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FULLY DISCRETE APPROXIMATIONS AND AN *A PRIORI* ERROR ANALYSIS OF A TWO–TEMPERATURE THERMO–ELASTIC MODEL WITH MICROTEMPERATURES

JACOBO BALDONEDO [a], JOSÉ R. FERNÁNDEZ [b,*], RAMÓN QUINTANILLA [c]

[a]Research Center in Technologies, Energy and Industrial Processes (CINTECX)
University of Vigo
Campus As Lagoas Marcosende s/n, 36310 Vigo, Spain
e-mail: jacobo.gonzalez.baldonedo@uvigo.es

[b]Department Applied Mathematics
Higher School of Telecommunications Engineering
Campus As Lagoas Marcosende s/n, 36310 Vigo, Spain
e-mail: jose.fernandez@uvigo.es

[c]Department of Mathematics, ESEIAAT
Polytechnic University of Catalonia
Colom 11, 08222 Terrassa, Barcelona, Spain
e-mail: ramon.quintanilla@upc.edu

In this paper, we consider, from a numerical point of view, a two-temperature poro-thermoelastic problem. The model is written as a coupled linear system of hyperbolic and elliptic partial differential equations. An existence result is proved and energy decay properties are recalled. Then we introduce a fully discrete approximation by using the finite element method and the implicit Euler scheme. Some *a priori* error estimates are obtained, from which the linear convergence of the approximation is deduced under an appropriate additional regularity. Finally, some numerical simulations are performed to demonstrate the accuracy of the approximation, the decay of the discrete energy and the behaviour of the solution depending on a constitutive parameter.

Keywords: two temperatures, poro-thermoelasticity, microtemperatures, finite elements, *a priori* error estimates.

1. Introduction

It is usually accepted that porous elasticity (also known as elasticity with voids) is the easiest generalization of the classical theory of elasticity (Cowin, 1985; Cowin and Nunziato, 1983; Nunziato and Cowin, 1979). In this situation, the existence of a skeleton where we can consider several holes (or voids) of the material is assumed. The existence of these voids implies an interdependence between the macrostructure and the microstructure of the material. In general, several theories (such as micropolar or micromorphic elasticity) have been developed trying to incorporate microstructural effects to understand the behaviour of different materials. The elasticity with voids has gained much interest over the last fifty years (Barabasz *et al.*, 2014; Feng and Apalara, 2019; Feng and Yin, 2019; Leseduarte *et al.*, 2010; Magaña and Quintanilla, 2021; Magaña *et al.*, 2020; Miranville and Quintanilla, 2019; 2020; Pamplona *et al.*, 2011). Of course, this theory has been extended to incorporate thermal effects.

One of the possibilities to introduce microstructural effects on the materials can be by means of microtemperatures. The theory of microtemperatures was firstly considered by Grot (1969) and Riha (1975; 1976). However, little attention was paid until the beginning of this century (Ieşan, 2007; Ieşan and Quintanilla, 2000). These two contributions were a starting point trying to understand the relevance of

*Corresponding author

the microtemperatures in the behaviour of the materials (see, e.g., Bazarra *et al*., 2019; Grot, 1969; Magaña and Quintanilla, 2018; Passarella *et al*., 2017) .

In the period between 1968 and 1973, Gurtin and several co-workers proposed and developed the so-called two-temperature thermoelasticity (Chen and Gurtin, 1968; Chen *et al*., 1968; 1969; Warren and Chen, 1973). In this theory, the heat equation is modified and two different temperatures (thermodynamical and inductive) are considered. This theory has been thoroughly studied (see, e.g., Abo-Dahab, 2020; Ali and Romano, 2017; D'Apice *et al*., 2020; Bazarra *et al*., 2020; Campo *et al*., 2022; Fernández and Quintanilla, 2021b; Gruais and Poliševski, 2017; Kumar *et al*., 2020; Makki *et al*., 2019; 2021; Miranville and Quintanilla, 2016; Mukhopadhyay *et al*., 2017; Sarkar and Mondal, 2020; Sellitto *et al*., 2021; Youssef and Elsibai, 2015).

Recently, Fernández and Quintanilla (2021a) proposed how to obtain a theory of porous thermoelasticity with two temperatures and microtemperatures for the one-dimensional case. The usual theory of thermoelasticity with microtemperatures was conveniently modified. Assumptions on the constitutive coefficients were imposed to guarantee several qualitative properties. In fact, existence, uniqueness and exponential decay of the solutions were obtained. To arrive at these results, the authors used the semigroup theory of linear operators as well as the characterization of exponentially stable semigroups obtained by Huang (also Pruss), recalled in the book by Liu and Zheng (1999).

In this paper, we want to continue the study of this theory, but from a numerical point of view. In this sense, a fully discrete approximation is introduced by using the classical finite element method for the spatial approximation and the implicit Euler scheme to discretize the time derivatives. *A priori* error estimates are proved from which the linear convergence is shown under some adequate additional regularity conditions. Finally, some numerical simulations are performed to demonstrate the accuracy of the approximation, the discrete energy decay and the behaviour of the solution with respect to some constitutive coefficients.

2. Model

Let u, ϕ, T, S, θ and ϑ be the displacement field, the porosity (or volume fraction), the thermodynamic microtemperature, the inductive microtemperature, the thermodynamic temperature and the inductive temperature, respectively. We note that the temperatures and microtemperatures satisfy the relations

$$\theta = \vartheta - \alpha\vartheta_{xx}, \quad T = S - \alpha S_{xx},$$

where α is a given positive constant. We denote by $(0,\ell)$ the one-dimensional domain occupied by the body, and we will study its deformation over the time interval $(0, T_f)$, with $T_f > 0$.

Therefore, the thermomechanical problem of a one-dimensional poro-elastic rod with two temperatures and microtemperatures is written in the following form (see Fernández and Quintanilla, 2021a).

Problem P. *Find the displacement field* $u : [0,\ell] \times [0,T_f] \to \mathbb{R}$*, the porosity field* $\phi : [0,\ell]\times[0,T_f] \to \mathbb{R}$*, the thermodynamic temperature* $\theta : [0,\ell] \times [0,T_f] \to \mathbb{R}$*, the inductive temperature* $\vartheta : [0,\ell]\times[0,T_f] \to \mathbb{R}$*, the thermodynamic microtemperature* $T : [0,\ell] \times [0,T_f] \to \mathbb{R}$ *and the inductive microtemperature* $S : [0,\ell] \times [0,T_f] \to \mathbb{R}$ *such that*

$$\begin{aligned}
\rho\ddot{u} &= \mu u_{xx} + \mu_0\phi_x - \beta_0\theta_x + \rho F,\\
J\ddot{\phi} &= a_0\phi_{xx} - \mu_0 u_x - \mu_2 T_x\\
&\quad + \beta_1\theta - \xi\phi + \rho L,\\
a\dot{\theta} &= -\beta_0\dot{u}_x - \beta_1\dot{\phi} + \kappa\vartheta_{xx}\\
&\quad + \kappa_1 S_x + \rho Q,\\
b\dot{T} &= -\mu_2\dot{\phi}_x + \kappa_4 S_{xx} - \kappa_2 S\\
&\quad - \kappa_3\vartheta_x - \rho G,\\
\theta &= \vartheta - \alpha\vartheta_{xx},\\
T &= S - \alpha S_{xx},
\end{aligned} \tag{1}$$

in $(0,\ell)\times(0,T_f)$,

$$\begin{aligned}
u(x,0) &= u^0(x), \quad \dot{u}(x,0) = v^0(x),\\
\theta(x,0) &= \theta^0(x), \quad \phi(x,0) = \phi^0(x),\\
\dot{\phi}(x,0) &= \varphi^0(x), \quad T(x,0) = T^0(x),
\end{aligned} \tag{2}$$

for a.e. $x \in (0,\ell)$,

$$\begin{aligned}
u(0,t) &= \phi(0,t) = \vartheta(0,t) = 0,\\
u(\ell,t) &= \phi(\ell,t) = \vartheta(\ell,t) = 0\\
S(0,t) &= S(\ell,t) = 0,
\end{aligned} \tag{3}$$

for a.e. $t \in (0,T_f)$.

