On iterative fixed point convergence in uniformly convex Banach space and Hilbert space

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Abstract

Some fixed point convergence properties are proved for compact and demicompact maps acting over closed, bounded and convex subsets of a real Hilbert space. We also show that for a generalized nonexpansive mapping in a uniformly convex Banach space the Ishikawa iterates converge to a fixed point. Finally, a convergence type result is established for multivalued contractive mappings acting on closed subsets of a complete metric space. These are extensions of results in Ciric, et. al. [7], Panyanak [2] and Agarwal, et. al. [9].

1 Introduction

Let $H$ be a Hilbert space and $K$ be a nonempty subset of $H$. A mapping $T : K \to H$ is said to be pseudo-contractive if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|(I - T)x - (I - T)y\|^2, \quad \text{for all } x, y \in K.$$ 

A mapping $T : K \to H$ is called hemicontractive if $F(T) = \{x \in K : Tx = x\} \neq \emptyset$ and

$$\|Tx - x^*\|^2 \leq \|x - x^*\|^2 + \|x - Tx\|^2, \quad \text{for all } x^* \in F(T) \text{ and for all } x \in K.$$

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It is easy to observe that each pseudo-contractive mapping with fixed points is hemicontractive. The reciprocal is not in general true; see [1],[4].

There are two well known methods of approximating a fixed point of a pseudo-contractive mapping, viz. Mann [11] iterative and Ishikawa [10] iterative processes. In 1991, Xu [3] introduced the following iteration process: For \( T : K \to E \), let a sequence \( \{x_n\} \) and \( x_0 \in K \), where \( K \) is a nonempty subset of a Banach space \( E \), defined iteratively as follows:

\[
\begin{align*}
x_{n+1} &= a_n x_n + b_n T y_n + c_n u_n, \\
y_n &= a'_n x_n + b'_n T x_n + c'_n v_n, & n &\geq 0,
\end{align*}
\]

(1)

where \( \{u_n\} \) and \( \{v_n\} \) are bounded sequences in \( K \) and \( \{a_n\}, \{b_n\}, \{c_n\}, \{a'_n\}, \{b'_n\} \) and \( \{c'_n\} \) are sequences in \([0,1]\), such that

\[
a_n + b_n + c_n = a'_n + b'_n + c'_n = 1,
\]

for all \( n \geq 1 \). If, in (1), \( b'_n = 0 = c'_n \), then we obtain the Mann iterative sequence in the sense of Xu. If \( c_n = 0 = c'_n \) in (1), then we obtain the Ishikawa iterative sequence.

In [7], Ciric, et al. have introduced and investigated the following modified Mann implicit iterative process. Let \( K \) be a closed convex subset of a real normed space \( N \) and \( T : K \to K \) be a mapping. Define \( \{x_n\} \) in \( K \) as follows:

\[
\begin{align*}
x_0 &\in K, \\
x_n &= a_n x_{n-1} + b_n T v_n + c_n u_n, & n &\geq 1,
\end{align*}
\]

(2)

where \( \{a_n\}, \{b_n\}, \{c_n\} \) are real sequences in \([0,1]\) such that \( a_n + b_n + c_n = 1 \), for each \( n \in \mathbb{N} \) and \( \{u_n\} \) and \( \{v_n\} \) are sequences in \( K \).

Let \( H \) be a Hilbert space and \( C \) a subset of \( H \). A mapping \( T : C \to H \) is called demicompact if it has the property that whenever \( \{u_n\} \) is a bounded sequence in \( H \) and \( \{Tu_n - u_n\} \) is strongly convergent, there exists a strongly convergent subsequence \( \{u_{n_k}\} \) of \( \{u_n\} \).

In section two of the present paper, we have shown that if \( K \) is closed, bounded and convex subset of a real Hilbert space \( H \), \( T : K \to K \) a compact hemicontractive map with \( x_0 \in T(K) \) and sequence \( \{x_n\} \) in \( T(K) \) be defined by (1) and \( \{b_n\}, \{c_n\} \) and \( \{v_n\} \) satisfy some appropriate conditions, then the sequence \( \{x_n\} \) converges strongly to a fixed point of \( T \). Also, we have investigated that if \( K \) is closed, bounded and convex subset of a real Hilbert space \( H \) and the mapping \( T : K \to K \) is continuous demicompact hemicontractive map and \( \{a_n\}, \{b_n\}, \{c_n\} \) are real sequences in \([0,1]\) such that \( a_n + b_n + c_n = 1 \), for each \( n \in \mathbb{N} \) and \( \{b_n\}, \{c_n\}, \{v_n\} \) satisfy some appropriate conditions, then the sequence \( \{x_n\} \), defined by (2), converges strongly to some fixed point of \( T \).
Let $E$ be a Banach space. A subset $K$ of $E$ is called proximinal if for each $x \in E$, there exists an element $k \in K$ such that

$$d(x, k) = \text{dist}(x, K) = \inf\{\|x - y\| : y \in K\}.$$ 

It is well known that every closed convex subset of a uniformly convex Banach space is proximinal. We shall denote by $P(K)$, the family of nonempty bounded proximinal subsets of $K$. We say that the mapping $T : E \to P(E)$ is generalized nonexpansive if

$$H(Tx, Ty) \leq a\|x - y\| + b\{d(x, Tx) + d(y, Ty)\} + c\{d(x, Ty) + d(y, Tx)\},$$

for all $x, y \in X$, where $a + 2b + 2c \leq 1$.

