Int. J. Appl. Math. Comput. Sci., 2022, Vol. 32, No. 2, 271–283
DOI: 10.34768/amcs-2022-0020

GLOBAL BEHAVIOR OF A MULTI–GROUP SEIR EPIDEMIC MODEL WITH SPATIAL DIFFUSION IN A HETEROGENEOUS ENVIRONMENT

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In this paper, we propose a multi-group SEIR epidemic model with spatial diffusion, where the model parameters are spatially heterogeneous. The positivity and ultimate boundedness of the solution, as well as the existence of a global attractor of the associated solution semiflow, are established. The definition of the basic reproduction number is given by utilizing the next generation operator approach, whereby threshold-type results on the global dynamics in terms of this number are established. That is, when the basic reproduction number is less than one, the disease-free steady state is globally asymptotically stable, while if it is greater than one, uniform persistence of this model is proved. Finally, the feasibility of the main theoretical results is shown with the aid of numerical examples for a model with two groups.

Keywords: global stability, multi-group epidemic model, spatial heterogeneity, spatial diffusion.

1. Introduction

Establishing mathematical models and studying them can help people solve many practical problems (Chaturantabut, 2020; El-Douh *et al.*, 2022). It has been widely recognized that spatial diffusion and environmental heterogeneity are ubiquitous in the real world and they have significant impact on the spread of infectious diseases, e.g., influenza. Indeed, very recently, these problems have attracted many researchers to study the impact of the spatial heterogeneity of the environment and the movement of individuals on the dynamical behaviors of a disease in an analytical aspect; see the works of Yang *et al*. (2020), Yang and Wang (2019), Song *et al*. (2019), Li *et al*. (2017), Allen *et al*. (2008) and the references therein.

Among the above-mentioned authors, Allen *et al*. (2008) proposed a frequency-dependent SIS (susceptible-infected-susceptible) reaction-diffusion epidemic model for a population inhabiting a continuous spatial habitat, and studied the properties of the basic reproduction number and threshold-type results on the global dynamics. Yang *et al*. (2020) proposed a diffusive SIRS (susceptible-infected-recovered-susceptible) model with a general incidence rate and a spatial heterogeneity, and established threshold dynamics, including global attractors of the disease-free equilibrium and uniform persistence. Song *et al*. (2019) considered a kind of SEIRS (susceptible-exposed-infected-recovered-susceptible) reaction-diffusion model where the disease transmission and recovery rates can be spatially heterogeneous, and investigated the asymptotic profiles of the basic reproduction number and the endemic equilibrium with respect to diffusion coefficients. Li *et al*. (2017) provided qualitative analysis of an SIS epidemic reaction-diffusion system with a linear source in a spatially heterogeneous environment, including the uniform bounds of solutions and the threshold dynamics in terms of the basic reproduction number.

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In addition, epidemic parameters are related to the spatial location, taking account of the fact that the total population should be classified into different groups according to different communities, cities or counties and the epidemic parameters should vary among different population groups, which can also make the description of complex disease dynamics more realistic. Therefore, it is necessary and reasonable to include multi-group into epidemic models. Some recent developments on the dynamical properties of multi-group effects epidemic models have been discussed by Liu and Li (2020b; 2020a), Luo *et al.* (2019), Zhao *et al.* (2018), Chen *et al.* (2016) and in the references cited therein. Luo *et al.* (2019) investigated the global dynamics of a reaction-diffusion multi-group SIR epidemic model with nonlinear incidence in a spatially heterogeneous and homogeneous environment. Zhao *et al.* (2018) incorporated constant recruitment in a two-group SIR epidemic model with time delay and showed that the existence of traveling waves is determined by the basic reproduction number. Chen *et al.* (2016) analyzed multi-group coupled systems on networks with multi-diffusion (MCSNMs) and analyzed the stability of the systems by using the graph-theoretic approach and the vertex Lyapunov function set to construct the appropriate global Lyapunov function.

However, almost all reported results for reaction-diffusion epidemic models with environmental heterogeneity or multiple groups are about the SIR (Susceptible-Infected-Recovered) or SIS models and rarely about the SEIR (Susceptible-Exposed-Infected-Recovered) model. These models did not include the class of exposed individuals and ignored the movement of exposed individuals. For many infectious diseases, infected individuals can experience incubation before showing symptoms, e.g., malaria, West Nile virus, HIV/AIDS. The travel of exposed individuals showing no symptoms can spread the disease geographically, which makes the disease harder to control. Therefore, it seems imperative to include the exposed subclass and explore the influences of exposed individuals' movement on the disease spread. The influence of the incubation period on the spread of infectious diseases has been widely discussed by Xing and Li (2021), Liu and Li (2020a), Song *et al.* (2019) and in the references cited therein.

Motivated by the above discussion, we extend the classic diffusive SEIR epidemic model to the situation in which all the parameters are functions of the location x and the population is divided into n groups according to different contact patterns. Let $S_k(t,x)$, $E_k(t,x)$, $I_k(t,x)$ and $R_k(t,x)$ be the densities of susceptible individuals, exposed individuals, infectious individuals and recovered individuals at time t and location $x \in \Omega$ in group $k \in \{1,2,\dots,n\}$, respectively, where the habitat Ω is bounded and connected. Hence, the n-group diffusive SEIR epidemic model has the following form:

$$\begin{cases} \dfrac{\partial S_k}{\partial t} =\nabla\cdot(d_{1k}(x)\nabla S_k)+\Lambda_k(x)-\mu_k(x)S_k \\ \qquad -\displaystyle\sum_{j=1}^{n}\beta_{kj}(x)S_kI_j, \\ \dfrac{\partial E_k}{\partial t} =\nabla\cdot(d_{2k}(x)\nabla E_k)+\displaystyle\sum_{j=1}^{n}\beta_{kj}(x)S_kI_j \\ \qquad -\sigma_k(x)E_k-(\mu_k(x)+\delta_{1k}(x))E_k, \\ \dfrac{\partial I_k}{\partial t} =\nabla\cdot(d_{3k}(x)\nabla I_k)+\sigma_k(x)E_k-(\mu_k(x) \\ \qquad +\delta_{2k}(x)+c_k(x))I_k+\gamma_k(x)R_k, \\ \dfrac{\partial R_k}{\partial t} =\nabla\cdot(d_{4k}(x)\nabla R_k)+c_k(x)I_k-\gamma_k(x)R_k \\ \qquad -\mu_k(x)R_k \end{cases} \tag{1}$$

for $x \in \Omega$, with the homogeneous Neumann boundary conditions

$$\frac{\partial S_k}{\partial \nu}=\frac{\partial E_k}{\partial \nu}=\frac{\partial I_k}{\partial \nu}=\frac{\partial R_k}{\partial \nu}=0, \quad x\in\partial\Omega,$$

and the initial conditions

$$\begin{cases} S_k(0,x)=S_k^0(x)>0, \quad E_k(0,x)=e_k^0(x)>0, \\ I_k(0,x)=I_k^0(x)>0, \quad R_k(0,x)=r_k^0(x)>0 \end{cases} \tag{2}$$

for $x \in \Omega$.

Here $d_{1k}(x)$, $d_{2k}(x)$, $d_{3k}(x)$, $d_{4k}(x)$ denote the diffusion coefficients of susceptible individuals, exposed individuals, infectious individuals and recovered individuals in group k at location x, respectively. $\Lambda_k(x)$, $\mu_k(x)$ and $c_k(x)$ denote the recruitment rate of the susceptible class, the per-capita natural death rate and the recovery rate from the infectious class in group k at location x, respectively. Furthermore, $\beta_{kj}(x)$ denotes the transmission rate of the disease between susceptible individuals in group k and infectious individuals in group j at location x; $\delta_{1k}(x)$ and $\delta_{2k}(x)$ denote the additional death rates of exposed and infectious individuals induced by the infectious diseases in group k at location x, respectively; $\sigma_k(x)$ denotes the conversional rate from the latent class in group k at location x and $\gamma_k(x)$ denotes the relapse rate from the recovered class into the infectious class in group k at location x. The homogeneous Neumann boundary conditions imply that there is no population flux across the boundary $\partial\Omega$. Throughout this paper, we assume that all the model parameters are continuous and positive functions on $\overline{\Omega}$.

The remainder of this paper is organized as follows. In Section 2, some basic properties, including the existence, uniqueness, positivity and ultimate boundedness of solution, are established. Section 3 is devoted to the threshold dynamics in terms of the basic reproduction number. In Section 4, the threshold criteria on the global stability of disease-free steady state and the uniform persistence of model are stated and proved. In Section 5, numerical simulations are reported to supplement our theoretical results. A brief conclusion is presented in Section 6.