We note that, in Problem P, u^0, v^0, ϕ^0, φ^0, θ^0 and T^0 are initial conditions for the variables, ρ, J, a, b, μ, μ_0, β_0, a_0, μ_2, β_1, ξ, κ, κ_1, κ_4, κ_2 and κ_3 are given positive constants, and F, L, Q and G are supply terms.

According to Fernández and Quintanilla (2021a) we will make the following assumptions on the constitutive coefficients:

$$\begin{gathered}
\rho > 0, \quad J > 0, \quad a > 0, \quad b > 0, \quad \mu > 0,\\
\mu\xi > \mu_0^2, \quad a_0 > 0, \quad \kappa > 0, \quad \kappa_4 > 0,\\
4\alpha\kappa\kappa_2\kappa_4 - \alpha\kappa_4(\kappa_1+\kappa_3)^2 - \alpha^2\kappa_2\kappa_3^2 > 0,\\
4\alpha\kappa(\kappa_4 + \alpha\kappa_2) > \alpha^2\kappa_1^2.
\end{gathered} \tag{4}$$

The following result was recently proved by Fernández and Quintanilla (2021a), and it states the existence of a unique solution to Problem P and an energy decay property.

Theorem 1. *Assume that the coefficients satisfy the conditions (4). If we denote by $(v, \varphi, \theta, \vartheta, T, S)$ the solution to Problem P and we suppose that the initial conditions have the following regularity:*

$$\begin{gathered} u^0,\ \phi^0 \in H_0^1(0,\ell), \\ v^0,\ T^0,\ \theta^0,\ \varphi^0 \in L^2(0,\ell), \\ \mu u_{xx}^0 - \beta_0\theta_x^0 \in L^2(0,\ell), \\ a_0\phi_{xx}^0 - \mu_2 T_x^0 \in L^2(0,\ell), \end{gathered}$$

then Problem P has a unique solution such that

$$\begin{gathered} u, \phi \in C^1([0,T_f]; H_0^1(0,\ell)) \cap C^2([0,T_f]; L^2(0,\ell)), \\ \theta, T \in C^1([0,T_f]; L^2(0,\ell)), \\ \vartheta, S \in C([0,T_f]; H_0^2(0,\ell)). \end{gathered}$$

Moreover, if we also assume that $\beta_0\mu_2 \neq 0$, then the energy decay of this solution is exponentially stable.

In order to obtain the variational formulation of the above thermomechanical problem, write $Y = L^2(0,\ell)$, $V = H_0^1(0,\ell)$ and $E = H_0^2(0,\ell)$. Moreover, let $(\cdot,\cdot)$ and $\|\cdot\|$ be the inner product and the norm defined in $L^2(0,\ell)$, respectively.

Integrating Eqns. (1) by parts and using the initial conditions (2) and the boundary conditions (3), we obtain the following weak formulation written using the velocity $v = \dot{u}$, the porosity speed $\varphi = \dot{\phi}$, the thermodynamic temperature θ, the inductive temperature ϑ, the thermodynamic microtemperature T and the inductive microtemperature S.

Problem VP. *Find the velocity $v : [0,T_f] \to V$, the porosity speed $\varphi : [0,T_f] \to V$, the thermodynamic temperature $\theta : [0,T_f] \to Y$, the inductive temperature $\vartheta : [0,T_f] \to E$, the thermodynamic microtemperature $T : [0,T_f] \to Y$ and the inductive microtemperature $S : [0,T_f] \to E$ such that $v(0) = v^0$, $\varphi(0) = \varphi^0$, $\theta(0) = \theta^0$, $T(0) = T^0$ and, for a.e. $t \in (0,T_f)$ and for all $w, m \in V$ and $r, l, z, s \in Y$,*

$$\begin{aligned} &\rho(\dot{v}(t), w) + \mu(u_x(t), w_x) \\ &\quad = \mu_0(\phi_x(t), w) + \beta_0(\theta(t), w_x) + \rho(F(t), w), \end{aligned} \tag{5}$$

$$\begin{aligned} &J(\dot{\varphi}(t), m) + a_0(\phi_x(t), m) + \xi(\phi(t), m) \\ &\quad = -\mu_0(u_x(t), m) + \mu_2(T(t), m_x) \\ &\qquad + \beta_1(\theta(t), m) + \rho(L(t), m), \end{aligned} \tag{6}$$

$$\begin{aligned} a(\dot{\theta}(t), r) = {} & -\beta_0(\dot{u}_x(t), r) - \beta_1(\dot{\phi}(t), r) \\ & + \kappa(\vartheta_{xx}(t), r) + \kappa_1(S_{xx}(t), r) \\ & + \rho(Q(t), r), \end{aligned} \tag{7}$$

$$(\theta(t), l) = (\vartheta(t) - \alpha\vartheta_{xx}(t), l), \tag{8}$$

$$\begin{aligned} b(\dot{T}(t), z) = {} & -\mu_2(\varphi_x(t), z) + \kappa_4(S_{xx}(t), z) \\ & - \kappa_2(S(t), z) - \kappa_3(\vartheta_x(t), z) \\ & - \rho(G(t), z), \end{aligned} \tag{9}$$

$$(T(t), s) = (S(t) - \alpha S_{xx}(t), s), \tag{10}$$

where the displacements and the porosity are then recovered from the relations

$$\begin{aligned} u(t) &= \int_0^t v(s)\,\mathrm{d}s + u^0, \\ \phi(t) &= \int_0^t \varphi(s)\,\mathrm{d}s + \phi^0. \end{aligned} \tag{11}$$

3. Numerical analysis: Fully discrete approximations and *a priori* error estimates

In this section, fully discrete approximations of Problem VP are introduced and numerically analyzed. In order to provide the spatial approximation, let the interval $[0,\ell]$ be partitioned into M subintervals denoted by $a_0 = 0 < a_1 < \ldots < a_M = \ell$ with a uniform length $h = a_{i+1} - a_i = \ell/M$. The variational spaces V, E and Y are then approximated by the finite dimensional spaces $V^h \subset V$, $E^h \subset E$ and $W^h \subset Y$ given by

$$\begin{aligned} V^h = {} & \{w^h \in C([0,\ell]) \ : \ w^h_{[a_i,a_{i+1}]} \in P_1([a_i,a_{i+1}], \\ & \quad i = 0,\ldots,M-1, \quad w^h(0) = w^h(\ell) = 0\}, \end{aligned} \tag{12}$$

$$\begin{aligned} E^h = {} & \{r^h \in C^1([0,\ell]) \cap H^2(0,\ell) : \\ & \quad r^h_{[a_i,a_{i+1}]} \in P_3([a_i,a_{i+1}]), \quad i = 0,\ldots,M-1, \\ & \quad r^h_x(0) = r^h_x(\ell) = r^h(0) = r^h(\ell) = 0\}, \end{aligned} \tag{13}$$

$$\begin{aligned} W^h = {} & \{l^h \in L^2([0,\ell]) \ : \ l^h_{[a_i,a_{i+1}]} \in P_1([a_i,a_{i+1}]), \\ & \quad i = 0,\ldots,M-1\}, \end{aligned} \tag{14}$$

where the set $P_r([a_i,a_{i+1}])$ denotes the space of a polynomials of degree less than or equal to r for each subinterval $[a_i,a_{i+1}]$, i.e., the finite element space V^h is composed of continuous and piecewise affine functions, E^h made of C^1 and piecewise cubic functions, and W^h by L^2 and piecewise affine functions.

In the above definitions, as usual $h > 0$ represents the spatial discretization parameter. Furthermore, we construct the discrete initial conditions u^{0h}, v^{0h}, ϕ^{0h}, φ^{0h}, θ^{0h} and T^{0h} as

$$\begin{aligned} u^{0h} &= \mathcal{P}_1^h u^0, \quad v^{0h} = \mathcal{P}_1^h v^0, \quad \phi^{0h} = \mathcal{P}_1^h \phi^0, \\ \varphi^{0h} &= \mathcal{P}_1^h \varphi^0, \quad \theta^{0h} = \mathcal{P}_2^h \theta^0, \quad T^{0h} = \mathcal{P}_2^h T^0, \end{aligned} \tag{15}$$

where $\mathcal{P}_1^h$ and $\mathcal{P}_2^h$ are the finite element projection operators over V^h and W^h, defined, for instance, in the paper by Clément (1975).