Bancha Panyanak proved the following Theorem in [2].

**Theorem 1.1.** Let $K$ be a nonempty compact convex subset of a uniformly convex Banach space $E$. Suppose $T : K \to P(K)$ is a nonexpansive map with a fixed point $p$. Let $\{x_n\}$ be the sequence of Ishikawa iterates defined by $x_0 \in K,$

$$y_n = (1 - \beta_n)x_n + \beta_n z_n \quad \beta_n \in [0, 1], \quad n \geq 0,$$

where $z_n \in Tx_n$ is such that $\|z_n - p\| = \text{dist}(p, Tx_n),$ and

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n z_n', \quad \alpha_n \in [0, 1],$$

where $z_n' \in Ty_n$ is such that $\|z_n' - p\| = \text{dist}(p, Ty_n).$ Assume that

(i) $0 \leq \alpha_n, \beta_n < 1$,

(ii) $\beta_n \to 0$ and

(iii) $\sum \alpha_n \beta_n = \infty$. Then the sequence $\{x_n\}$ converges to a fixed point of $T$.

In section three, we generalize the above theorem by taking generalized nonexpansive map in place of nonexpansive map in which the sequence of Ishikawa iterates converges to the fixed point of $T$.

Let $X$ be a complete metric space and $C(X)$ is collection of all nonempty closed subsets of $X$, $CB(X)$ is the collection of all nonempty closed bounded subsets of $X$. Let $H$ be a Hausdorff metric on $C(X)$, that is

$$H(A, B) = \max \{\sup_{x \in A} d(x, B), \sup_{x \in B} d(x, A)\},$$

for any $A, B \in C(X)$, where $d(x, B) = \inf\{\|x - y\| : y \in B\}$.

A function $f : X \to \mathbb{R}$ is called lower semi-continuous, if for any sequence $\{x_n\}$ in $X$ and $x \in X$,

$$x_n \to x \implies f(x) \leq \liminf_{n \to \infty} f(x_n).$$

In section four, we generalize the following result (cf. Theorem 4.2.11 in [9]) by taking $C(X)$ in place of $CB(X)$. 


Theorem 1.2. [9]. Let \( X \) be a complete metric space and let \( T_n : X \to CB(X)(n = 0, 1, 2, 3, \ldots) \) be contraction mappings each having Lipschitz constant \( k < 1 \), i.e.,
\[
H(T_n x, T_n y) \leq kd(x, y),
\]
for all \( x, y \in X \) and \( n \in (0, 1, 2, 3, \ldots) \). If \( \lim_{n \to \infty} H(T_n x, T_0 x) = 0 \) uniformly for \( x \in X \), then \( \lim_{n \to \infty} H(F(T_n), F(T_0)) = 0 \).

2 Fixed point theorems for hemicontractive map

We shall make use of the following Lemmas.

Lemma 2.1. [8]. Let \( H \) be a Hilbert space, then for all \( x, y, z \in H \),
\[
\|ax + by + cz\|^2 = a\|x\|^2 + b\|y\|^2 + c\|z\|^2 - ab\|x - y\|^2 - bc\|y - z\|^2 - ca\|z - x\|^2,
\]
where \( a, b, c \in [0, 1] \) and \( a + b + c = 1 \).

Lemma 2.2. [5]. Suppose that \( \{\rho_n\}, \{\sigma_n\} \) are two sequences of nonnegative numbers such that for some real number \( N_0 \geq 1 \),
\[
\rho_{n+1} \leq \rho_n + \sigma_n, \quad \forall \ n \geq N_0.
\]
(a) If \( \sum_{n=1}^{\infty} \sigma_n < \infty \), then \( \lim \{\rho_n\} \) exists.
(b) If \( \sum_{n=1}^{\infty} \sigma_n < \infty \) and \( \{\rho_n\} \) has a subsequence converging to zero, then \( \lim_{n \to \infty} \rho_n = 0 \).