2. Basic properties

We first recall a stability theorem for the the reaction-diffusion equation, which is of importance in proving our main results. Consider the following scalar reaction-diffusion equation:

$$\begin{cases} \dfrac{\partial \omega}{\partial t} = \nabla \cdot (d(x)\nabla \omega) + \alpha(x) \\ \qquad - \beta(x)\omega, \quad x \in \Omega, \quad t > 0, \\ \dfrac{\partial \omega}{\partial \nu} = 0, \quad x \in \partial\Omega, \quad t > 0, \end{cases} \tag{3}$$

where $d(x)$, $\alpha(x)$ and $\beta(x)$ are continuous and positive functions on $\overline{\Omega}$.

Lemma 1. (Guo *et al.*, 2012, Lemma 2.2) *The system (3) admits a unique positive steady state $\omega^*(x)$ satisfying the equation*

$$\nabla \cdot (d(x)\nabla \omega^*(x)) + \alpha(x) - \beta(x)\omega^*(x) = 0,$$

subject to

$$\frac{\partial \omega^*(x)}{\partial \nu} = 0 \quad \text{for } x \in \partial\Omega,$$

which is globally asymptotically stable in $C(\overline{\Omega}, \mathbb{R}_+)$. Furthermore, if $d(\cdot) \equiv d$, $\alpha(\cdot) \equiv \alpha$ and $\beta(\cdot) \equiv \beta$ are positive constants, then $\omega^*(x) \equiv \alpha/\beta$, $\forall x \in \Omega$.

Define the functional space $X = C(\overline{\Omega}, \mathbb{R}^{4n})$ equipped with the norm

$$|\psi|_X = \max_i \sup_{x\in\overline{\Omega}} |\psi_i(x)|,$$

for $\psi = (\psi_1, \psi_2, \dots, \psi_{4n}) \in X$ and $X_+ = C(\overline{\Omega}, \mathbb{R}^{4n}_+)$. Let $T_{1k}, T_{2k}, T_{3k}, T_{4k} : C(\overline{\Omega}, \mathbb{R}) \to C(\overline{\Omega}, \mathbb{R})$ be the C_0-semigroups generated by $\nabla \cdot (d_{1k}(x)\nabla) - \mu_k(x)$, $\nabla \cdot (d_{2k}(x)\nabla) - (\sigma_k(x) + \mu_k(x) + \delta_{1k}(x))$, $\nabla \cdot (d_{3k}(x)\nabla) - (\mu_k(x) + \delta_{2k}(x) + c_k(x))$, $\nabla \cdot (d_{4k}(x)\nabla) - (\gamma_k(x) + \mu_k(x))$ subject to the Neumann boundary condition in group k, respectively. Then, for all $t > 0$ and $\varphi \in C(\overline{\Omega}, \mathbb{R})$, we have

$$(T_{ik}(t)\varphi)(x) = \int_\Omega \Gamma_{ik}(t, x, y)\varphi(y)\,\mathrm{d}y,$$

where $\Gamma_{ik}(t, x, y)$ is the Green function.

We have that T_{ik}, $i = 1, 2, 3, 4$, $k = 1, 2, \dots, n$ are compact and strongly positive for each $t > 0$ by the Corollary 7.2.3 of Smith (1995). Let $\mathcal{A}_{ik} : D(\mathcal{A}_{ik}) \to C(\overline{\Omega}, \mathbb{R})$ be the generator of T_{ik}, $i = 1, 2, 3, 4$, $k = 1, 2, \dots, n$. Then $T(t) = (T_1, T_2, T_3, T_4)$ is a semigroup generated by the operator $\mathcal{A} = (\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4)$, where $T_i = (T_{i1}, T_{i2}, \dots, T_{in})$ and $\mathcal{A}_i = (\mathcal{A}_{i1}, \mathcal{A}_{i2}, \dots, \mathcal{A}_{in})$ for $i = 1, 2, 3, 4$.

For all $x \in \overline{\Omega}$ and $\phi = (\phi_1, \phi_2, \phi_3, \phi_4) \in X_+$, where $\phi_i = (\phi_{i1}, \phi_{i2}, \dots, \phi_{in})$, let $\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \mathcal{F}_4) : X_+ \to X$, where $\mathcal{F}_i = (\mathcal{F}_{i1}, \mathcal{F}_{i2}, \dots, \mathcal{F}_{in})$ and

$$\begin{aligned} \mathcal{F}_{1k}(\phi)(x) &= \Lambda_k(x) - \sum_{j=1}^n \beta_{kj}(x)\phi_{1k}(x)\phi_{2j}(x), \\ \mathcal{F}_{2k}(\phi)(x) &= \sum_{j=1}^n \beta_{kj}(x)\phi_{1k}(x)\phi_{2j}(x), \\ \mathcal{F}_{3k}(\phi)(x) &= \sigma_k(x)\phi_{3k}(x) + \gamma_k(x)\phi_{4k}(x), \\ \mathcal{F}_{4k}(\phi)(x) &= c_k(x)\phi_{3k}(x). \end{aligned}$$

We denote by

$$\mathbf{u}(t, \cdot, \phi) = (S(t, \cdot, \phi), E(t, \cdot, \phi), I(t, \cdot, \phi), R(t, \cdot, \phi))$$

the solution of (1) with an initial-value function ϕ, where $S = (S_1, S_2, \dots, S_n)$, $E = (E_1, E_2, \dots, E_n)$, $I = (I_1, I_2, \dots, I_n)$ and $R = (R_1, R_2, \dots, R_n)$. Then we can rewrite system (1) as follows:

$$\begin{cases} \dfrac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = \mathcal{A}\mathbf{u} + \mathcal{F}(\mathbf{u}), \quad t > 0, \\ \mathbf{u}(0, \cdot, \phi) = \phi \in X_+. \end{cases} \tag{4}$$

Let $B_\theta(\phi)$ denote the open ball with center ϕ and radius θ.

Definition 1. (*Lyapunov stability*) A solution $\mathbf{u}(t, \cdot, \phi)$ to system (4) is said to be Lyapunov stable (stable) if, for each $\varepsilon > 0$, there exists $\theta(\varepsilon) > 0$ such that for every $\varphi \in B_\theta(\phi)$ the relation

$$\|u(t, \cdot, \phi) - u(t, \cdot, \phi)\| < \varepsilon$$

holds for all $t \geq 0$.

Definition 2. (*Asymptotic Lyapunov stability*) A solution $\mathbf{u}(t, \cdot, \phi)$ to system (4) is said to be asymptotically Lyapunov stable (asymptotically stable) if it is Lyapunov stable and there exists $\eta > 0$ such that for every $\varphi \in B_\eta(\phi)$, one has the relation

$$\|u(t, \cdot, \phi) - u(t, \cdot, \phi)\| \to 0, \quad \text{as } t \to \infty.$$

For convenience, we shall write $(y_1, y_2, \dots, y_n)^T \geq (z_1, z_2, \dots, z_n)^T$ whenever $y_i \geq z_i$ for all $i = 1, 2, \dots, n$.

Let $\alpha(x) = \max_k \sum_{j=1}^{n} \beta_{kj}(x)\phi_{2j}(x)$. We get

$$\phi(x) + h\mathcal{F}(\phi)(x) = \begin{pmatrix} \phi_1(x) + h\mathcal{F}_1(\phi)(x) \\ \phi_2(x) + h\mathcal{F}_2(\phi)(x) \\ \phi_3(x) + h\mathcal{F}_3(\phi)(x) \\ \phi_4(x) + h\mathcal{F}_4(\phi)(x) \end{pmatrix}^T \geq \begin{pmatrix} \phi_1(x)(1 - h\alpha(x)) \\ \phi_2(x) \\ \phi_3(x) \\ \phi_4(x) \end{pmatrix}^T$$

for $h > 0$. This implies that $\phi + h\mathcal{F}(\phi) \in X_+$ for all small $h > 0$.

Let $\phi^* = \phi + h\mathcal{F}(\phi) + h^2\phi$. Then $\phi^* \in X_+$. Furthermore,

$$\frac{1}{h}|\phi + h\mathcal{F}(\phi) - \phi^*|_X = h|\phi|_X \to 0,$$

as $h \to 0^+$. Thus, we have

$$\lim_{h\to 0^+} \frac{1}{h}\text{dist}(\phi + h\mathcal{F}(\phi), X_+) = 0, \quad \forall \phi \in X_+.$$

It is obvious that the subtangential conditions in Corollary 4 of Martin and Smith (1990) are satisfied. Let $[0, \tau_\phi)$ denote the maximum interval of existence for solution $\mathbf{u}(t,\cdot,\phi)$. Then the following lemma is valid according to Corollary 4 of Martin and Smith (1990).