Now, to obtain the discretization of the time derivatives, we consider a uniform partition of the time interval $[0,T]$, denoted by $0 = t_0 < t_1 < \ldots < t_N = T$, with time step size $k = T/N$ and nodes $t_n = n\,k$ for $n = 0, 1, \ldots, N$. Here, n is the time step index.

Therefore, using the implicit Euler scheme, we obtain the fully discrete approximations of Problem VP, which leads to the following discrete problem.

Problem VP^{hk}. *Find the discrete velocity $v^{hk} = \{v_n^{hk}\}_{n=0}^N \subset V^h$, the discrete porosity speed $\varphi^{hk} = \{\varphi_n^{hk}\}_{n=0}^N \subset V^h$, the discrete thermodynamic temperature $\theta^{hk} = \{\theta_n^{hk}\}_{n=0}^N \subset W^h$, the discrete inductive temperature $\vartheta^{hk} = \{\vartheta_n^{hk}\}_{n=0}^N \subset E^h$, the discrete thermodynamic microtemperature $T^{hk} = \{T_n^{hk}\}_{n=0}^N \subset W^h$ and the discrete inductive microtemperature $S^{hk} = \{S_n^{hk}\}_{n=0}^N \subset E^h$ such that $v_0^{hk} = v^{0h}$, $\varphi_0^{hk} = \varphi^{0h}$, $\theta_0^{hk} = \theta^{0h}$, $T_0^{hk} = T^{0h}$, and, for all $w^h, m^h \in V^h$ and r^h, r^h, l^h, $s^h \in W^h$, and $n = 1, \ldots, N$,*

$$\begin{aligned}\rho < (\delta v_n^{hk}, w^h) &+ \mu((u_n^{hk})_x, w_x^h)\\ &= \mu_0((\phi_n^{hk})_x, w^h)\\ &\quad + \beta_0(\theta_n^{hk}, w_x^h) + \rho(F_n, w^h),\end{aligned} \tag{16}$$

$$\begin{aligned}&J(\delta\varphi_n^{hk}, m^h) + a_0((\phi_n^{hk})_x, m^h) + \xi(\phi_n^{hk}, m^h)\\ &= -\mu_0((u_n^{hk})_x, m^h) + \mu_2(T_n^{hk}, m_x^h) + \beta_1(\theta_n^{hk}, m^h)\\ &\quad + \rho(L_n, m),\end{aligned} \tag{17}$$

$$\begin{aligned}a(\delta\theta_n^{hk}, r^h) &= -\beta_0((v_n^{hk})_x, r^h) - \beta_1(\varphi_n^{hk}, r^h)\\ &\quad + \kappa((\vartheta_n^{hk})_{xx}, r^h) + \kappa_1((S_n^{hk})_{xx}, r^h)\\ &\quad + \rho(Q_n, r),\end{aligned} \tag{18}$$

$$(\theta_n^{hk}, l^h) = (\vartheta_n^{hk} - \alpha(\vartheta_n^{hk})_{xx}, l^h), \tag{19}$$

$$\begin{aligned}b(\delta T_n^{hk}, z^h) &= -\mu_2((\varphi_n^{hk})_x, z^h) + \kappa_4((S_n^{hk})_{xx}, z^h)\\ &\quad - \kappa_2(S_n^{hk}, z^h) - \kappa_3((\vartheta_n^{hk})_x, z^h)\\ &\quad - \rho(G_n, z^h),\end{aligned} \tag{20}$$

$$(T_n^{hk}, s^h) = (S_n^{hk} - \alpha(S_n^{hk})_{xx}, s^h), \tag{21}$$

where, for a continuous function $z(t)$, we use the notation $z_n = z(t_n)$ and, for a sequence $\{z_n\}_{n=0}^N$, let $\delta z_n = (z_n - z_{n-1})/k$ be its divided differences. Moreover, the discrete displacements u_n^{hk} and the discrete porosity ϕ_n^{hk} are now recovered from the relations

$$u_n^{hk} = k\sum_{j=1}^n v_j^{hk} + u^{0h}, \quad \phi_n^{hk} = k\sum_{j=1}^n \varphi_j^{hk} + \phi^{0h}. \tag{22}$$

Applying the classical Lax–Milgram lemma, we can easily deduce that Problem VP^{hk} admits a unique solution under the assumptions (4).

In what follows, we aim to prove some *a priori* error estimates on the numerical approximations provided in Problem VP^{hk}. We note that, for the sake of simplicity in the calculations, we assume that the supply terms F, L, Q and G vanish.

First, let us obtain the error estimates for the velocity. Thus, subtracting the variational equation (5) at time $t = t_n$ for a test function $w = w^h \in V^h \subset V$ and the discrete variational equation (16), we have, for all $w^h \in V^h$,

$$\begin{aligned}&\rho(\dot v_n - \delta v_n^{hk}, w^h) + \mu((u_n - u_n^{hk})_x, w_x^h)\\ &\qquad -\mu_0((\phi_n - \phi_n^{hk})_x, w^h) - \beta_0(\theta_n - \theta_n^{hk}, w_x^h) = 0,\end{aligned}$$

and, therefore, we have, for all $w^h \in V^h$,

$$\begin{aligned}&\rho(\dot v_n - \delta v_n^{hk}, v_n - v_n^{hk})\\ &\quad + \mu((u_n - u_n^{hk})_x, (v_n - v_n^{hk})_x)\\ &\quad - \mu_0((\phi_n - \phi_n^{hk})_x, v_n - v_n^{hk})\\ &\quad - \beta_0(\theta_n - \theta_n^{hk}, (v_n - v_n^{hk})_x)\\ &\qquad = \rho(\dot v_n - \delta v_n^{hk}, v_n - w^h)\\ &\qquad\quad + \mu((u_n - u_n^{hk})_x, (v_n - w^h)_x)\\ &\qquad\quad - \mu_0((\phi_n - \phi_n^{hk})_x, v_n - w^h)\\ &\qquad\quad - \beta_0(\theta_n - \theta_n^{hk}, (v_n - w^h)_x).\end{aligned}$$

By using the estimates

$$\begin{aligned}(\dot v_n - \delta v_n^{hk}, v_n - v_n^{hk}) &= (\dot v_n - \delta v_n, v_n - v_n^{hk})\\ &\quad + (\delta v_n - \delta v_n^{hk}, v_n - v_n^{hk}),\end{aligned}$$

$$\begin{aligned}&\big(\delta v_n - \delta v_n^{hk}, v_n - v_n^{hk}\big)\\ &\quad \ge \frac{1}{2k}\Big\{\|v_n - v_n^{hk}\|^2 - \|v_{n-1} - v_{n-1}^{hk}\|^2\Big\},\end{aligned}$$

$$\begin{aligned}&((u_n - u_n^{hk})_x, (v_n - v_n^{hk})_x)\\ &\quad \ge ((u_n - u_n^{hk})_x, (\dot u_n - \delta u_n)_x)\\ &\qquad + \frac{1}{2k}\Big\{\|(u_n - u_n^{hk})_x\|^2 - \|(u_{n-1} - u_{n-1}^{hk})_x\|^2\Big\},\end{aligned}$$

where $\delta v_n = (v_n - v_{n-1})/k$ and $\delta u_n = (u_n - u_{n-1})/k$, applying Cauchy's inequality

$$ab \le \epsilon a^2 + \frac{1}{4\epsilon}b^2, \quad a, b, \epsilon \in \mathbb{R}, \quad \epsilon > 0, \tag{23}$$

we find that, for all $w^h \in V^h$,

$$\begin{aligned}&\frac{\rho}{2k}\Big\{\|v_n - v_n^{hk}\|^2 - \|v_{n-1} - v_{n-1}^{hk}\|^2\Big\}\\ &\qquad -\beta_0(\theta_n - \theta_n^{hk}, (v_n - v_n^{hk})_x)\\ &\qquad +\frac{\mu}{2k}\Big\{\|(u_n - u_n^{hk})_x\|^2 - \|(u_{n-1} - u_{n-1}^{hk})_x\|^2\Big\}\\ &\quad \le C\Big(\|\dot v_n - \delta v_n\|^2 + \|(\dot u_n - \delta u_n)_x\|^2\\ &\qquad +\|v_n - w^h\|_V^2 + \|(u_n - u_n^{hk})_x\|^2\\ &\qquad +\|(\phi_n - \phi_n^{hk})_x\|^2 + \|\theta_n - \theta_n^{hk}\|^2\\ &\qquad +(\delta v_n - \delta v_n^{hk}, v_n - w^h) + \|v_n - v_n^{hk}\|^2\Big).\end{aligned} \tag{24}$$