Now we prove our main results in this section which is generalization of [7], Theorem 4

Theorem 2.3. Let \( K \) be a closed bounded convex subset of a real Hilbert space \( H \) and \( T : K \to K \) a compact, hemicontractive map. Let \( \{a_n\}, \{b_n\}, \{c_n\} \) be real sequences in \([0, 1]\) such that \( a_n + b_n + c_n = 1 \), for each \( n \in \mathbb{N} \) and satisfying:
(i) \( \{b_n\} \subset [\delta, 1 - \delta] \) for some \( \delta \in (0, 1/2] \),
(ii) \( \sum_{n=1}^{\infty} c_n < \infty \).
For arbitrary \( x_0 \in T(K) \), let a sequence \( \{x_n\} \) in \( T(K) \) be iteratively defined by
\[
x_n = a_n x_{n-1} + b_n T v_n + c_n u_n, \quad n \geq 1,
\]
where \( v_n \in T(K) \) are chosen such that \( \sum_{n=1}^{\infty} \|v_n - x_n\| < \infty \) and \( \{u_n\}_{n=1}^{\infty} \) is arbitrary sequence in \( K \). Then \( \{x_n\}_{n=1}^{\infty} \) converges strongly to some fixed point of \( T \).
Proof. Let $T : K \to K$ be a continuous map, where $K$ is a closed bounded convex subset of a real Hilbert space $H$. Then $T(K)$ is closed subset of $K$ and $\overline{T(K)}$ is compact. Hence $T(K)$ is compact. Let $A = \overline{r(T(K))}$, convex closure of $T(K)$. Then $A \subset K$. Since $T(K)$ is a relatively compact subset of $K$, by Mazur’s theorem $\overline{r(T(K))}$ is compact and convex. Furthermore, $T(A) \subset A$. Now we have to show that in restriction $T : A \to A$, $\{x_n\}_{n=1}^\infty$ converges strongly to some fixed point of $T$. Let $x^* \in T(K)$ be a fixed point of $T$ and $M = \text{diam}(T(K))$, diameter of $T(K)$. Since $T$ is hemicontractive,

$$
\|Tv_n - x^*\|^2 \leq \|v_n - x^*\|^2 + \|v_n - Tv_n\|^2,
$$

for each $n \in \mathbb{N}$. By virtue of (3), Lemma 2.1 and (4), we obtain

$$
\|x_n - x^*\|^2 = \|a_n x_{n-1} + b_nTv_n + c_nu_n - x^*\|^2
= \|a_n (x_{n-1} - x^*) + b_n(Tv_n - x^*) + c_n(u_n - x^*)\|^2
= a_n \|x_{n-1} - x^*\|^2 + b_n \|Tv_n - x^*\|^2 + c_n \|u_n - x^*\|^2
- a_n b_n \|x_{n-1} -Tv_n\|^2 - b_n c_n \|Tv_n - u_n\|^2
- a_n c_n \|x_{n-1} - u_n\|^2
\leq a_n \|x_{n-1} - x^*\|^2 + b_n \|Tv_n - x^*\|^2
+ c_n \|u_n - x^*\|^2 - a_n b_n \|x_{n-1} -Tv_n\|^2
(1 - b_n) \|x_{n-1} - x^*\|^2 + b_n (\|v_n - x^*\|^2
+ \|v_n - Tv_n\|^2) + c_n M^2 - a_n b_n \|x_{n-1} -Tv_n\|^2.
$$

Also, we have

$$
\|v_n - x^*\|^2 \leq \|v_n - x_n\|^2 + \|x_n - x^*\|^2 + 2 \|x_n - x^*\| \|v_n - x_n\|
\leq \|v_n - x_n\|^2 + \|x_n - x^*\|^2 + 2M \|v_n - x_n\|,
$$

and

$$
\|v_n - Tv_n\|^2 \leq \|v_n - x_n\|^2 + \|x_n - Tv_n\|^2 + 2 \|x_n - Tv_n\| \|v_n - x_n\|
\leq \|v_n - x_n\|^2 + \|x_n - Tv_n\|^2 + 2M \|v_n - x_n\|
$$

and

$$
\|x_n - Tv_n\|^2 = \|(a_n x_{n-1} + b_nTv_n + c_nu_n) - Tv_n\|^2
= \|[(1 - b_n) (x_{n-1} + b_nTv_n + c_n(u_n - Tv_n)]\|^2
\leq [(1 - b_n) \|x_{n-1} -Tv_n\| + c_n \|u_n - x_{n-1}\|]^2
\leq [(1 - b_n) \|Tv_n\| + M c_n]^2
\leq (1 - b_n)^2 \|Tv_n\|^2 + 3M^2 c_n.
$$
In view of (7) and (8), (5) takes the form

$$
\|x_n - x^*\|^2 \leq (1 - b_n)^2 \|x_{n-1} - x^*\|^2 \\
+ b_n \|x_n - x^*\|^2 + 2b_n \|v_n - x_n\|^2 + 4Mb_n \|v_n - x_n\| \\
+ 4M^2c_n - b_n[a_n - (1 - b_n)^2] \|x_{n-1} - Tv_n\|^2.
$$

(9)

Using $a_n + b_n + c_n = 1$ in condition $(i)$, we have

$$
a_n - (1 - b_n)^2 = 1 - b_n - c_n - (1 - b_n)^2 \\
= b_n(1 - b_n) - c_n \\
\geq \delta^2 - c_n. \quad (10)
$$