Lemma 2. *For each initial function $\phi \in X_+$, the system (1) has a unique mild solution $\mathbf{u}(t,\cdot,\phi)$ on $[0,\tau_\phi)$, $\tau_\phi \leq \infty$. Moreover, $\mathbf{u}(t,\cdot,\phi) \in X_+$ for all $t \in [0,\tau_\phi)$ and this solution is a classical solution.*

Next, we establish the following results on the existence and ultimate boundedness of the global solution, and the existence of a global attractor for model (1). For convenience, we write

$$\overline{m} = \max_{x\in\overline{\Omega}} m(x), \quad \underline{m} = \min_{x\in\overline{\Omega}} m(x)$$

for any function $m(x)$ defined on $\overline{\Omega}$.

Theorem 1. *The system (1) has a unique solution $\mathbf{u}(t,\cdot,\phi) \in X_+$ on $[0,\infty)$ with $\phi \in X_+$ and this solution is also ultimately bounded.*

Proof. We first prove the existence of a global solution. Suppose that $\tau_\phi < \infty$. We have

$$\|\mathbf{u}(t,\cdot,\phi)\|_X \to \infty \quad \text{as } t \to \tau_\phi \tag{5}$$

by Theorem 2 of Martin and Smith (1990).

We define the total population size in group k at time t and location x as

$$N_k = S_k + E_k + I_k + R_k.$$

Adding the equations of system (1), we get

$$\begin{cases} \dfrac{\partial N_k}{\partial t} \leq \nabla\cdot(D_k(x)\nabla N_k) + \Lambda_k(x) \\ \qquad - \mu_k(x)N_k, \quad x \in \Omega, \quad t > 0, \\ \dfrac{\partial N_k}{\partial \nu} = 0, \; x \in \partial\Omega, \quad t > 0, \end{cases} \tag{6}$$

where $D_k(x) = \max\{d_{1k}(x), d_{2k}(x), d_{3k}(x), d_{4k}(x)\}$ for $x \in \Omega$ and $k = 1, 2, \dots, n$. The standard parabolic comparison theorem (Smith, 1995, Theorem 7.3.4) implies that N_k is uniformly bounded, and so are S_k, E_k, I_k and R_k, which leads to a contradiction to (5). Therefore, $\tau_\phi = \infty$ and the global existence of $\mathbf{u}(t,\cdot,\phi)$ is derived.

We now show that the solution is also ultimately bounded. It follows from (6), Lemma 1 and the standard parabolic comparison theorem (Smith, 1995, Theorem 7.3.4) that

$$\limsup_{t\to\infty} N_k \leq \frac{\overline{\Lambda}_k}{\underline{\mu}_k} \quad \text{uniformly for } x \in \overline{\Omega}.$$

Then there exists a $t_1 > 0$ such that

$$N_k \leq 2\frac{\overline{\Lambda}_k}{\underline{\mu}_k}, \quad \forall t \geq t_1.$$

Thus, we get

$$S_k, E_k, I_k, R_k \leq 2\frac{\overline{\Lambda}_k}{\underline{\mu}_k}, \quad \forall t \geq t_1,$$

which implies that S_k, E_k, I_k, R_k are ultimately bounded. ■

Corollary 1. *The solution semiflow $\Phi(t) : X_+ \to X_+$ of (1), defined by*

$$\Phi(t)\phi = \mathbf{u}(t,\cdot,\phi), \quad t \geq 0,$$

admits a global attractor.

Proof. According to Theorem 1, we know that the solution to system (1) is ultimately bounded, which implies that the solution semiflow $\Phi(t)$ is point dissipative on X_+. By Theorem 2.2.6 of Wu (1996), we can get that $\Phi(t)$ is compact for any $t > 0$. Thus, from Theorem 3.4.8 of Hale (1988), we further obtain that $\Phi(t)$ has a global attractor. ■

3. Basic reproduction number

We consider the following susceptible subsystem of model (1)

$$\begin{cases} \dfrac{\partial S_k}{\partial t} = \nabla\cdot(d_{1k}(x)\nabla S_k) + \Lambda_k(x) \\ \qquad - \mu_k(x)S_k, \quad x \in \Omega, \quad t > 0, \\ \dfrac{\partial S_k}{\partial \nu} = 0, \; x \in \partial\Omega, \quad t > 0. \end{cases}$$

By Lemma 1, system (1) admits a unique positive steady state $S_k^0(x)$, satisfying the equation

$$\nabla \cdot (d_{1k}(x)\nabla S_k^0(x)) + \Lambda_k(x) - \mu_k(x)S_k^0(x) = 0,$$

subject to

$$\frac{\partial S_k^0(x)}{\partial \nu} = 0, \quad x \in \partial\Omega,$$

which is globally asymptotically stable in $C(\overline{\Omega}, \mathbb{R}_+)$. Moreover, if $\Lambda_k(\cdot) \equiv \Lambda_k$ and $\mu_k(\cdot) \equiv \mu_k$ are positive constants, then $S_k^0(x) = \frac{\Lambda_k}{\mu_k}$. Therefore, model (1) has a unique disease-free steady state $E_0(x) = (S^0(x), \mathbf{0}, \mathbf{0}, \mathbf{0})$ with $S^0(x) = (S_1^0(x), S_2^0(x), \dots, S_n^0(x))$ and $\mathbf{0} = (0, 0, \dots, 0)_n$.

We now utilize the next-generation operator approach developed by Wang and Zhao (2012) to derive the basic reproduction number of system (1). Let $w = (E_1, \dots, E_n, I_1, \dots, I_n, R_1, \dots, R_n)^T$. Then the last $3n$ equations of system (1) could be rewritten in the following form:

$$\frac{\partial w}{\partial t} = \nabla \cdot (D(x)\nabla w) + \mathscr{F}(x, w) - \mathscr{V}(x, w),$$

where

$$D(x) = \operatorname{diag}(d_{21}(x), \dots, d_{2n}(x), d_{31}(x), \dots, d_{3n}(x), d_{41}(x), \dots, d_{4n}(x)),$$

and

$$\mathscr{F}(x, w) = \begin{pmatrix} \sum_{j=1}^{n} \beta_{1j}(x) S_1 I_j \\ \vdots \\ \sum_{j=1}^{n} \beta_{nj}(x) S_n I_j \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

$$\mathscr{V}(x, w) = \begin{pmatrix} (\sigma_1(x) + \mu_1(x) + \delta_{11}(x))E_1 \\ \vdots \\ (\sigma_n(x) + \mu_n(x) + \delta_{1n}(x))E_n \\ -(\sigma_1(x)E_1 + (\mu_1(x) + \delta_{21}(x) + c_1(x))I_1 \\ -\gamma_1(x)R_1) \\ \vdots \\ -(\sigma_n(x)E_n + (\mu_n(x) + \delta_{2n}(x) + c_n(x))I_n \\ -\gamma_n(x)R_n) \\ -c_1(x)I_1 + (\gamma_1(x) + \mu_1(x))R_1 \\ \vdots \\ -c_n(x)I_n + (\gamma_n(x) + \mu_n(x))R_n \end{pmatrix}.$$

Here $\mathscr{F}(x, w)$ accounts for new infections and $\mathscr{V}(x, w)$ accounts for other transfers into and out of compartment. Linearizing $\mathscr{F}(x, w) - \mathscr{V}(x, w)$ about the disease-free steady state $E_0(x)$ gives the matrix $F(x) - V(x)$, where

$$F(x) = \left(\frac{\partial \mathscr{F}(x, E_0(x))}{\partial w} \right), \quad V(x) = \frac{\partial \mathscr{V}(x, E_0(x))}{\partial w}.$$

Moreover, $3n \times 3n$ matrices $F(x)$ and $V(x)$ are given by

$$F(x) = \begin{pmatrix} 0 & F_{12} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$V(x) = \begin{pmatrix} V_{11} & 0 & 0 \\ V_{21} & V_{22} & V_{23} \\ 0 & V_{32} & V_{33} \end{pmatrix},$$

where

$$F_{12} = \begin{pmatrix} \beta_{11}(x)S_1^0(x) & \beta_{12}(x)S_1^0(x) & \cdots & \beta_{1n}(x)S_1^0(x) \\ \beta_{21}(x)S_2^0(x) & \beta_{22}(x)S_2^0(x) & \cdots & \beta_{2n}(x)S_2^0(x) \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{n1}(x)S_n^0(x) & \beta_{n2}(x)S_n^0(x) & \cdots & \beta_{nn}(x)S_n^0(x) \end{pmatrix},$$

$$V_{11} = \operatorname{diag}(\sigma_1(x) + \mu_1(x) + \delta_{11}(x), \sigma_2(x) + \mu_2(x) + \delta_{12}(x), \dots, \sigma_n(x) + \mu_n(x) + \delta_{1n}(x)),$$

$$V_{21} = \operatorname{diag}(-\sigma_1(x), -\sigma_2(x), \dots, -\sigma_n(x)),$$

$$V_{22} = \operatorname{diag}(c_1(x) + \mu_1(x) + \delta_{21}(x), c_2(x) + \mu_2(x) + \delta_{22}(x), \dots, c_n(x) + \mu_n(x) + \delta_{2n}(x)),$$

$$V_{23} = \operatorname{diag}(-\gamma_1(x), -\gamma_2(x), \dots, -\gamma_n(x)),$$

$$V_{32} = \operatorname{diag}(-c_1(x), -c_2(x), \dots, -c_n(x)),$$

$$V_{33} = \operatorname{diag}(\gamma_1(x) + \mu_1(x), \gamma_2(x) + \mu_2(x), \dots, \gamma_n(x) + \mu_n(x)).$$