Proceeding in a similar form, we derive the error estimates on the porosity speed. Therefore, keeping in mind that

$$\begin{aligned}\xi(\phi_n - \phi_n^{hk}, \varphi_n - \varphi_n^{hk}) &\geq \xi(\phi_n - \phi_n^{hk}, \dot{\phi}_n - \delta\phi_n) \\ &+ \frac{\xi}{2k}\Big\{\|\phi_n - \phi_n^{hk}\|^2 - \|\phi_{n-1} - \phi_{n-1}^{hk}\|^2\Big\},\end{aligned}$$

we obtain now, for all $m^h \in V^h$,

$$\begin{aligned}&\frac{J}{2k}\Big\{\|\varphi_n - \varphi_n^{hk}\|^2 - \|\varphi_{n-1} - \varphi_{n-1}^{hk}\|^2\Big\} \\ &+\frac{\xi}{2k}\Big\{\|\phi_n - \phi_n^{hk}\|^2 - \|\phi_{n-1} - \phi_{n-1}^{hk}\|^2\Big\} \\ &+\frac{a_0}{2k}\Big\{\|(\phi_n - \phi_n^{hk})_x\|^2 - \|(\phi_{n-1} - \phi_{n-1}^{hk})_x\|^2\Big\} \\ &-\mu_2(T_n - T_n^{hk}, (\varphi_n - \varphi_n^{hk})_x) \\ &\leq C\Big(\|\dot{\varphi}_n - \delta\varphi_n\|^2 + \|(\dot{\phi}_n - \delta\phi_n)_x\|^2 \\ &+\|\varphi_n - m^h\|_V^2 + \|(\phi_n - \phi_n^{hk})_x\|^2 \\ &+\|\theta_n - \theta_n^{hk}\|^2 + \|\varphi_n - \varphi_n^{hk}\|^2 \\ &+\|(u_n - u_n^{hk})_x\|^2 + \|T_n - T_n^{hk}\|^2 \\ &+(\delta\varphi_n - \delta\varphi_n^{hk}, \varphi_n - m^h)\Big), \qquad (25)\end{aligned}$$

where $\delta\varphi_n = (\varphi_n - \varphi_{n-1})/k$ and $\delta\phi_n = (\phi_n - \phi_{n-1})/k$.

Now, we obtain the error estimates on the inductive temperature. Therefore, subtracting the variational equation (8) at time $t = t_n$ for a test function $l = l^h \in W^h \subset Y$ and the discrete variational equation (19), we obtain

$$\begin{aligned}&(\theta_n - \theta_n^{hk}, l^h) \\ &= (\vartheta_n - \vartheta_n^{hk} - \alpha(\vartheta_n - \vartheta_n^{hk})_{xx}, l^h), \quad \forall l^h \in W^h,\end{aligned}$$

so that, we have, for all $\xi^h \in E^h$ (because $\xi^h_{xx} \in W^h$),

$$\begin{aligned}&(\theta_n - \theta_n^{hk}, (\vartheta_n - \vartheta_n^{hk})_{xx}) \\ &\quad -(\vartheta_n - \vartheta_n^{hk} - \alpha(\vartheta_n - \theta_n^{hk})_{xx}, (\vartheta_n - \vartheta_n^{hk})_{xx}) \\ &= (\theta_n - \theta_n^{hk}, (\vartheta_n - \xi^h)_{xx}) \\ &\quad -(\vartheta_n - \vartheta_n^{hk} - \alpha(\vartheta_n - \vartheta_n^{hk})_{xx}, (\vartheta_n - \xi^h)_{xx}).\end{aligned}$$

Taking into account that

$$\begin{aligned}&-(\vartheta_n - \vartheta_n^{hk}, (\vartheta_n - \vartheta_n^{hk})_{xx}) \\ &\quad = ((\vartheta_n - \vartheta_n^{hk})_x, (\vartheta_n - \vartheta_n^{hk})_x), \\ &-(\vartheta_n - \vartheta_n^{hk}, (\vartheta_n - \xi^h)_{xx}) \\ &\quad = ((\vartheta_n - \vartheta_n^{hk})_x, (\vartheta_n - \xi^h)_x),\end{aligned}$$

using Cauchy's inequality (23) several times we find that, for all $\xi^h \in E^h$,

$$\begin{aligned}&\|(\vartheta_n - \vartheta_n^{hk})_x\|^2 + \|(\vartheta_n - \vartheta_n^{hk})_{xx}\|^2 \\ &\quad \leq C\Big(\|(\vartheta_n - \xi^h)_{xx}\|^2 + \|\theta_n - \theta_n^{hk}\|^2 \qquad (26) \\ &\quad + \|(\vartheta_n - \xi^h)_x\|^2\Big).\end{aligned}$$

Proceeding in an analogous way, we get the following estimates for the inductive microtemperature, for all $\Xi^h \in E^h$:

$$\begin{aligned}&\|(S_n - S_n^{hk})_x\|^2 + \|(S_n - S_n^{hk})_{xx}\|^2 \\ &\quad \leq C\Big(\|(S_n - \Xi^h)_{xx}\|^2 + \|T_n - T_n^{hk}\|^2 \\ &\quad + \|(S_n - \Xi^h)_x\|^2\Big).\end{aligned} \qquad (27)$$

Finally, we obtain the estimates on the thermodynamic temperature and the thermodynamic microtemperature. We subtract the variational equation (7) at time $t = t_n$ for a test function $r = r^h \in W^h \subset Y$ and the discrete variational equation (18) to obtain

$$\begin{aligned}&a(\dot{\theta}_n - \delta\theta_n^{hk}, r^h) + \beta_0((v_n - v_n^{hk})_x, r^h) \\ &+ \beta_1(\varphi_n - \varphi_n^{hk}, r^h) - \kappa((\vartheta_n - \vartheta_n^{hk})_{xx}, r^h) \\ &- \kappa_1((S_n - S_n^{hk})_{xx}, r^h) = 0,\end{aligned}$$

and therefore, we have, for all $r^h \in W^h$,

$$\begin{aligned}&a(\dot{\theta}_n - \delta\theta_n^{hk}, \theta_n - \theta_n^{hk}) + \beta_0((v_n - v_n^{hk})_x, \theta_n - \theta_n^{hk}) \\ &+\beta_1(\varphi_n - \varphi_n^{hk}, \theta_n - \theta_n^{hk}) \\ &-\kappa((\vartheta_n - \vartheta_n^{hk})_{xx}, \theta_n - \theta_n^{hk}) \\ &-\kappa_1((S_n - S_n^{hk})_{xx}, \theta_n - \theta_n^{hk}) \\ &= a(\dot{\theta}_n - \delta\theta_n^{hk}, \theta_n - r^h) + \beta_0((v_n - v_n^{hk})_x, \theta_n - r^h) \\ &+\beta_1(\varphi_n - \varphi_n^{hk}, \theta_n - r^h) \\ &-\kappa((\vartheta_n - \vartheta_n^{hk})_{xx}, \theta_n - r^h) \\ &-\kappa_1((S_n - S_n^{hk})_{xx}, \theta_n - r^h).\end{aligned}$$