From condition $(ii)$, it follows that there exists a positive integer $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, we have $c_n \leq \delta^3$, i.e. $\delta^2 - c_n \geq \delta^2 - \delta^3 = \delta^2(1 - \delta)$. Thus, from (10), we obtain

$$
a_n - (1 - b_n)^2 \geq \delta^2(1 - \delta). \quad (11)
$$

From (9) and (11), we have, for all $n \geq n_0$

$$(1 - b_n) \|x_n - x^*\|^2 \leq (1 - b_n) \|x_{n-1} - x^*\|^2 + 2b_n \|v_n - x_n\|^2 \\
+ 4Mb_n \|v_n - x_n\| + 4M^2c_n \\
- b_n\delta^2(1 - \delta) \|x_{n-1} - Tv_n\|^2.
$$

or

$$
\|x_n - x^*\|^2 \leq \|x_{n+1} - x^*\|^2 + \frac{2b_n}{(1 - b_n)} \|v_n - x_n\|^2 \\
+ \frac{4M - b_n}{(1 - b_n)} \|v_n - x_n\| + \frac{4M^2c_n}{(1 - b_n)} \\
+ b_n\frac{\delta^2(1 - \delta)}{(1 - b_n)} \|x_{n-1} - Tv_n\|^2.
$$

(12)

Since $\frac{1}{1 - \delta^3} \leq \frac{1}{\delta}$ and $\frac{-1}{1 - \delta} \leq \frac{-\delta}{\delta^2}$; $\delta \leq b_n \leq 1 - \delta$, we have $\frac{b_n}{1 - b_n} \leq \frac{1 - \delta}{\delta}$
\[ \frac{1}{\delta} - 1 < \frac{1}{\delta}. \]

Hence from (12), we have
\[ \|x_n - x^*\|^2 \leq \|x_{n-1} - x^*\|^2 + \frac{2}{\delta} \|v_n - x_n\|^2 + \frac{4M}{\delta} \|v_n - x_n\| + \frac{4M^2 c_n}{\delta} \]
\[ - \frac{\delta^3 (1 - \delta)}{(1 - b_n)} \|x_{n-1} - Tv_n\|^2 \]
\[ \leq \|x_{n-1} - x^*\|^2 + \frac{2}{\delta} \|v_n - x_n\|^2 + \frac{4M}{\delta} \|v_n - x_n\| + \frac{4M^2 c_n}{\delta} \]
\[ - \frac{\delta^3 (1 - \delta)}{(1 - \delta)} \|x_{n-1} - Tv_n\|^2 \]
\[ \leq \|x_{n-1} - x^*\|^2 + \frac{2}{\delta} \|v_n - x_n\|^2 + \frac{4M}{\delta} \|v_n - x_n\| + \frac{4M^2 c_n}{\delta} \]
\[ - \delta^3 \|x_{n-1} - Tv_n\|^2, \]
i.e. \[ \|x_n - x^*\|^2 \leq \|x_{n-1} - x^*\|^2 - \delta^3 \|x_{n-1} - Tv_n\|^2 + \sigma_n, \]
for all \( n \geq n_0 \).

(13)

where
\[ \sigma_n = \left[ \frac{4M}{\delta} \|v_n - x_n\|^2 + \frac{4M^2}{\delta} \right] \]
\[ = \frac{1}{\delta} \left[ \frac{2}{\delta} \|v_n - x_n\|^2 + 4M \|v_n - x_n\| + 4M^2 c_n \right]. \]

(14)

By the hypothesis of the theorem, we obtain
\[ \sum_{j=n_0}^{\infty} \sigma_j < +\infty. \]

(15)

From (14), we get
\[ \delta^3 \|x_{n-1} - Tv_n\|^2 \leq \|x_{n-1} - x^*\|^2 - \|x_n - x^*\|^2 + \sigma_n, \]
and hence
\[ \delta^3 \sum_{j=n_0}^{\infty} \|x_{j-1} - Tv_j\|^2 \leq \sum_{j=n_0}^{\infty} \sigma_j + \|x_{n_0-1} - x^*\|^2. \]

By (15) we get \( \sum_{j=n_0}^{\infty} \|x_{j-1} - Tv_j\|^2 < +\infty \). This implies \( \lim_{n \to \infty} \|x_{n-1} - Tv_n\| = 0 \). From (8) and condition (ii), it further implies that \( \lim_{n \to \infty} \|x_n - Tv_n\| = 0 \). Also the condition \( \sum_{j=n_0}^{\infty} \|v_n - x_n\| < \infty \) implies \( \lim_{n \to \infty} \|v_n - x_n\| = 0 \).