For each initial function $\phi \in X_+$, we denote by $\bar{\phi} = (\phi_2(x), \phi_3(x), \phi_4(x))$ the distribution of initial infection. As time evolves, the distribution of new infected individuals becomes $F(x)(\bar{T}(t)\bar{\phi})(x)$ at time t, where $\bar{T}(t) = (T_2(t), T_3(t), T_4(t))$. Then, the total distribution of the new infected individuals is

$$\int_0^\infty F(x)(\bar{T}(t)\phi)(x)\,\mathrm{d}t = \int_0^\infty F_{12}(x)(T_3(t)\phi_3)(x)\,\mathrm{d}t.$$

Therefore, the next-generation operator L is given by

$$L(\varphi)(x) = F_{12}(x)\int_0^\infty (T_3(x)\varphi)(x)\,\mathrm{d}t, \qquad \varphi \in C(\overline{\Omega}, \mathbb{R}_+).$$

Motivated by Wang and Zhao (2012), we define the basic reproduction number R_0 of model (1) as the spectral radius of L,

$$R_0 := r(L).$$

Next we study the stability of $E_0(x)$ in terms of R_0. We first consider the eigenvalue problem

$$\begin{cases} \nabla\cdot(d_{2k}(x)\nabla\varphi_{2k}) + S_k^0(x)\sum_{j=1}^n \beta_{kj}(x)\varphi_{3j} \\ \qquad -(\sigma_k(x)+\mu_k(x)+\delta_{1k}(x))\varphi_{2k} = \lambda\varphi_{2k}, \\ \nabla\cdot(d_{3k}(x)\nabla\varphi_{3k}) + \sigma_k(x)\varphi_{2k} - (\mu_k(x) \\ \qquad +\delta_{2k}(x)+c_k(x))\varphi_{3k} + \gamma_k(x)\varphi_{4k} = \lambda\varphi_{3k}, \\ \nabla\cdot(d_{4k}(x)\nabla\varphi_{4k}) + c_k(x)\varphi_{3k} - (\gamma_k(x) \\ \qquad +\mu_k(x))\varphi_{4k} = \lambda\varphi_{4k}, \\ \dfrac{\partial\varphi_{2k}}{\partial\nu} = \dfrac{\partial\varphi_{3k}}{\partial\nu} = \dfrac{\partial\varphi_{4k}}{\partial\nu} = 0, \quad x\in\partial\Omega. \end{cases} \tag{7}$$

By the Krein–Rutman theorem (Du, 2006, Theorem 1.2), the eigenvalue problem (7) has a unique principal eigenvalue λ_0. Furthermore, according to Theorem 3.1 of Wang and Zhao (2012), we get the following result.

Theorem 2.

(i) $R_0 - 1$ *has the same sign as the principal eigenvalue.*

(ii) If $R_0 < 1$*, then the disease-free steady state* $E_0(x)$ *is locally asymptotically stable for model (1).*

(iii) If $R_0 > 1$*, then* $E_0(x)$ *is unstable.*

The biological meaning of the basic reproduction number R_0 is the effective number of secondary infections caused by a typical infectious individual during his entire period of infectiousness. From Theorem 2, we get that when an infectious individual spreads the disease to more than one susceptible individual over his expected lifetime, the disease will reach a disease-free steady state which means the disease will be eliminated in local time. In turn, when an infectious individual spreads the disease to less than one susceptible individual over his expected lifetime, the disease-free steady state is unstable and the disease will not be eliminated.

4. Asymptotic behaviors

4.1. Stability of disease-free steady state.

Theorem 3. *If* $R_0 < 1$*, then the disease-free steady state* $E_0(x)$ *of model (1) is globally asymptotically stable in* X_+*.*

Proof. We denote by $(\eta_2(x), \eta_3(x), \eta_4(x))$ the eigenfunction corresponding to the principal eigenvalue λ_0 associated with the eigenvalue problem (7), where $\eta_2 = (\eta_{21}, \eta_{22}, \dots, \eta_{2n})$, $\eta_3 = (\eta_{31}, \eta_{32}, \dots, \eta_{3n})$, $\eta_4 = (\eta_{41}, \eta_{42}, \dots, \eta_{4n})$. Define the functional

$$L(\mathbf{u}) = \int_\Omega \sum_{k=1}^n (\eta_{2k}E_k + \eta_{3k}I_k + \eta_{4k}R_k)\,\mathrm{d}x.$$

Now we prove that $L(\mathbf{u})$ is a Lyapunov functional for system (1). For an arbitrary solution $\mathbf{u} = (S, E, I, R)$ of system (1), we have

$$\begin{aligned} \frac{\mathrm{d}L(\mathbf{u})}{\mathrm{d}t} &= \int_\Omega \sum_{k=1}^n \Big\{ \eta_{2k}\big[\nabla\cdot(d_{2k}(x)\nabla E_k) \\ &\quad + \sum_{j=1}^n \beta_{kj}(x)S_kI_j \\ &\quad - (\sigma_k(x)+\mu_k(x)+\delta_{1k}(x))E_k\big] \\ &\quad + \eta_{3k}\big[\nabla\cdot(d_{3k}(x)\nabla I_k) + \sigma_k(x)E_k \\ &\quad - (\mu_k(x)+\delta_{2k}(x)+c_k(x))I_k + \gamma_k(x)R_k\big] \\ &\quad + \eta_{4k}\big[\nabla\cdot(d_{4k}(x)\nabla R_k) + c_k(x)I_k \\ &\quad - (\gamma_k(x)+\mu_k(x))R_k\big]\Big\}\,\mathrm{d}x \\ &= \int_\Omega \sum_{k=1}^n \Big[(S_k - S_k^0(x))\sum_{j=1}^n \beta_{kj}(x)I_j \\ &\quad + \lambda_0(\eta_{2k}E_k + \eta_{3k}I_k + \eta_{4k}R_k)\Big]\,\mathrm{d}x. \end{aligned} \tag{8}$$

From the first equation of model (1), we have

$$\begin{cases} \dfrac{\partial S_k}{\partial t} \le \nabla\cdot(d_{1k}(x)\nabla S_k) + \Lambda_k(x) \\ \qquad - \mu_k(x)S_k, \quad x\in\Omega, \quad t>0, \\ \dfrac{\partial S_k}{\partial\nu} = 0, \quad x\in\partial\Omega, \quad t>0. \end{cases}$$

By the standard parabolic comparison theorem (Smith, 1995, Theorem 7.3.4) and Lemma 1,

$$\limsup_{t\to\infty} S_k \le S_k^0(x)$$

uniformly for $x\in\overline{\Omega}$, $\quad 1\le k\le n$.

Within loss of generality, we assume that $S_k \le S_k^0(x)(1\le k\le n)$ for all $t>0$ and $x\in\Omega$. By Theorem 2, $R_0<1$ yields $\lambda_0<0$. Besides, S_k, E_k, I_k, R_k and β_{kj}, η_{2k}, η_{3k}, η_{4k} are positive for all $1\le k,j\le n$, which implies $\mathrm{d}L(\mathbf{u})/\mathrm{d}t<0$.

Next define

$$\dot{L}_0(\mathbf{u}) := \left.\frac{\mathrm{d}L(\mathbf{u})}{\mathrm{d}t}\right|_{t=0},$$

$$\mathcal{M} = \{\phi \in X_+ : \dot{L}_0(\mathbf{u}) = 0\},$$

where $\phi = (\phi_1, \phi_2, \phi_3, \phi_4) \in X_+$ is the initial function of (1). By (8), we have $\mathcal{M} = \{(\phi_1, \phi_2, \phi_3, \phi_4) \in X_+ : \phi_{2i} = \phi_{3i} = \phi_{4i} = 0, 1 \leq i \leq n\}$ if $\lambda_0 < 0$. It follows from (1) that for $\lambda_0 < 0$, the maximal invariant set in $\mathcal{M}$ is given by

$$\hat{\mathcal{M}} := \{(\phi_1, \phi_2, \phi_3, \phi_4) \in X_+ : \phi_{2i} = \phi_{3i} = \phi_{4i} = 0, 1 \leq i \leq n\}.$$

Therefore, by the LaSalle invariant principle (Hale, 1969, Theorem 1), we obtain

$$(E_k, I_k, R_k) \to (0, 0, 0) \quad \text{as } t \to \infty.$$

Then we obtain $S_k \to S_k^0(x)$ as $t \to \infty$. ∎

4.2. Uniform persistence.

Theorem 4. *If $R_0 > 1$, there exists a constant ϵ_0 such that for any initial value $\phi = (\phi_1, \phi_2, \phi_3, \phi_4) \in X_+$ with $\phi_{2i} \neq 0$ and $\phi_{3i} \neq 0$ $(1 \leq i \leq n)$, solution $\mathbf{u}(t, x, \phi) = (S(t, x, \phi), E(t, x, \phi), I(t, x, \phi), R(t, x, \phi))$ of model (1) satisfies*

$$\liminf_{t\to\infty} S_k(t, x, \phi) \geq \epsilon_0,$$
$$\liminf_{t\to\infty} E_k(t, x, \phi) \geq \epsilon_0,$$
$$\liminf_{t\to\infty} I_k(t, x, \phi) \geq \epsilon_0,$$
$$\liminf_{t\to\infty} R_k(t, x, \phi) \geq \epsilon_0,$$

uniformly for $x \in \overline{\Omega}$ and $1 \leq k \leq n$. Moreover, system (1) admits at least one endemic equilibrium.