Keeping in mind that

$$\begin{aligned}a(\dot{\theta}_n - \delta\theta_n^{hk}, \theta_n - \theta_n^{hk}) &\geq a(\dot{\theta}_n - \delta\theta_n, \theta_n - \theta_n^{hk}) \\ &+ \frac{a}{2k}\Big\{\|\theta_n - \theta_n^{hk}\|^2 - \|\theta_{n-1} - \theta_{n-1}^{hk}\|^2\Big\},\end{aligned}$$

it follows, for all $r^h \in W^h$, that

$$\begin{aligned}&\frac{a}{2k}\Big\{\|\theta_n - \theta_n^{hk}\|^2 - \|\theta_{n-1} - \theta_{n-1}^{hk}\|^2\Big\} \\ &\quad + \beta_0((v_n - v_n^{hk})_x, \theta_n - \theta_n^{hk}) \\ &\leq C\Big(\|\dot{\theta}_n - \delta\theta_n\|^2 + \|\theta_n - r^h\|^2 + \|\varphi_n - \varphi_n^{hk}\|^2 \\ &\quad + \|(S_n - S_n^{hk})_{xx}\|^2 + ((\delta u_n - \delta u_n^{hk})_x, \theta_n - r^h) \\ &\quad + \|(\vartheta_n - \vartheta_n^{hk})_{xx}\|^2 + ((\delta\theta_n - \delta\theta_n^{hk})_x, \theta_n - r^h) \\ &\quad + \|(\dot{u}_n - \delta u_n)_x\|^2\Big).\end{aligned} \qquad (28)$$

Similarly, we also find, for all $z^h \in W^h$, that

$$\begin{aligned}
&\frac{b}{2k}\Big\{\|T_n - T_n^{hk}\|^2 - \|T_{n-1} - T_{n-1}^{hk}\|^2\Big\}\\
&\quad +\mu_2((\varphi_n - \varphi_n^{hk})_x, T_n - T_n^{hk})\\
&\leq C\Big(\|\dot{T}_n - \delta T_n\|^2 + \|T_n - z^h\|^2 + \|\varphi_n - \varphi_n^{hk}\|^2\\
&\quad +\|(\vartheta_n - \vartheta_n^{hk})_{xx}\|^2 + \|S_n - S_n^{hk}\|^2\\
&\quad +((\delta u_n - \delta u_n^{hk})_x, T_n - z^h) + \|(S_n - S_n^{hk})_{xx}\|^2\\
&\quad +(\delta T_n - \delta T_n^{hk}, T_n - z^h)\Big). \qquad (29)
\end{aligned}$$

Combining the estimates (24), (25), (28) and (29), it follows that

$$\begin{aligned}
&\frac{\rho}{2k}\Big\{\|v_n - v_n^{hk}\|^2 - \|v_{n-1} - v_{n-1}^{hk}\|^2\Big\}\\
&\quad +\frac{\mu}{2k}\Big\{\|(u_n - u_n^{hk})_x\|^2 - \|(u_{n-1} - u_{n-1}^{hk})_x\|^2\Big\}\\
&\quad +\frac{J}{2k}\Big\{\|\varphi_n - \varphi_n^{hk}\|^2 - \|\varphi_{n-1} - \varphi_{n-1}^{hk}\|^2\Big\}\\
&\quad +\frac{\xi}{2k}\Big\{\|\phi_n - \phi_n^{hk}\|^2 - \|\phi_{n-1} - \phi_{n-1}^{hk}\|^2\Big\}\\
&\quad +\frac{a_0}{2k}\Big\{\|(\phi_n - \phi_n^{hk})_x\|^2 - \|(\phi_{n-1} - \phi_{n-1}^{hk})_x\|^2\Big\}\\
&\quad +\frac{a}{2k}\Big\{\|\theta_n - \theta_n^{hk}\|^2 - \|\theta_{n-1} - \theta_{n-1}^{hk}\|^2\Big\}\\
&\quad +\frac{b}{2k}\Big\{\|T_n - T_n^{hk}\|^2 - \|T_{n-1} - T_{n-1}^{hk}\|^2\Big\}\\
&\leq C\Big(\|\dot{v}_n - \delta v_n\|^2 + \|(\dot{u}_n - \delta u_n)_x\|^2 + \|v_n - w^h\|_V^2\\
&\quad +\|(u_n - u_n^{hk})_x\|^2 + \|\theta_n - \theta_n^{hk}\|^2 + \|v_n - v_n^{hk}\|^2\\
&\quad +\|(\phi_n - \phi_n^{hk})_x\|^2 + (\delta v_n - \delta v_n^{hk}, v_n - w^h)\\
&\quad +\|\dot{\varphi}_n - \delta\varphi_n\|^2 + \|(\dot{\phi}_n - \delta\phi_n)_x\|^2 + \|\varphi_n - m^h\|_V^2\\
&\quad +((\delta u_n - \delta u_n^{hk})_x, T_n - z^h) + \|\varphi_n - \varphi_n^{hk}\|^2\\
&\quad +\|\dot{\theta}_n - \delta\theta_n\|^2 + \|\theta_n - r^h\|^2 + \|T_n - z^h\|^2\\
&\quad +(\delta\varphi_n - \delta\varphi_n^{hk}, \varphi_n - m^h) + \|(\vartheta_n - \vartheta_n^{hk})_{xx}\|^2\\
&\quad +\|(S_n - S_n^{hk})_{xx}\|^2 + ((\delta u_n - \delta u_n^{hk})_x, \theta_n - r^h)\\
&\quad +\|\dot{T}_n - \delta T_n\|^2 + (\delta\theta_n - \delta\theta_n^{hk}, \theta_n - r^h)\\
&\quad +\|S_n - S_n^{hk}\|^2 + \|T_n - T_n^{hk}\|^2\\
&\quad +(\delta T_n - \delta T_n^{hk}, T_n - z^h)\Big).
\end{aligned}$$

Multiplying the above estimates by k and summing up to n, we have

$$\begin{aligned}
&\|v_n - v_n^{hk}\|^2 + \|(u_n - u_n^{hk})_x\|^2 + \|\varphi_n - \varphi_n^{hk}\|^2\\
&\quad +\|\phi_n - \phi_n^{hk}\|^2 + \|(\phi_n - \phi_n^{hk})_x\|^2 + \|\theta_n - \theta_n^{hk}\|^2\\
&\quad +\|T_n - T_n^{hk}\|^2\\
&\leq Ck\sum_{j=1}^{n}\Big(\|\dot{v}_j - \delta v_j\|^2 + \|(\dot{u}_j - \delta u_j)_x\|^2\\
&\quad +\|v_j - w_j^h\|_V^2 + \|(u_j - u_j^{hk})_x\|^2\\
&\quad +\|\theta_j - \theta_j^{hk}\|^2 + \|v_j - v_j^{hk}\|^2\\
&\quad +\|(\phi_j - \phi_j^{hk})_x\|^2 + (\delta v_j - \delta v_j^{hk}, v_j - w_j^h)\\
&\quad +\|\dot{\varphi}_j - \delta\varphi_j\|^2 + \|(\dot{\phi}_j - \delta\phi_j)_x\|^2 + \|\varphi_j - m_j^h\|_V^2\\
&\quad +\|\varphi_j - \varphi_j^{hk}\|^2 + \|T_j - T_j^{hk}\|^2 + \|\dot{\theta}_j - \delta\theta_j\|^2\\
&\quad +\|\theta_j - r_j^h\|^2 + (\delta\varphi_j - \delta\varphi_j^{hk}, \varphi_j - m_j^h)\\
&\quad +\|(\vartheta_j - \vartheta_j^{hk})_{xx}\|^2 + (\delta\theta_j - \delta\theta_j^{hk}, \theta_j - r_j^h)\\
&\quad +\|(S_j - S_j^{hk})_{xx}\|^2 + ((\delta u_j - \delta u_j^{hk})_x, \theta_j - r_j^h)\\
&\quad +\|\dot{T}_j - \delta T_j\|^2 + \|T_j - z_j^h\|^2\\
&\quad +((\delta u_j - \delta u_j^{hk})_x, T_j - z_j^h)\\
&\quad +\|S_j - S_j^{hk}\|^2 + (\delta T_j - \delta T_j^{hk}, T_j - z_j^h)\Big)\\
&\quad +C\Big(\|v^0 - v^{0h}\|^2 + \|u^0 - u^{0h}\|_V^2 + \|\varphi^0 - \varphi^{0h}\|^2\\
&\quad +\|\phi^0 - \phi^{0h}\|_V^2 + \|\theta^0 - \theta^{0h}\|^2 + \|T^0 - T^{0h}\|^2\Big).
\end{aligned}$$