Thus from (7), we have
\[ \lim_{n \to \infty} \|v_n - Tv_n\| = 0. \]

(16)
By compactness of $\overline{T(K)}$, there is a convergent subsequence $\{v_{n_j}\}$ of $\{v_n\}$, such that it converges to some point $z \in \overline{T(K)} \subset \overline{rn(T(K))} = A$. By continuity of $T$, $\{Tv_{n_j}\}$ converges to $Tz$. Therefore, from (16), we conclude that $Tz = z$. Further, $\lim_{n \to \infty} \|v_n - x_n\| = 0$ implies
\[ \lim_{j \to \infty} \|x_{n_j} - z\| = 0. \tag{17} \]

Since (13) holds for any fixed points of $T$, we have
\[ \|x_n - z\|^2 \leq \|x_{n-1} - z\|^2 - \delta^3 \|x_{n-1} - Tv_n\|^2 + \sigma_n \]
and in view of (15), (17) and Lemma 2.2, we conclude that $\|x_n - z\| \to 0$ as $n \to \infty$ i.e. $x_n \to z$ as $n \to \infty$. Thus, we have proved that a sequence $\{x_n\}$ converges strongly to some fixed point of $T$. This sequence in $K$ automatically converges strongly to a fixed point of $T$.

**Theorem 2.4.** Let $K$ be a closed bounded convex subset of a real Hilbert space $H$ and $T : K \to K$ a continuous demicompact and hemicontractive map. Let $\{a_n\}, \{b_n\}$ and $\{c_n\}$ be a real sequences in $[0,1]$ such that $a_n + b_n + c_n = 1$ for each $n \in \mathbb{N}$ and satisfying:

(i) $\{b_n\} \subset [\delta, 1 - \delta]$, for some $\delta \in (0, \frac{1}{2}]$,

(ii) $\sum_{n=1}^{\infty} c_n < \infty$.

For arbitrary $x_0 \in K$, let a sequence $x_n \in K$ be iteratively defined by
\[ x_n = a_n x_{n-1} + b_n Tv_n + c_n u_n, \quad n \geq 1, \tag{18} \]
where $v_n \in K$ are chosen such that $\sum_{n=1}^{\infty} \|v_n - x_n\| < \infty$. Then $\{x_n\}_{n=1}^{\infty}$ converges strongly to some fixed point of $T$.

**Proof.** Let $x^* \in K$ be a fixed point of hemicontractive map $T$ and $M = \text{diam}(K)$. Using inequality (4) as in the proof of Theorem 2.3 and proceeding in the similar manner we arrive at (16) which implies that the sequence $\{v_n - Tv_n\}_{n \in \mathbb{N}}$ converges strongly to zero. As $T$ is demicompact, it results that there exists a strongly convergent subsequence $\{v_{n_j}\}$ of $\{v_n\}$, such that $v_{n_j} \to z \in K$. By continuity of $T$, $Tv_{n_j}$ converges to $Tz$. Therefore, from (16), we conclude that $Tz = z$. Further, $\lim_{n \to \infty} \|v_n - x_n\| = 0$ implies
\[ \|x_{n_j} - z\| = 0. \tag{19} \]

Since (13) holds for any fixed points of $T$, we have
\[ \|x_n - z\|^2 \leq \|x_{n-1} - z\|^2 - \delta^3 \|x_{n-1} - Tv_n\|^2 + \sigma_n. \tag{20} \]

In view of (15), (19) and Lemma 2.2, we conclude that $\|x_n - z\| \to 0$ as $n \to \infty$ i.e. $x_n \to z$ as $n \to \infty$. Thus, we have proved that $\{x_n\}$ converges strongly to some fixed point of $T$. \qed
3 Ishikawa iteration for multivalued generalized nonexpansive map

To prove the main theorem of this section, we need the following Lemmas:

Lemma 3.1. [3]. Let E be a Banach space. Then E is uniformly convex if and only if for any given number ρ > 0, the square norm ∥∥^2 of E is uniformly convex on Bρ, the closed ball centered at the origin with radius ρ; namely, there exists a continuous strictly increasing function φ : [0, ∞) → [0, ∞) with φ(0) = 0 such that

\[ \|αx + (1 − α)y\|^2 \leq α \|x\|^2 + (1 − α) \|y\|^2 − α(1 − α)φ(∥x − y∥), \]

for all x, y ∈ Bρ, α ∈ [0, 1].

Lemma 3.2. [2]. Let \{α_n\}, \{β_n\} be two real sequences such that
(i) 0 ≤ α_n, β_n < 1,
(ii) β_n → 0 as n → ∞ and
(iii) \sum_{n=1}^{∞} α_n β_n = ∞.
Let \{γ_n\} be a nonnegative real sequence such that \sum_{n=1}^{∞} α_n β_n (1 − β_n) γ_n is bounded. Then \{γ_n\} has a subsequence which converges to zero.