Proof. Firstly, we show the uniform persistence of $S_k(t, x, \phi)$. The ultimate boundedness obtained in Theorem 1 implies that there exists a constant $M > 0$, for any solution $(S(t, x, \phi), E(t, x, \phi), I(t, x, \phi), R(t, x, \phi))$ of model (1), and there exists $t_0 > 0$ such that $I_j(t, x, \phi) \leq M$ for all $x \in \overline{\Omega}$, $t \geq t_0$ and $1 \geq j \geq n$. Thus, from the first equation of model (1) we have

$$\frac{\partial S_k}{\partial t} \geq \nabla \cdot (d_{1k}(x)\nabla S_k) + \Lambda_k(x) - (\mu_k(x) + M\sum_{j=1}^{n}\beta_{kj}(x))S_k,$$

for any $t \geq t_0$, which implies that

$$\liminf_{t\to\infty} S_k(t, x, \phi) \geq \frac{\underline{\Lambda}_k}{\overline{\mu}_k + M\sum_{j=1}^{n}\overline{\beta}_{kj}},$$

by Lemma 1 and the standard parabolic comparison principle (Smith, 1995, Theorem 7.3.4).

Define

$$\mathcal{W} = \{(\phi_1, \phi_2, \phi_3, \phi_4) \in X_+ : \phi_{2i} \neq 0 \text{ and } \phi_{3i} \neq 0 \text{ and } \phi_{4i} \neq 0, 1 \leq i \leq n\},$$

and

$$\partial\mathcal{W} = \{(\phi_1, \phi_2, \phi_3, \phi_4) \in X_+ : \phi_{2i} = 0 \text{ or } \phi_{3i} = 0 \text{ or } \phi_{4i} = 0, 1 \leq i \leq n\}.$$

Clearly, $\mathcal{W} \bigcup \partial\mathcal{W} = X_+$. From the strong maximum principle for parabolic equations, we have $\Phi(t)\mathcal{W} \subset \mathcal{W}$ for all $t > 0$, where $\Phi(t)$ is the solution semiflow of (1). Write

$$\mathcal{M}_\partial = \{\phi = (\phi_1, \phi_2, \phi_3, \phi_4) \in X_+ : \Phi(t)\phi \in \partial\mathcal{W}, t \geq 0\}.$$

It is easy to verify that $\mathcal{M}_\partial = \{\phi = (\phi_1, \phi_2, \phi_3, \phi_4) \in X_+ : \phi_{2i} = \phi_{3i} = \phi_{4i} = 0, 1 \leq i \leq n\}$.

Let $\omega(\phi)$ be the ω-limit set of $\Phi(t)\phi$ and

$$\hat{\mathcal{M}}_\partial = \bigcup_{\phi\in\mathcal{M}_\partial} \omega(\phi).$$

Then, we prove that $\hat{\mathcal{M}}_\partial = \{E_0(x)\}$. For any $\phi \in \mathcal{M}_\partial$, i.e., $\Phi(t)\phi \in \partial\mathcal{W}$ for all $t \geq 0$, we have $E_k(t, x, \phi) = I_k(t, x, \phi) = R_k(t, x, \phi) = 0$ for all $x \in \overline{\Omega}$, $t \geq 0$ and $1 \leq k \leq n$. From model (1), we have the following subsystem:

$$\begin{cases} \dfrac{\partial S_k}{\partial t} = \nabla \cdot (d_{1k}(x)\nabla S_k) + \Lambda_k(x) - \mu_k(x)S_k, \quad x \in \Omega, \quad t > 0, \\ \dfrac{\partial S_k}{\partial \nu} = 0, \quad x \in \partial\Omega, \quad t > 0. \end{cases}$$

By Lemma 1, we have $\lim_{t\to\infty} S_k(t, x, \phi) \to S_k^0(x)$ uniformly for $x \in \overline{\Omega}$. Hence, $\hat{\mathcal{M}}_\partial = \{E_0(x)\}$. Therefore, $\{E_0(x)\}$ is an isolated invariant set for $\Phi(t)$ restricted in $\mathcal{M}_\partial$. Next, we show that there exists some constant ϵ independent of initial values such that

$$\limsup_{t\to\infty} |\Phi(t)\phi - E_0(x)|_X \geq \epsilon, \quad \forall\phi \in \mathcal{W}. \tag{9}$$

From $R_0 > 1$ and Theorem 2, we have $\lambda_0 > 0$, where λ_0 is the principal eigenvalue of the eigenvalue problem (7). Consider the following eigenvalue problem:

$$\begin{cases} \nabla \cdot (d_{2k}(x)\nabla\xi_{2k}) + (S_k^0(x) - \epsilon)\sum_{j=1}^{n}\beta_{kj}(x)\xi_{3j} \\ \qquad -(\sigma_k(x) + \mu_k(x) + \delta_{1k}(x))\xi_{2k} = \lambda\xi_{2k}, \\ \nabla \cdot (d_{3k}(x)\nabla\xi_{3k}) + \sigma_k(x)\xi_{2k} - (\mu_k(x) \\ \qquad +\delta_{2k}(x) + c_k(x))\xi_{3k} + \gamma_k(x)\xi_{4k} = \lambda\xi_{3k}, \\ \nabla \cdot (d_{4k}(x)\nabla\xi_{4k}) + c_k(x)\xi_{3k} - (\gamma_k(x) \\ \qquad +\mu_k(x))\xi_{4k} = \lambda\xi_{4k}, \\ \dfrac{\partial\xi_{2k}}{\partial\nu} = \dfrac{\partial\xi_{3k}}{\partial\nu} = \dfrac{\partial\xi_{4k}}{\partial\nu} = 0, \quad x \in \partial\Omega, \end{cases}$$

and denote by $\lambda_0(\epsilon)$ its principle eigenvalue. Note that $\lim_{\epsilon\to 0}\lambda_0(\epsilon)=\lambda_0>0$.

Then there exists a sufficiently small constant $\epsilon_1>0$ such that $\lambda_0(\epsilon_1)>0$ and $S_k^0(x)-\epsilon_1>0$ for all $x\in\overline{\Omega}$. Moreover, from the Krein–Rutman theorem (Du, 2006, Theorem 1.2), the eigenfunction $(\xi_2(x),\xi_3(x),\xi_4(x))$ corresponding to $\lambda_0(\epsilon_1)$ is also strictly positive for $x\in\overline{\Omega}$.