Now, from the above estimates as well as (26) and (27), we find that

$$\begin{aligned}
&\|v_n - v_n^{hk}\|^2 + \|(u_n - u_n^{hk})_x\|^2 + \|\varphi_n - \varphi_n^{hk}\|^2\\
&\quad +\|\phi_n - \phi_n^{hk}\|^2 + \|(\phi_n - \phi_n^{hk})_x\|^2 + \|\theta_n - \theta_n^{hk}\|^2\\
&\quad +\|T_n - T_n^{hk}\|^2 + \|(S_n - S_n^{hk})_x\|^2\\
&\quad +\|(S_n - S_n^{hk})_{xx}\|^2 + \|(\vartheta_n - \vartheta_n^{hk})_x\|^2\\
&\quad +\|(\vartheta_n - \vartheta_n^{hk})_{xx}\|^2\\
&\leq Ck\sum_{j=1}^{n}\Big(\|\dot{v}_j - \delta v_j\|^2 + \|(\dot{u}_j - \delta u_j)_x\|^2\\
&\quad +\|v_j - w_j^h\|_V^2 + \|(u_j - u_j^{hk})_x\|^2\\
&\quad +\|\theta_j - \theta_j^{hk}\|^2 + \|v_j - v_j^{hk}\|^2\\
&\quad +(\delta v_j - \delta v_j^{hk}, v_j - w_j^h) + \|\dot{\varphi}_j - \delta\varphi_j\|^2\\
&\quad +\|\varphi_j - m_j^h\|_V^2 + \|\varphi_j - \varphi_j^{hk}\|^2 + \|T_j - T_j^{hk}\|^2\\
&\quad +\|\dot{\theta}_j - \delta\theta_j\|^2 + \|\theta_j - r_j^h\|^2\\
&\quad +(\delta\varphi_j - \delta\varphi_j^{hk}, \varphi_j - m_j^h)\\
&\quad +\|(\vartheta_j - \vartheta_j^{hk})_{xx}\|^2 + \|(S_j - S_j^{hk})_{xx}\|^2\\
&\quad +\|(\phi_j - \phi_j^{hk})_x\|^2 + ((\delta u_j - \delta u_j^{hk})_x, \theta_j - r_j^h)\\
&\quad +(\delta\theta_j - \delta\theta_j^{hk}, \theta_j - r_j^h) + \|\dot{T}_j - \delta T_j\|^2\\
&\quad +\|T_j - z_j^h\|^2 + ((\delta u_j - \delta u_j^{hk})_x, T_j - z_j^h)\\
&\quad +\|S_j - S_j^{hk}\|^2 + (\delta T_j - \delta T_j^{hk}, T_j - z_j^h)\\
&\quad +\|(\dot{\phi}_j - \delta\phi_j)_x\|^2\Big) + C\Big(\|v^0 - v^{0h}\|^2 + \|u^0 - u^{0h}\|_V^2\\
&\quad +\|\varphi^0 - \varphi^{0h}\|^2 + \|\phi^0 - \phi^{0h}\|_V^2 + \|\theta^0 - \theta^{0h}\|^2\\
&\quad +\|T^0 - T^{0h}\|^2\Big) + C\Big(\|S_n - \Xi^h\|_E^2\\
&\quad +\|\vartheta_n - \xi^h\|_E^2\Big).
\end{aligned}$$

Keeping in mind that

$$\sum_{j=1}^{n}(v_j - v_j^{hk} - (v_{j-1} - v_{j-1}^{hk}), v_j - w_j^h)$$
$$= (v_n - v_n^{hk}, v_n - w_n^h) + (v_0^h - v_0, v_1 - w_1^h)$$
$$+ \sum_{j=1}^{n-1}(v_j - v_j^{hk}, v_j - w_j^h - (v_{j+1} - w_{j+1}^h)),$$

$$\sum_{j=1}^{n}(\varphi_j - \varphi_j^{hk} - (\varphi_{j-1} - \varphi_{j-1}^{hk}), \varphi_j - m_j^h)$$
$$= (\varphi_n - \varphi_n^{hk}, \varphi_n - m_n^h) + (\varphi^{0h} - \varphi^0, \varphi_1 - m_1^h)$$
$$+ \sum_{j=1}^{n-1}(\varphi_j - \varphi_j^{hk}, \varphi_j - m_j^h - (\varphi_{j+1} - m_{j+1}^h)),$$

$$\sum_{j=1}^{n}(\theta_j - \theta_j^{hk} - (\theta_{j-1} - \theta_{j-1}^{hk}), \theta_j - r_j^h)$$
$$= (\theta_n - \theta_n^{hk}, \theta_n - r_n^h) + (\theta^{0h} - \theta^0, \theta_1 - r_1^h)$$
$$+ \sum_{j=1}^{n-1}(\theta_j - \theta_j^{hk}, \theta_j - r_j^h - (\theta_{j+1} - r_{j+1}^h)),$$

$$\sum_{j=1}^{n}(T_j - T_j^{hk} - (T_{j-1} - T_{j-1}^{hk}), T_j - z_j^h)$$
$$= (T_n - T_n^{hk}, T_n - z_n^h) + (T^{0h} - T^0, T_1 - z_1^h)$$
$$+ \sum_{j=1}^{n-1}(T_j - T_j^{hk}, T_j - z_j^h - (T_{j+1} - z_{j+1}^h)),$$

$$\sum_{j=1}^{n}((u_j - u_j^{hk} - (u_{j-1} - u_{j-1}^{hk}))_x, \theta_j - r_j^h)$$
$$= ((u_n - u_n^{hk})_x, \theta_n - r_n^h)$$
$$+((u^{0h} - u^0)_x, \theta_1 - r_1^h)$$
$$+ \sum_{j=1}^{n-1}((u_j - u_j^{hk})_x, \theta_j - r_j^h - (\theta_{j+1} - r_{j+1}^h)),$$

$$\sum_{j=1}^{n}((u_j - u_j^{hk} - (u_{j-1} - u_{j-1}^{hk}))_x, T_j - z_j^h)$$
$$= ((u_n - u_n^{hk})_x, T_n - z_n^h)$$
$$+((u^{0h} - u^0)_x, T_1 - z_1^h)$$
$$+ \sum_{j=1}^{n-1}((u_j - u_j^{hk})_x, T_j - z_j^h - (T_{j+1} - z_{j+1}^h)),$$

and applying a discrete version of Gronwall's inequality (see, e.g., Campo *et al.*, 2006) we have thus proved our main *a priori* error estimates result.

Theorem 2. *Let the assumptions of Theorem 1 still hold. If we denote by $(u, v, \phi, \varphi, \theta, \vartheta, T, S)$ the solution to the problems (5)–(11) and by $(u^{hk}, v^{hk}, \phi^{hk}, \varphi^{hk}, \theta^{hk}, \vartheta, T, S^{hk})$ the solution to the problems (16)–(22) , then we have the following* a priori *error estimates, for all $w^h = \{w_j^h\}_{j=0}^N$, $m^h = \{m_j^h\}_{j=0}^N \subset V^h$, $r^h = \{r_j^h\}_{j=0}^N$, $z^h = \{z_j^h\}_{j=0}^N \subset W^h$ and ξ^h, $\Xi^h \in E^h$,*