Theorem 3.3. Let K be a nonempty compact convex subset of a uniformly convex Banach space E. Suppose T : K → P(K) is a generalized nonexpansive map with a fixed point p. Let \{x_n\} be the sequence of Ishikawa iterates defined by x_0 ∈ K,

\[ y_n = (1 − β_n)x_n + β_n z_n, \quad β_n ∈ [0, 1], \quad n ≥ 0, \]

where z_n ∈ Tx_n is such that \|z_n − p\| = dist(p, Tx_n), and

\[ x_{n+1} = (1 − α_n)x_n + α_n z_n', \quad α_n ∈ [0, 1], \]

where z_n' ∈ Ty_n is such that \|z_n' − p\| = dist(p, Ty_n). Assume that
(i) 0 ≤ α_n, β_n < 1
(ii) β_n → 0 and
(iii) \sum_{n=1}^{∞} α_n β_n = ∞. Then the sequence \{x_n\} converges to a fixed point of T.

Proof. By using Lemma 3.1, we have

\[ \|x_{n+1} − p\|^2 = \|(1 − α_n)x_n + α_n z_n' − p\|^2 \]
\[ \leq (1 − α_n) \|x_n − p\|^2 + α_n \|z_n' − p\|^2 − α_n(1 − α_n)φ(\|x_n − z_n\|) \]
\[ \leq (1 − α_n) \|x_n − p\|^2 + α_n H^2(Ty_n, Tp) \]
\[ − α_n(1 − α_n)φ(\|x_n − z_n'\|). \quad (21) \]
By generalized nonexpansive property of $T$, we have

\[
H(Ty_n, Ty_n) \leq a\|y_n - p\| + b\|y_n, Ty_n\| + c\{d(p, Ty_n) + d(y_n, Tp)\} \\
\leq a\|y_n - p\| + b(\|y_n - p\| + d(p, Ty_n)) + c\{d(p, Ty_n) + d(y_n, Tp)\} \\
\leq (a + b + c)\|y_n - p\| + (b + c)d(p, Ty_n) \\
\leq (a + b + c)\|y_n - p\| + (b + c)H(Ty_n, Ty_n)
\]

Since $\frac{a + b + c}{1 - (b + c)} \leq 1$, it follows that

\[
H(Ty_n, Ty_n) \leq \|y_n - p\| 
\]

From (21) and (23), we get

\[
\|x_{n+1} - p\|^2 \leq (1 - \alpha_n)\|x_n - p\|^2 + \alpha_n\|y_n - p\|^2 \\
- \alpha_n(1 - \alpha_n)\phi(\|x_n - z_n\|).
\]

Now

\[
\|y_n - p\|^2 = \|(1 - \beta_n)x_n + \beta_nz_n - p\|^2 \\
\leq (1 - \beta_n)\|x_n - p\|^2 + \beta_n\|z_n - p\|^2 - \beta_n(1 - \beta_n)\phi(\|x_n - z_n\|) \\
\leq (1 - \beta_n)\|x_n - p\|^2 + \beta_nH^2(Tx_n, Tp) - \beta_n(1 - \beta_n)\phi(\|x_n - z_n\|) \\
\leq \|x_n - p\|^2 - \beta_n(1 - \beta_n)\phi(\|x_n - z_n\|).
\]

From (24) and (25), we get

\[
\|x_{n+1} - p\|^2 \leq \|x_n - p\|^2 - \alpha_n\beta_n(1 - \beta_n)\phi(\|x_n - z_n\|).
\]

Therefore

\[
\alpha_n\beta_n(1 - \beta_n)\phi(\|x_n - z_n\|) \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2.
\]

This implies

\[
\sum_{n=1}^{\infty} \alpha_n\beta_n(1 - \beta_n)\phi(\|x_n - z_n\|) \leq \|x_1 - p\|^2 < \infty.
\]

By Lemma 3.2, there exists a subsequence $\{x_{n_k} - z_{n_k}\}$ of $\{x_n - z_n\}$ such that $\phi(\|x_{n_k} - z_{n_k}\|) \to 0$ as $k \to \infty$ and hence $\|x_{n_k} - z_{n_k}\| \to 0$, by continuity and
strictly increasing nature of $\phi$. By compactness of $K$, we may assume that $x_{n_k} \to q$, for some $q \in K$. Thus,

$$
\text{dist}(q, Tx) \leq \|q - x_{n_k}\| + \text{dist}(x_{n_k}, Tx_{n_k}) + H(Tx_{n_k}, Tq)
$$

$$
\leq \|q - x_{n_k}\| + \|x_{n_k} - z_{n_k}\| + \|x_{n_k} - q\| \to 0 \text{ as } k \to \infty
$$

(27)

Hence $q$ is a fixed point of $T$. Now on taking $q$ in place of $p$, we get $\|x_n - q\|$ as a decreasing sequence by (26). Since $\|x_{n_k} - q\| \to 0$ as $k \to \infty$, it follows that $\{\|x_n - q\|\}$ converges to zero, so that the conclusion of the theorem follows.