Contrary to (9), we assume that there exists some initial value ϕ^* such that

$$\limsup_{t\to\infty}|\Phi(t)\phi^*-E_0(x)|_X<\epsilon_1.$$

Then there exists a sufficiently large t_1 such that

$$S_k(t,x,\phi^*)>S_k^0(x)-\epsilon_1$$

for all $x\in\overline{\Omega}$, $t>t_1$ and $1\le k\le n$. By the strong maximum principle of parabolic equations,

$$E_k(t,x,\phi^*),I_k(t,x,\phi^*),R_k(t,x,\phi^*)>0$$

for all $t>0$. Then we can find a small positive constant c_0 such that $E_k(t_1,x,\phi^*)\ge c_0\xi_{2k}(x)$, $I_k(t_1,x,\phi^*)\ge c_0\xi_{3k}(x)$ and $R_k(t_1,x,\phi^*)\ge c_0\xi_{4k}(x)$ for $x\in\overline{\Omega}$. It is easy to verify that $(E_k(t,x,\phi^*),I_k(t,x,\phi^*),R_k(t,x,\phi^*))$ is a supersolution of the problem

$$\begin{cases}\dfrac{\partial E_k}{\partial t}=\nabla\cdot(d_{2k}(x)\nabla E_k)+(S_k^0(x)-\epsilon_1)\\ \qquad\times\displaystyle\sum_{j=1}^{n}\beta_{kj}(x)I_j-(\sigma_k(x)+\mu_k(x)\\ \qquad+\delta_{1k}(x))E_k,\\ \dfrac{\partial I_k}{\partial t}=\nabla\cdot(d_{3k}(x)\nabla I_k)+\sigma_k(x)E_k-(\mu_k(x)\\ \qquad+\delta_{2k}(x)+c_k(x))I_k+\gamma_k(x)R_k,\\ \dfrac{\partial R_k}{\partial t}=\nabla\cdot(d_{4k}(x)\nabla R_k)+c_k(x)I_k-(\gamma_k(x)\\ \qquad+\mu_k(x))R_k,\\ \dfrac{\partial E_k}{\partial\nu}=\dfrac{\partial I_k}{\partial\nu}=\dfrac{\partial R_k}{\partial\nu}=0,\quad x\in\partial\Omega,\end{cases}\tag{10}$$

for $t>t_1$ and

$$E_k(t_1,x)=c_0\xi_{2k}(x),$$
$$I_k(t_1,x)=c_0\xi_{3k}(x),$$
$$R_k(t_1,x)=c_0\xi_{4k}(x),$$

where

$$(c_0e^{\lambda_0(\epsilon_1)(t-t_1)}\xi_2(x),c_0e^{\lambda_0(\epsilon_1)(t-t_1)}\xi_3(x),\ c_0e^{\lambda_0(\epsilon_1)(t-t_1)}\xi_4(x))$$

is the unique solution to system (10). Note that $\lambda_0(\epsilon_1)>0$; therefore,

$$E_k(t,x,\phi^*)\ge c_0e^{\lambda_0(\epsilon_1)(t-t_1)}\xi_{2k}(x),$$

$$I_k(t,x,\phi^*)\ge c_0e^{\lambda_0(\epsilon_1)(t-t_1)}\xi_{3k}(x),$$

$$R_k(t,x,\phi^*)\ge c_0e^{\lambda_0(\epsilon_1)(t-t_1)}\xi_{4k}(x)\to\infty$$

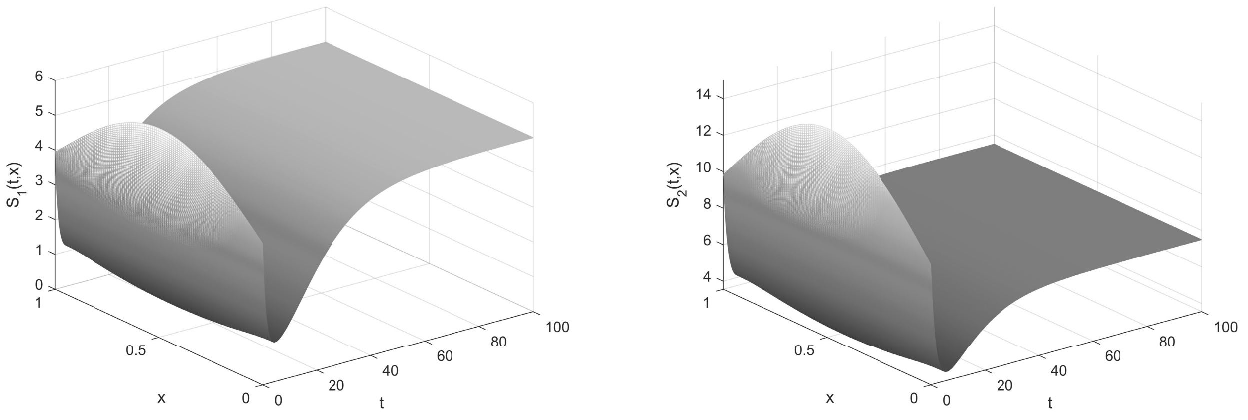
uniformly in $\overline{\Omega}$ as $t\to\infty$. This contradiction completes the proof of (9), which implies that $W^S(\{E_0(x)\})\bigcap\mathcal{W}$ is an empty set, where $W^S(\{E_0(x)\})$ is the stable set of $\{E_0(x)\}$ for $\Phi(t)$. Then, from Theorem 1.3.1 of Zhao (2003) and together with the fact that $\{E_0(x)\}$ is an isolated invariant set for $\Phi(t)$ in X_+, $\Phi(t)$ is uniformly persistent. Moreover, by Theorem 1.3.7 of Zhao (2003), (1) admits at least one endemic equilibrium. ■

5. Numerical simulations

To proceed, we implement numerical simulations in order to demonstrate our theoretical findings and explore the effect of population diffusion and environmental heterogeneity on disease transmission. By adopting the finite difference method to solve sets of partial differential equations in Matlab, graphs of the solution to model (1) are drawn as the numerical results in this section. We focus on the following 2-group SEIR epidemic model with spatial diffusion in a heterogeneous environment, which is a special case of system (1)

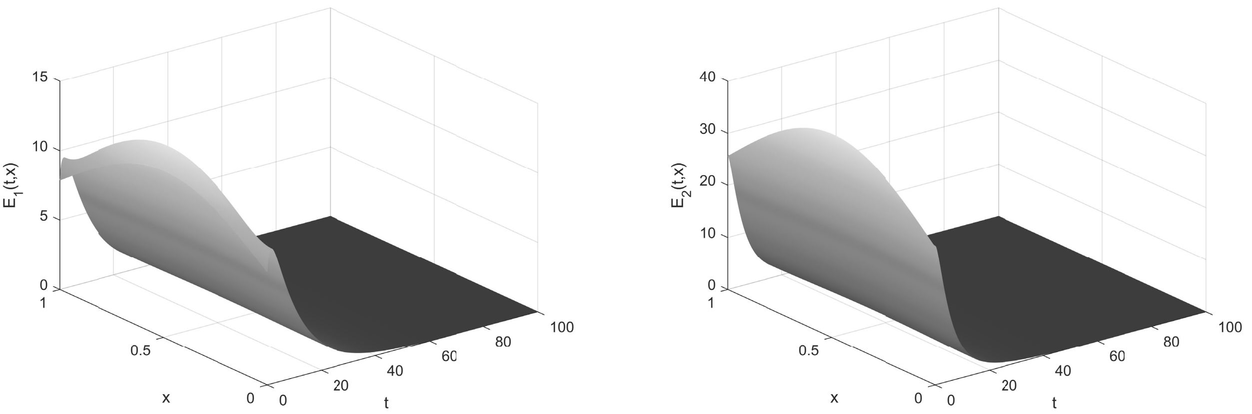
$$\begin{cases}\dfrac{\partial S_1}{\partial t}=\nabla\cdot(d_{11}(x)\nabla S_1)+\Lambda_1(x)-(\mu_1(x)\\ \qquad+\beta_{11}(x)I_1+\beta_{12}(x)I_2)S_1,\\ \dfrac{\partial S_2}{\partial t}=\nabla\cdot(d_{12}(x)\nabla S_2)+\Lambda_2(x)-(\mu_2(x)\\ \qquad+\beta_{21}(x)I_1+\beta_{22}(x)I_2)S_2,\\ \dfrac{\partial E_1}{\partial t}=\nabla\cdot(d_{21}(x)\nabla E_1)+\beta_{11}(x)S_1I_1+\beta_{12}(x)\\ \qquad\times S_1I_2-(\sigma_1(x)+\mu_1(x)+\delta_{11}(x))E_1,\\ \dfrac{\partial E_2}{\partial t}=\nabla\cdot(d_{22}(x)\nabla E_2)+\beta_{21}(x)S_2I_1+\beta_{22}(x)\\ \qquad\times S_2I_2-(\sigma_2(x)+\mu_2(x)+\delta_{12}(x))E_2,\\ \dfrac{\partial I_1}{\partial t}=\nabla\cdot(d_{31}(x)\nabla I_1)+\sigma_1(x)E_1-(\mu_1(x)\\ \qquad+\delta_{21}(x)+c_1(x))I_1+\gamma_1(x)R_1,\\ \dfrac{\partial I_2}{\partial t}=\nabla\cdot(d_{32}(x)\nabla I_2)+\sigma_2(x)E_2-(\mu_2(x)\\ \qquad+\delta_{22}(x)+c_2(x))I_2+\gamma_2(x)R_2,\\ \dfrac{\partial R_1}{\partial t}=\nabla\cdot(d_{41}(x)\nabla R_1)+c_1(x)I_1-(\gamma_1(x)\\ \qquad+\mu_1(x))R_1,\\ \dfrac{\partial R_2}{\partial t}=\nabla\cdot(d_{42}(x)\nabla R_2)+c_2(x)I_2-(\gamma_2(x)\\ \qquad+\mu_2(x))R_2.\end{cases}\tag{11}$$

For the sake of convenience, in both the cases, we



(a) state trajectories for $S_1(t,x)$ (b) state trajectories for $S_2(t,x)$

Fig. 1. State trajectories for $R_0 < 1$.