$$\max_{0\le n\le N}\Big\{\|v_n - v_n^{hk}\|^2 + \|u_n - u_n^{hk}\|_V^2 + \|\varphi_n - \varphi_n^{hk}\|^2$$
$$+ \|\phi_n - \phi_n^{hk}\|_V^2 + \|\theta_n - \theta_n^{hk}\|^2 + \|T_n - T_n^{hk}\|^2$$
$$+ \|\vartheta_n - \vartheta_n^{hk}\|_E^2 + \|S_n - S_n^{hk}\|_E^2\Big\}$$
$$\le Ck\sum_{j=1}^{N}\Big(\|\dot v_j - \delta v_j\|^2 + \|\dot u_j - \delta u_j\|_V^2 + \|v_j - w_j^h\|_V^2$$
$$+ \|\dot\varphi_j - \delta\varphi_j\|^2 + \|\dot\phi_j - \delta\phi_j\|_V^2 + \|\varphi_j - m_j^h\|_V^2$$
$$+ \|\dot\theta_j - \delta\theta_j\|^2 + \|\theta_j - r_j^h\|^2 + \|\dot T_j - \delta T_j\|^2$$
$$+ \|T_j - z_j^h\|^2\Big)$$
$$+ \frac{C}{k}\sum_{j=1}^{N-1}\Big(\|v_j - w_j^h - (v_{j+1} - w_{j+1}^h)\|^2$$
$$+ \|\varphi_j - m_j^h - (\varphi_{j+1} - m_{j+1}^h)\|^2$$
$$+ \|\theta_j - r_j^h - (\theta_{j+1} - r_{j+1}^h)\|^2$$
$$+ \|T_j - z_j^h - (T_{j+1} - z_{j+1}^h)\|^2\Big)$$
$$+ C\Big(\|v^0 - v^{0h}\|^2 + \|u^0 - u^{0h}\|_V^2$$
$$+ \|\varphi^0 - \varphi^{0h}\|^2 + \|\phi^0 - \phi^{0h}\|_V^2 + \|\theta^0 - \theta^{0h}\|^2$$
$$+ \|T^0 - T^{0h}\|^2\Big) + C\max_{0\le n\le N}\Big(\|v_n - w_n^h\|^2$$
$$+ \|\varphi_n - m_n^h\|^2 + \|\theta_n - r_n^h\|^2 + \|T_n - z_n^h\|^2\Big)$$
$$+ C\Big(\|S_n - \Xi^h\|_E^2 + \|\vartheta_n - \xi^h\|_E^2\Big),$$

where C is a positive constant which does not depend on the discretization parameters h and k.

By using the above estimates, we can derive the convergence order of the approximations given by the discrete problems (16)–(22). As an example, if we assume the following additional regularity:

$$u, \phi \in H^3(0, T_f; Y) \cap H^2(0, T_f; V)$$
$$\cap\, C^1([0, T_f]; H^2(0, \ell)),$$

$$\theta, T \in H^2(0, T_f; Y) \cap H^1([0, T_f]; H^1(0, \ell)),$$
$$S, \vartheta \in C^1([0, T_f]; H^3(0, \ell)),$$

we have that convergence of the algorithm is linear. This can be proved by applying some well-known results on the approximation by finite elements (see, e.g., Ciarlet, 1993) and some estimates already used by Campo *et al.* (2006). Therefore, we can conclude that there exists a positive constant $C > 0$ such that

$$\max_{0\le n\le N}\Big\{\|v_n - v_n^{hk}\| + \|u_n - u_n^{hk}\|_V + \|\varphi_n - \varphi_n^{hk}\|$$
$$+\|\phi_n - \phi_n^{hk}\|_V + \|\theta_n - \theta_n^{hk}\| + \|T_n - T_n^{hk}\|$$
$$+\|\vartheta_n - \vartheta_n^{hk}\|_E + \|S_n - S_n^{hk}\|_E\Big\} \le C(h + k).$$

4. Numerical results

In this section, we present several numerical simulations to show that the exponential decay predicted theoretically as well as the linear convergence of the approximation are achieved. We also perform a parametric study to show different behaviours of the solution, depending on the model parameters.

All simulations were computed on a PC with a 1.8 GHz processor using MATLAB. A typical run ($h = k = 10^{-2}$) with a final time $T_f = 1$ and length 1 took 1.5 seconds of CPU time.

4.1. Approximation accuracy. To show numerically the accuracy of the approximation, we performed a test with a known analytical solution. We manufacture the following analytical function for v, φ, ϑ and S, for all $(x,t) \in (0,1) \times (0,0.5)$:

$$v(x,t) = \varphi(x,t) = \vartheta(x,t) = S(x,t) = x^3(1-x)^3\, e^t.$$

Then, given the following model parameters:

$$\begin{gathered} J=1, \quad a=1, \quad a_0=1, \quad \alpha=1, \quad b=1, \\ \beta_0=1, \quad \beta_1=1, \quad \kappa=5, \quad \kappa_1=1, \quad \kappa_2=1, \\ \kappa_3=1, \quad \kappa_4=1, \quad \mu=10, \quad \mu_0=1, \quad \mu_2=1, \\ \rho=1, \quad \xi=1, \end{gathered}$$

we compute variables θ and T, for all $(x,t) \in (0,1) \times (0,0.5)$,

$$\begin{aligned} \theta(x,t) &= T(x,t) \\ &= x\, e^t(-x^5 + 3x^4 + 27x^3 - 59x^2 + 36x - 6), \end{aligned}$$

as well as the supply terms F, L, Q and G using Eqn. (1). The initial conditions for the simulation are obtained from those manufactured functions (at $t = 0$).

We run the simulation up to a final time of $T_f = 0.5$ with a domain of unit length, and compute the error between the numerical approximation and the analytic solution by using the expression

$$\begin{aligned} \max_{0\le n\le N} \Big\{ &\|v_n - v_n^{hk}\| + \|u_n - u_n^{hk}\|_V + \|\varphi_n - \varphi_n^{hk}\| \\ &+\|\phi_n - \phi_n^{hk}\|_V + \|\theta_n - \theta_n^{hk}\| + \|T_n - T_n^{hk}\| \\ &+\|\vartheta_n - \vartheta_n^{hk}\|_E + \|S_n - S_n^{hk}\|_E \Big\}. \end{aligned}$$

The errors for different timesteps and element sizes are summarized in Table 1. In Fig. 1, the diagonal of the table is plotted against $h+k$. Here, the linear convergence of the algorithm shown in the previous section is clearly seen.

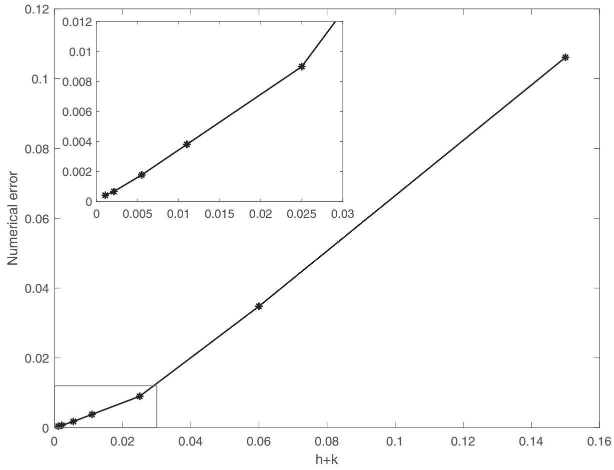

Fig. 1. Convergence of the numerical error depending on parameter $h + k$.

4.2. Exponential decay. To show the exponential decay of the solution, we perform some simulations with the following parameters:

$$\begin{gathered} J=1, \quad a=1, \quad \alpha=1, \quad b=1, \quad \beta_0=1, \\ \beta_1=1, \quad \kappa=10, \quad \kappa_1=1, \quad \kappa_2=1, \quad \kappa_3=1, \\ \kappa_4=1, \quad \mu=10, \quad \mu_0=1, \quad \mu_2=1, \quad \rho=1, \\ \xi=1, \quad k=0.0001, \quad h=0.01, \\ T_f=100, \quad \ell=1. \end{gathered}$$

Following the continuous case (see Fernández and Quintanilla, 2021a), we define the discrete energy in the following form:

$$\begin{aligned} E_n^{hk} = \frac{1}{2}\Big(&\rho\|v_n^{hk}\|^2 + J\|\varphi_n^{hk}\|^2 + \mu\|u_n^{hk}\|_V^2 \\ &+ \xi\|\phi_n^{hk}\|^2 + 2\mu_0((u_n^{hk})_x, \phi_n^{hk}) + a_0\|\phi_n^{hk}\|_V^2 \\ &+ c\|\theta_n^{hk}\|^2 + b\|T_n^{hk}\|^2 \Big). \end{aligned}$$

The results of those simulations, regarding the energy of the system, are shown in Fig. 2 for different values of a_0. After an initial fast decay, when the variables stabilize from the initial conditions to the oscillatory state (as shown in the next section), the exponential decay is achieved. This decay is clearly seen in the semi-logarithmic graph (right). Here, after a certain time (in this example it depends on the value of a_0), all the lines become straight.

