1}{\varphi_l^2}\frac{\partial h_l}{\partial \boldsymbol{x}}\begin{bmatrix} \boldsymbol{A}_h\boldsymbol{P}_h + \boldsymbol{B}_h\boldsymbol{F}_h & \boldsymbol{E}_h \end{bmatrix}(*)^T \geq 0,$$

pre and post-multiplying the result by $\begin{bmatrix} \boldsymbol{x}^T & \boldsymbol{\phi}^T \end{bmatrix}$, we get

$$\frac{1}{1+\delta^2}\left(\boldsymbol{x}^T\boldsymbol{P}_h^{-1}\boldsymbol{x} + \boldsymbol{\phi}^T\boldsymbol{\phi}\right) \geq \frac{1}{\varphi_l^2}\left(\dot{h}_l\right)^2. \tag{A1}$$

As $\boldsymbol{x}^T\boldsymbol{P}_h^{-1}\boldsymbol{x} \leq 1$ and $\boldsymbol{\phi}^T\boldsymbol{\phi} \leq \delta^2$, we have

$$1 \geq \frac{1}{\varphi_l^2}\left(\dot{h}_l\right)^2 \tag{A2}$$

which completes the proof. ■

Lemma A2. (LMI bounds for $|\dot{w}_{i_k}^k|$ inside $\mathcal{E}(\boldsymbol{P}_h^{-1})$) *LMIs* (106) *and* (107) *guarantee* $|\dot{w}_{i_k}^k| \leq \theta_k$ *inside* $\mathcal{E}(\boldsymbol{P}_h^{-1})$.

Proof. Follow the same procedure as in the proof of Lemma A1 with LMIs (106) and (107) and taking into account that

$$\dot{w}_0^k = \frac{\partial w_0^k}{\partial z_k}\dot{z}_k = \frac{\partial w_0^k}{\partial z_k}\boldsymbol{L}_k\dot{\boldsymbol{x}},$$
$$\frac{\partial w_0^k}{\partial z_k} \in [\tau_{\min}^k, \tau_{\max}^k].$$

■

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