$\square$

4 Fixed point theorem for multivalued contractive mappings

The main result of this section is as follows:

**Proposition 4.1.** Let $X$ be a complete metric space and let $S, T : X \to C(X)$ be a multivalued mapping. If there exists a constant $c \in (0, 1)$ such that for any $x \in X$ there is $y \in I_b^{(S)x}$ and $I_b^{(T)x}$ satisfying $d(y, S(y)) \leq cd(x, y)$ and $d(y, Ty) \leq cd(x, y)$ with $c < b$ and $f$ is lower semi-continuous, then

$$
H(F(s), F(T)) \leq (b - c)^{-1}\sup_{x \in X}H(Sx, Tx),
$$

(28)

where the following have been taken from [12], for mapping $f : X \to R$, $f(x)$ is defined as $f(x) = d(x, Tx)$ and for mapping $S$, $f(x)$ is defined as $f(x) = d(x, Sx)$,

$$
I_b^{(S)x} = \{y \in S(x) : bd(x, y) \leq d(x, Sx)\}
$$

and

$$
I_b^{(T)x} = \{y \in T(x) : bd(x, y) \leq d(x, Tx)\}.
$$

**Proof.** Since $S(x), T(x) \in C(X)$ for any $x \in X$, $I_b^{(S)x}$ and $I_b^{(T)x}$ are nonempty for any constant $b \in (0, 1)$. Let $x_0 \in F(S)$ implies $x_0 \in S(x_0)$. Then there is another point $x_1 \in S(x_0)$ such that for any initial point $x_0 \in X$, there exists $x_1 \in I_b^{(x_0)}$. For $x_1$, there exists $Sx_1$ such that

$$
d(x_1, Sx_1) \leq cd(x_0, x_1),
$$

and for any $x_0 \in X$, there exists $x_1 \in I_b^{(T)x_0}$ i.e. $\{x_1 \in T(x_0) : bd(x_0, x_1) \leq d(x_0, Tx_1)\}$ satisfying

$$
d(x_1, Tx_1) \leq cd(x_0, x_1),
$$
and for $x_1 \in X$, there is $x_2 \in \mathcal{I}_b^{(T)x_1}$ satisfying
\[ d(x_2, Tx_2) \leq cd(x_1, x_2). \]

Continuing this process, we can get an iterative sequence \( \{x_n\}_{n=0}^{\infty} \), where
\[ x_{n+1} \in \mathcal{I}_b^{(T)x_n} \]
and \[ d(x_{n+1}, Tx_{n+1}) \leq cd(x_n, x_{n+1}), \quad n = 0, 1, 2, .... \] (29)

On the other hand, $x_{n+1} \in \mathcal{I}_b^{(T)x_n}$ implies
\[ bd(x_n, x_{n+1}) \leq d(x_n, Tx_n), \quad n = 0, 1, 2, .... \] (30)

From (30) and (31), we have
\[ d(x_{n+1}, Tx_{n+1}) \leq \frac{c}{b} d(x_n, Tx_n), \quad n = 0, 1, 2, ... \]
and
\[ d(x_{n+1}, x_{n+2}) \leq \frac{c}{b} d(x_n, x_{n+1}), \quad n = 0, 1, 2, .... \]

Observe that
\[
\begin{align*}
  d(x_n, x_{n+1}) & \leq \frac{c}{b} d(x_{n-1}, x_n) \\
  & \leq \frac{c}{b} \left[ \frac{c}{b} d(x_{n-2}, x_{n-1}) \right] \\
  & = \frac{c^2}{b^2} d(x_{n-2}, x_{n-1}) \\
  & \quad \vdots \\
  & \quad \vdots \\
  & = \frac{c^n}{b^n} d(x_0, x_1).
\end{align*}
\] (31)

Since $c < b, \frac{c}{b} < 1$, therefore $\lim_{n \to \infty} \left( \frac{c}{b} \right)^n \to 0$, which means that \( \{x_n\}_{n=0}^{\infty} \) is a Cauchy sequence. By the completeness of $X$, there exists $v \in X$ such that \( \{x_n\}_{n=0}^{\infty} \) converges to $v$.

Now we have to show that $v \in F(T)$. We have given \( \{f(x_n)\}_{n=0}^{\infty} = \{d(x_n, Tx_n)\}_{n=0}^{\infty} \) to be a decreasing sequence and hence it converges to zero. Since $f$ is lower semi-continuous, as $x_n \to v$, we have $0 \leq f(v) \leq \lim_{n \to \infty} f(x_n) = 0$. Hence $f(v) = 0$. Finally the closeness of $T(v)$ implies $v \in T(v)$. Hence
Now, we observe that
\[
d(x_0, v) \leq \sum_{n=0}^{\infty} d(x_n, x_{n+1}) \\
\leq \sum_{n=0}^{\infty} \left( \frac{c}{b} \right)^n d(x_0, x_1) \\
\leq \left( \frac{1 - \frac{c}{b}}{1 - \frac{c}{b}} \right) d(x_0, x_1) \\
\leq \left( 1 - \frac{c}{b} \right)^{-1} \frac{1}{b} d(x_0, Tx_0).
\]
(32)

Now
\[
d(x_0, Tx_0) \leq \sup_{x \in Sx_0} d(x, Tx_0) \\
\leq \max \{ \sup_{x \in Sx_0} d(x, Tx_0), \sup_{x \in Tx_0} d(x, Sx_0) \} \\
= H(Sx_0, Tx_0).
\]
(33)