4.3. Parameric study. We complete the numerical experiments with a parametric study depending on parameter J (which corresponds to the equilibrated inertia). The simulation is done using the same parameters as in the previous case, with $a_0 = 10$. In Fig. 3 we show the evolution of the energy (top) and the evolution of the H^1-norm of variable ϕ (bottom). The nature of the equation for the evolution of ϕ (second order in time)

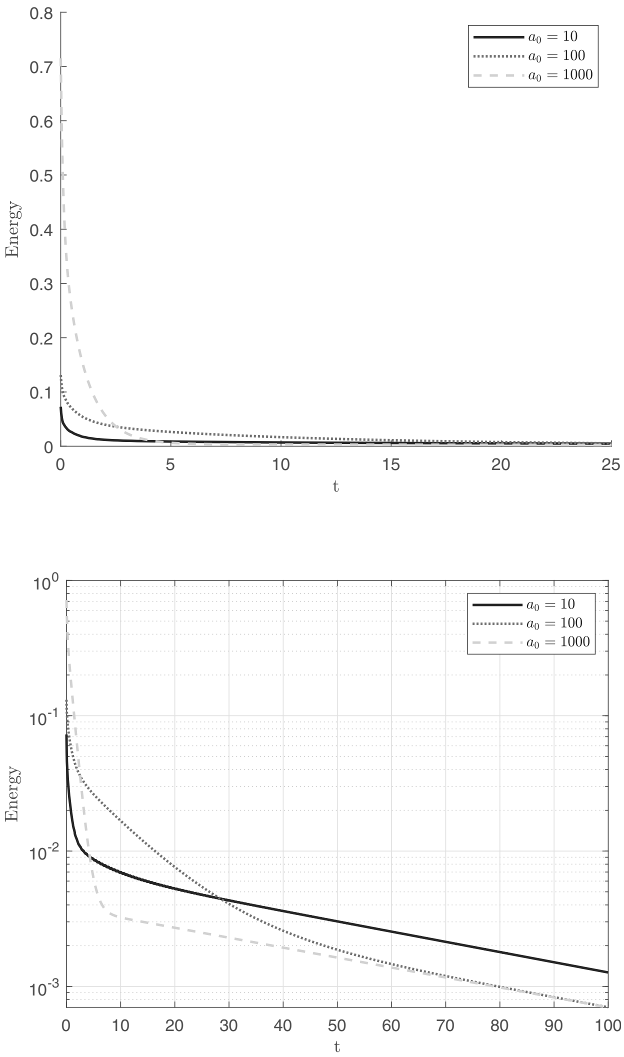

Fig. 2. Exponential decay of the energy for different values of a_0.

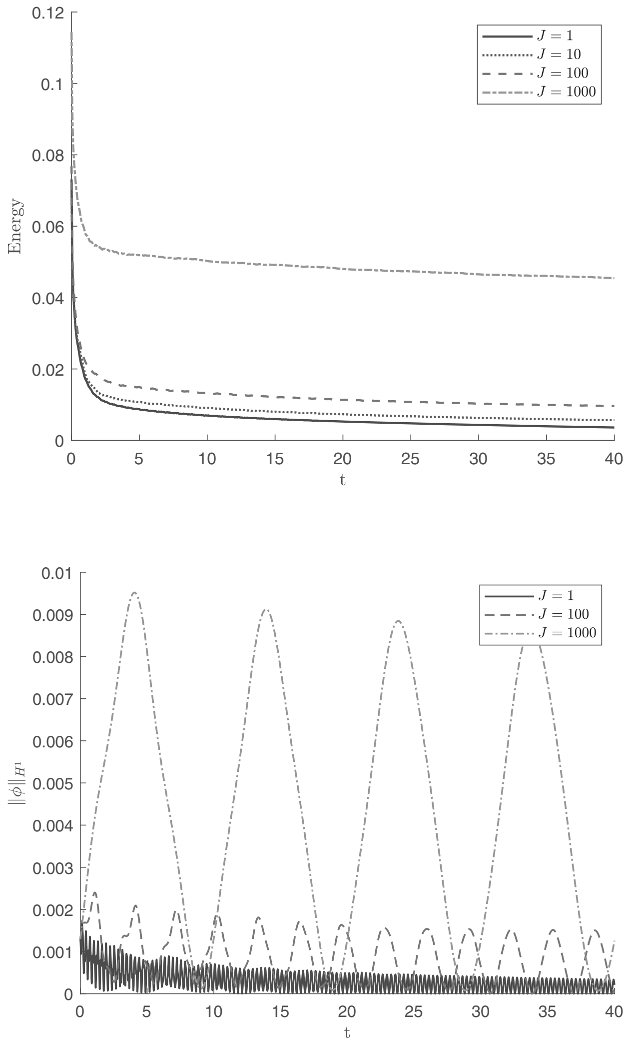

Fig. 3. Different behaviours of the discrete energy (top) and the solution (bottom) depending on J.

produces oscillations as time evolves. The amplitude and frequency of these oscillations are affected by the parameter J, but the energy depends on the mean value of the oscillations.

5. Conclusions

In this work, we studied, from the numerical point of view, a new two-temperature thermoelastic model, including the so-called microtemperatures, which was recently introduced by Fernández and Quintanilla (2021a). The approximations were obtained by using the finite element method for the spatial variable and the implicit Euler scheme for the time discretization, although the coupling among the inductive and usual thermal variables required the use of piecewise constant functions. An *a priori* error analysis was provided, obtaining the linear convergence with respect to the time step and mesh size under adequate regularity conditions. Some numerical simulations were presented to demonstrate the accuracy of the approximation (first example), the behavior of the discrete energy for different values of the constitutive coefficient a_0 (second example) and the dependence of the solution on the porosity function (third example). In particular, it is worth noting how the energy rate varies when the equilibrated inertia increases, maybe due to the oscillations of the porosity.

Even if we had used the well-known implicit Euler scheme for the time discretization, another time discretization scheme, such as the Crank–Nicolson method, could have been applied. We note that the *a priori* error analysis should be modified accordingly.

Table 1. Numerical errors for different values of h and k.

$h \downarrow\ k \rightarrow$	5×10^{-2}	1×10^{-2}	5×10^{-3}	1×10^{-3}	5×10^{-4}	1×10^{-4}	5×10^{-5}
1×10^{-1}	0.106101	0.114777	0.117589	0.120745	0.121229	0.121641	0.121695
5×10^{-2}	0.033437	0.034783	0.035505	0.036322	0.036450	0.036561	0.036575
2×10^{-2}	0.013247	0.009044	0.008995	0.009089	0.009111	0.009130	0.009133
1×10^{-2}	0.010828	0.004392	0.003947	0.003806	0.003806	0.003808	0.003809
5×10^{-3}	0.010043	0.002852	0.002138	0.001784	0.001769	0.001763	0.001762
2×10^{-3}	0.009671	0.002174	0.001303	0.000739	0.000681	0.000659	0.000674

Finally, although undoubtedly this is a theoretical numerical analysis, we think that there will be real-world applications which will can be simulated with this type of models; however, we also recognize that there is a need to obtain experimentally the numerous constitutive coefficients. Anyway, this two-temperature theory is really new and, from our point of view, it will gain a great interest over the next years.

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Jacobo Baldonedo is an assistant professor in the School of Industrial Engineering at the University of Vigo. His current research interests include solving problems in thermomechanics and biomechanics by applying the finite element method.

José R. Fernández is a full professor of applied mathematics in the School of Industrial Engineering at the University of Vigo. His research interests focus on the numerical analysis of different issues in solid mechanics as contact or thermomechanical problems.

Ramón Quintanilla is a full professor in the Department of Mathematics at the Polytechnic University of Catalonia. His research work focuses on the study of the partial/integral differential equations that determine the behavior of different classical and non-classical thermoelastic materials.

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