(a) state trajectories for $E_1(t,x)$ (b) state trajectories for $E_2(t,x)$

Fig. 2. State trajectories for $R_0 < 1$.

take $\Omega = (0,1)$ and the initial functions are given by

$$\begin{aligned} S_1(0,x) &= 2 \cdot (2 + \sin \pi x), \\ S_2(0,x) &= 5 \cdot (2 + \sin \pi x), \\ E_1(0,x) &= 4 \cdot (2 + \sin \pi x), \\ E_2(0,x) &= 13 \cdot (2 + \sin \pi x), \\ I_1(0,x) &= 3 \cdot (2 + \sin \pi x), \\ I_2(0,x) &= 2 \cdot (2 + \sin \pi x), \\ R_1(0,x) &= 20 \cdot (2 + \sin \pi x), \\ R_2(0,x) &= 10 \cdot (2 + \sin \pi x). \end{aligned}$$

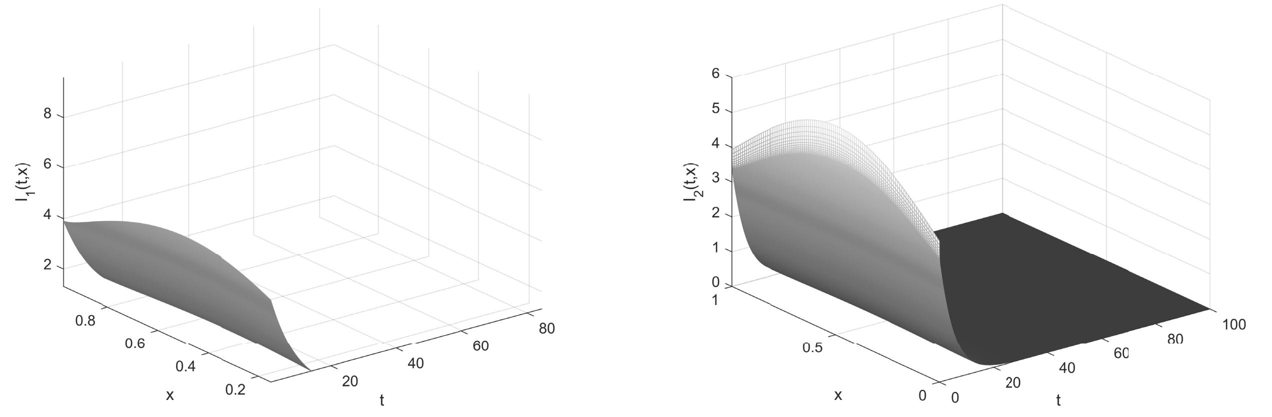
In addition, we fix some parameters of the model as follows

$$\begin{aligned} d_{11}(x) &= 0.3 \cdot 10^{-3} x, & d_{21}(x) &= 0.2 \cdot 10^{-2} x, \\ d_{12}(x) &= 0.1 \cdot 10^{-3} x, & d_{22}(x) &= 0.1 \cdot 10^{-2} x, \end{aligned}$$

$$\begin{aligned} d_{31}(x) &= 0.3 \cdot 10^{-2} x, & d_{41}(x) &= 0.5 \cdot 10^{-3} x, \\ d_{32}(x) &= 0.2 \cdot 10^{-2} x, & d_{42}(x) &= 0.6 \cdot 10^{-3} x, \end{aligned}$$

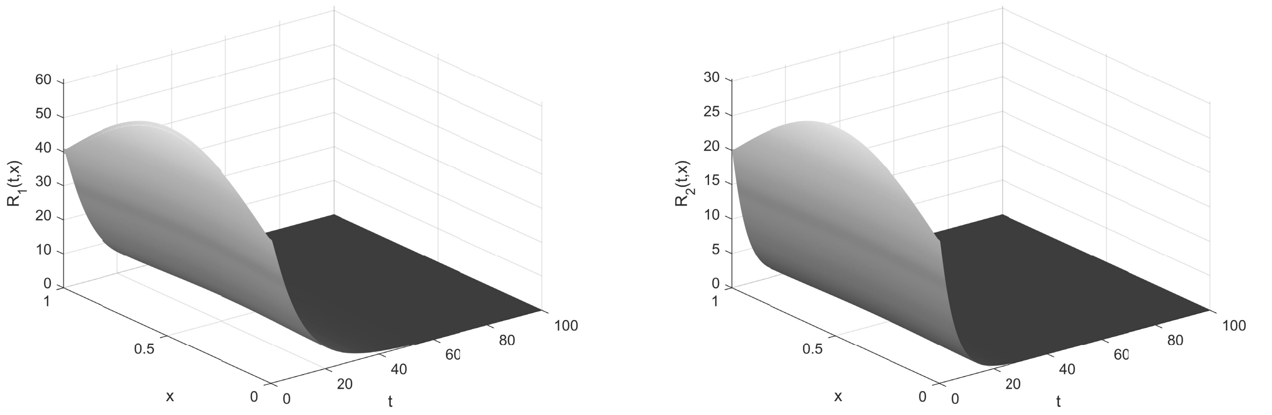
and

$$\begin{aligned} \Lambda_1(x) &= 1 + 0.5 \cos \pi x, \\ \Lambda_2(x) &= 2 + 0.5 \cos \pi x, \end{aligned}$$



(a) state trajectories for $I_1(t,x)$ (b) state trajectories for $I_2(t,x)$

Fig. 3. Case of $R_0 < 1$, state trajectories for $I_1(t,x)$ and $I_2(t,x)$.



(a) state trajectories for $R_1(t,x)$ (b) state trajectories for $R_2(t,x)$

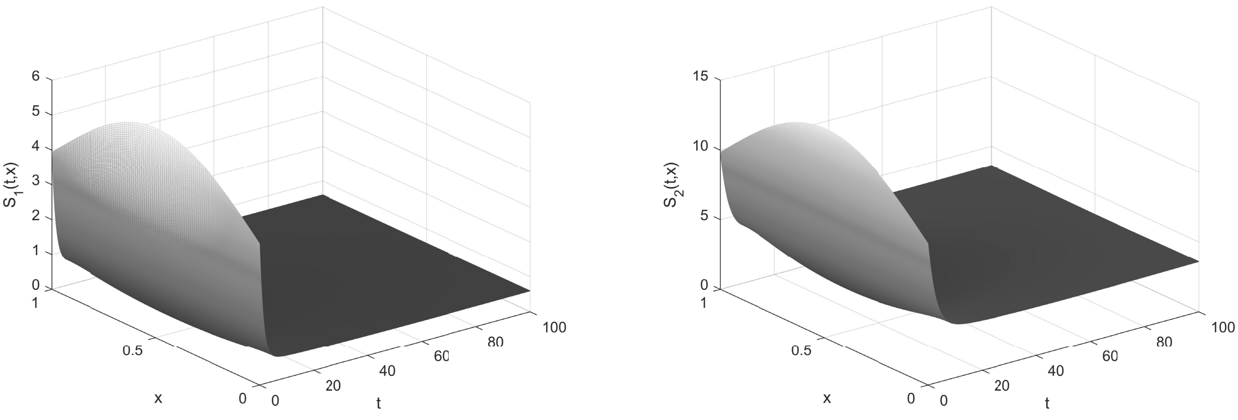
Fig. 4. Case of $R_0 < 1$, state trajectories for $R_1(t,x)$ and $R_2(t,x)$.

$$
\begin{aligned}
\sigma_1(x) &= 0.1 \cdot 10^{-3}(1 + 0.3\cos \pi x),\\
\sigma_2(x) &= 0.2 \cdot 10^{-3}(1 + 0.2\cos \pi x),\\
\beta_{11}(x) &= 0.3 \cdot 10^{-1}(1 + 0.2\cos \pi x),\\
\beta_{21}(x) &= 0.2 \cdot 10^{-1}(2 + 0.1\cos \pi x),\\
\beta_{12}(x) &= 0.9 \cdot 10^{-1}(2 + 0.1\cos \pi x),\\
\beta_{22}(x) &= 0.6 \cdot 10^{-1}(1 + 0.1\cos \pi x),\\
\delta_{11}(x) &= 0.5 \cdot 10^{-1}(1 + 0.2\cos \pi x),\\
\delta_{21}(x) &= 0.4 \cdot 10^{-1}(1 + 0.6\cos \pi x),\\
\delta_{12}(x) &= 0.4 \cdot 10^{-1}(1 + 0.1\cos \pi x),\\
\delta_{22}(x) &= 0.5 \cdot 10^{-1}(1 + 0.9\cos \pi x),\\
c_1(x) &= 75 \cdot (1 + 0.8\cos \pi x),\\
c_2(x) &= 55 \cdot (1 + 0.7\cos \pi x),\\
\gamma_1(x) &= 2 \cdot (1 + 0.2\cos \pi x),\\
\gamma_2(x) &= 3 \cdot (1 + 0.1\cos \pi x).
\end{aligned}
$$

Example 1. We choose $\mu_1(x) = 0.2 \cdot (1 + 0.5\cos \pi x)$ and $\mu_2(x) = 0.4 \cdot (1 + 0.5\cos \pi x)$; then $R_0 < 1$. Hence, system (1) is globally asymptotically stable, and all solutions of the system converge to the disease-free steady state $E_0(x)$, see Figs. 1–4. ♦

From Theorem 3, the disease-free steady state $E_0(x)$ of model (1) is globally asymptotically stable if $R_0 < 1$. Figures 1–4 show that the solutions for model (1) tend to a disease-free steady state, i.e., the infectious disease extincts. Hence, Example 1 in which parameters meet the condition $R_0 < 1$ is an illustration of Theorem 3.