Hence we get
\[
d(x_0, v) \leq b(b - c)^{-1} \frac{1}{b} d(x_0, Tx_0) \\
\leq (b - c)^{-1} H(Sx_0, Tx_0).
\]
(34)

Interchanging the roles of \( S \) and \( T \), for each \( y_0 \in F(T) \) and \( y_1 \in Sy_0 \), for any \( y_0 \in X \) and \( u \in F(S) \), we have
\[
d(y_0, u) \leq (b - c)^{-1} H(Sy_0, Ty_0).
\]

Thus, we have
\[
H(F(S), F(T)) \leq (b - c)^{-1} \sup_{x \in X} H(Sx, Tx).
\]

**Example** Let \( X = \left\{ \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots, \frac{1}{2^n}, \ldots \right\} \cup \{0, 1\} \), \( d(x, y) = |x - y| \) for any \( x, y \in X \), be a complete metric space. Define the mappings \( S, T : X \to C(X) \) as and
\[
S(x) = \begin{cases} 
\left\{ \frac{1}{2^{n+2}}, 1 \right\}, & \text{if } x = \frac{1}{2^n}, \ n = 0, 1, 2, \ldots \\
\left\{ 0, \frac{1}{2} \right\}, & \text{if } x = 0.
\end{cases}
\]
Now
\[ f(x) = d(x, Tx) = \begin{cases} \frac{1}{2^{n+1}}, & \text{if } x = \frac{1}{2^n}, n = 1, 2, \ldots \\ 0, & \text{if } x = 0, 1 \end{cases} \]
and
\[ f(x) = d(x, Sx) = \begin{cases} \frac{3}{2^{n+2}}, & \text{if } x = \frac{1}{2^n}, n = 1, 2, \ldots \\ 0, & \text{if } x = 0, 1 \end{cases} \]
Hence \( f \) is continuous for both mappings \( S \) and \( T \). Obviously, \( S \) and \( T \) are not contractive mappings. It is clear that
\[ H \left( T \left( \frac{1}{2^n} \right), T(0) \right) = \frac{1}{2}. \]

Hence
\[ H \left( T \left( \frac{1}{2^n} \right), T(0) \right) = \frac{1}{2} \geq \frac{1}{2^n} = \left| \frac{1}{2^n} - 0 \right| = d \left( \frac{1}{2^n}, 0 \right) \quad n = 1, 2, 3, \ldots. \]

For mapping \( S : X \rightarrow C(X) \)
\[ H \left( S \left( \frac{1}{2^n} \right), S(0) \right) = \frac{1}{2}. \]

Hence
\[ H \left( S \left( \frac{1}{2^n} \right), S(0) \right) = \frac{1}{2} \geq \frac{1}{2^n} = \left| \frac{1}{2^n} - 0 \right| = d \left( \frac{1}{2^n}, 0 \right), \quad n = 1, 2, 3, \ldots. \]

Furthermore, there exists \( y \in I^x_{\frac{1}{2^n}} \), for any \( x \in X \), such that \( d(y, T(y)) = \frac{1}{2} d(x, y) \) and \( d(y, S(y)) < \frac{1}{2} d(x, y) \), then
\[ H(F(S), F(T)) = 0 \]
and
\[ \text{Sup}_{x \in X} H(Sx, Tx) = \frac{1}{4}. \]

Hence, we get \( H(F(S), F(T)) \leq (b - c)^{-1} \text{Sup}_{x \in X} H(Sx, Tx) \).

**Theorem 4.2.** Let \( X \) be a complete metric space and let \( T_n : X \rightarrow C(X) \) \((n = 0, 1, 2, 3, \ldots)\) be multivalued mappings. If there exists a constant \( c \in (0, 1) \) such that for any \( x \in X \), there is \( y \in I^{(n)x}_{\frac{1}{2^n}} \) satisfying
\[ d(y, T_0 y) \leq cd(x, y), \quad \text{for } n = 1, 2, 3, 4, \ldots. \]
If \( \lim_{n \rightarrow \infty} H(T_n x, T_0 x) = 0 \) uniformly for \( x \in X \), then \( \lim_{n \rightarrow \infty} H(F(T_n), F(T_0)) = 0 \).
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Proof. Since

$$\lim_{n \to \infty} H(T_n(x), T_0(x)) = 0$$

uniformly for $x \in X$, it is possible to select $n_0 \in \mathbb{N}$, such that

$$\sup_{x \in X} H(T_n x, T_0 x) \leq (b - c) \epsilon, \quad \text{for all } n \geq n_0.$$ 

By proposition 4.1, we have

$$H(F(T_n), F(T_0)) < \epsilon, \quad \text{for all } n \geq n_0.$$ 

Hence

$$\lim_{n \to \infty} H(F(T_n), F(T_0)) = 0.$$ 

\[\square\]

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References