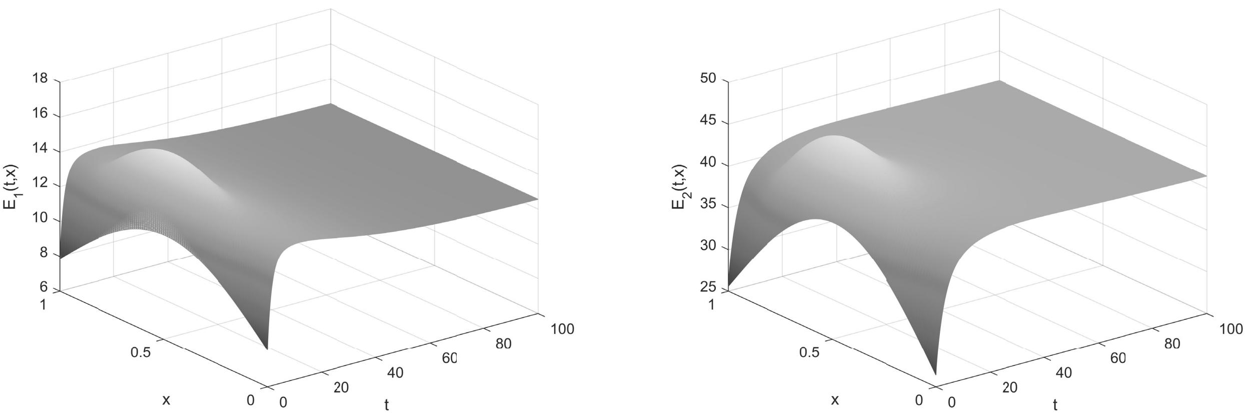
Example 2. We choose $\mu_1(x) = 0.001\cdot(0.4+0.5\cos \pi x)$



(a) state trajectories for $S_1(t, x)$

(b) state trajectories for $S_2(t, x)$

Fig. 5. Case of $R_0 > 1$, state trajectories for $S_1(t, x)$ and $S_2(t, x)$.



(a) state trajectories for $E_1(t, x)$

(b) state trajectories for $E_2(t, x)$

Fig. 6. Case of $R_0 > 1$, state trajectories for $E_1(t, x)$ and $E_2(t, x)$.

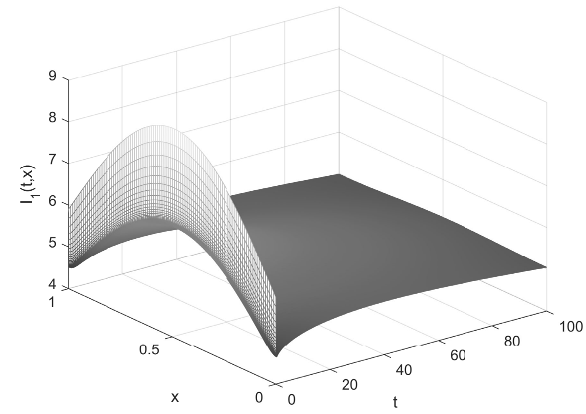
and $\mu_2(x) = 0.25 \cdot (0.5 + 0.5 \cos \pi x)$; then $R_0 > 1$. Hence, system (1) is uniformly persistent, see Figs. 5–8.

Theorem 4 shows that system (1) is uniformly persistent and admits at least one endemic equilibrium if $R_0 > 1$. It can be seen from Figs. 5–8 that the solutions of model (1) tend to an endemic steady state, i.e., the infectious disease will break out. Therefore, Example 1 in which parameters meet the condition $R_0 > 1$ demonstrates the validity of Theorem 4. ♦

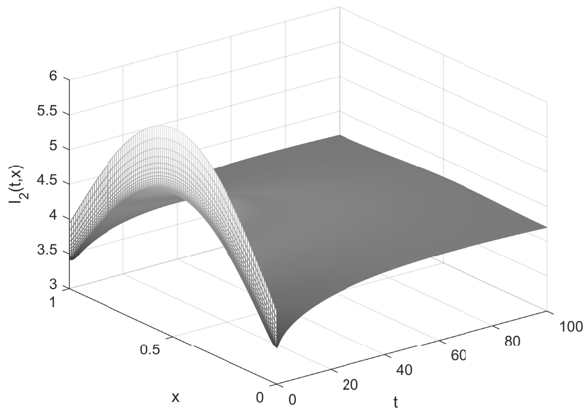
6. Conclusion

Treating the environmental heterogeneity as an important factor in disease dynamics may not only give insights into disease spread and control in reality, but also suggest new aspects and considerations for modeling spatial-temporal dynamics of infectious diseases. In order to understand the impact of the spatial heterogeneity of the environment and the movement of individuals on the spread of infectious diseases in an analytical aspect, in this paper, we incorporate the diffusion terms and environmental heterogeneity into the classical SEIR epidemic model to drive a multi-group SEIR epidemic model with spatial diffusion in a heterogeneous environment. For this model, the basic reproduction number R_0 is defined by applying the next-generation operator.

With the help of the comparison principle of reaction-diffusion equations, global asymptotical stability of the disease-free steady state $E_0(x)$ is proved when $R_0 < 1$, which means that the disease will extinct. Moreover, when $R_0 > 1$, the uniform persistence of the model and the existence of endemic steady state are

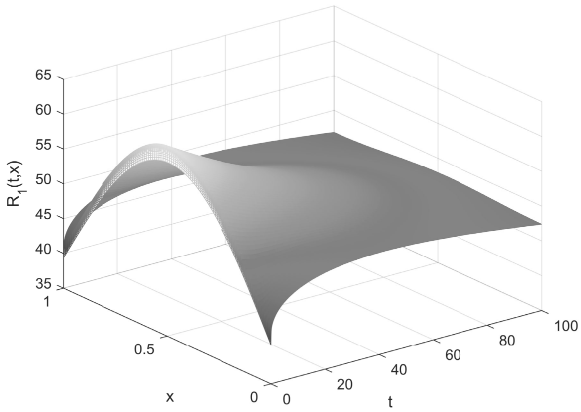


(a) state trajectories for $I_1(t, x)$

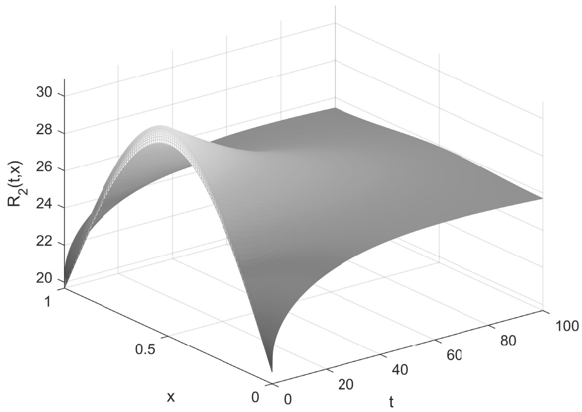


(b) state trajectories for $I_2(t, x)$

Fig. 7. Case of $R_0 > 1$, state trajectories for $I_1(t, x)$ and $I_2(t, x)$.



(a) state trajectories for $R_1(t, x)$



(b) state trajectories for $R_2(t, x)$

Fig. 8. Case of $R_0 > 1$, state trajectories for $R_1(t, x)$ and $R_2(t, x)$.

obtained by using the theory of persistence of dynamical systems. However, it is difficult to establish a criterion for the global asymptotic stability of the endemic equilibrium when $R_0 > 1$ for model (1) in spatially heterogeneous environments. We leave this as an open problem.

Acknowledgment

The authors would like to thank the anonymous referees and the editor for their valuable comments which can improve the presentation and the quality of the earlier version of this paper. This work was partly supported by the National Natural Science Foundation of China (grant no. 11971329) and partly by the Scientific Research Project of the Tianjin Municipal Education Commission (grant no. 2021KJ007).

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Received: 14 November 2021
Revised: 16 March 2022
Accepted: 18 March 2022