Altering distances, some generalizations of Meir - Keeler theorems and applications

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Abstract

The purpose of this paper is to prove a general theorem of Meir - Keeler type using the notion of altering distance for occasionally weakly compatible mappings satisfying an implicit relation.

As application, a problem of Meir - Keeler type satisfying a condition of integral type becomes a special case of a problem of Meir - Keeler type with an altering distance.

1 Introduction

Let $f$ and $g$ be self mappings of a metric space $(X,d)$. We say that $x \in X$ is a coincidence point of $f$ and $g$ if $fx = gx$.

We denote by $C(f,g)$ the set of all coincidence points of $f$ and $g$.

A point $w$ is a point of coincidence of $f$ and $g$ if there exists an $x \in X$ such that $w = fx = gx$.

Jungck [10] defined $f$ and $g$ to be compatible if $\lim d(fgx_n, gfx_n) = 0$ whenever $\{x_n\}$ is a sequence in $X$ such that $\lim fx_n = \lim gx_n = t$ for some $t \in X$.

In 1994, Pant introduced the notion of pointwise $R$ - weakly commuting mappings. It is proved in [25] that pointwise $R$ - weakly commuting is equivalent to commutativity in coincidence points.

Key Words: Fixed point, Meir - Keeler type, altering distance, integral type, occasionally weakly compatible.

2010 Mathematics Subject Classification: Primary 54H25; Secondary 47H10.

Received: January 2013

Accepted: February 2013

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Definition 1.1 ([11]). Two self mappings of a metric space \((X, d)\) are said to be weakly compatible if \(f gu = g fu\) for each \(u \in C(f, g)\).


Definition 1.2. Two self mappings \(f\) and \(g\) of a metric space \((X, d)\) are said to be occasionally weakly compatible (owc) mappings if there exists a point \(x \in X\) which is a coincidence point of \(f\) and \(g\) at which \(f\) and \(g\) commute.

Remark 1.1. Two weakly compatible mappings having coincidence points are owc. The converse is not true, as shown in the Example of [3].

Some fixed point theorems for occasionally weakly compatible mappings are proved in [2], [12], [34] and in other papers.

Lemma 1.1 ([12]). Let \(X\) be a nonempty set and let \(f\) and \(g\) be owc self maps of \(X\). If \(f\) and \(g\) have a unique point of coincidence \(w = fx = gx\), then \(w\) is the unique common fixed point of \(f\) and \(g\).

2 Preliminaries

In 1969, Meir and Keeler [19], established a fixed point theorem for self mappings of a metric space \((X, d)\) satisfying the following condition:

for each \(\varepsilon > 0\), there exists a \(\delta > 0\) such that

\[
\varepsilon < d(x, y) < \varepsilon + \delta \text{ implies } d(f x, f y) < \varepsilon.
\]  \quad (2.1)

There exists a vast literature which generalizes the result of Meir-Keeler. In [18], Mati and Pal proved a fixed point theorem for a self mapping of a metric space \((X, d)\) satisfying the following condition which is a generalization of (2.1):

for \(\varepsilon > 0\), there exists \(\delta > 0\) such that

\[
\varepsilon < \max\{d(x, y), d(x, f x), d(y, f y)\} < \varepsilon + \delta \text{ implies } d(f x, f y) < \varepsilon.
\]  \quad (2.2)

In [29] and [35], Park-Rhoades, respectively Rao-Rao extend this result for two self mappings \(f\) and \(g\) of a metric space \((X, d)\) satisfying the following condition:

\[
\varepsilon < \max\left\{d(f x, f y), d(f x, g x), d(f y, g y), \frac{1}{2} [d(f x, g y) + d(f y, g x)]\right\} < \varepsilon + \delta
\]  \quad (2.3)

implies \(d(g x, g y) < \varepsilon\).
In 1986, Jungck [10] and Pant [21] extend these results for four mappings. It is known by Jungck [10], Pant [22], [24], [25] and other papers that, in the case of theorems for four mappings $A, B, S$ and $T : (X, d) \to (X, d)$, a condition of Meir-Keeler type does not assure the existence of a fixed point. The following theorem is stated in [9].

**Theorem 2.1.** Let $(A, S)$ and $(B, T)$ be compatible pairs of self mappings of a complete metric space $(X, d)$ such that

1) $AX \subset TX$ and $BX \subset SX$,

2) given an $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $x, y \in X$,

$$\varepsilon \leq M(x, y) < \varepsilon + \delta$$

implies $d(Ax, By) < \varepsilon$,

where

$$M(x, y) = \max \left\{ d(Sx, Ty), d(Sx, Ax), d(Ty, By), \right. \frac{1}{2} \left[ d(Sx, By) + d(Ty, Ax) \right], \left. \frac{1}{2} \left[ d(Sx, By) + d(Ty, Ax) \right] \right\}$$

and

3) $d(Ax, By) < k(d(Sx, Ty) + d(Sx, Ax) + d(Ty, By) + d(Sx, By) + d(Ty, Ax))$

for all $x, y \in X$, where $k \in [0, \frac{1}{3})$.

If one of mappings $A, B, S$ and $T$ is continuous then $A, B, S$ and $T$ have a unique common fixed point.

Some similar theorems are proved in [8], [27], [28] and in other papers. Recently, Theorem 2.1 was improved and extended for weakly compatible pairs in [4].

**Theorem 2.2.** Let $(A, S)$ and $(B, T)$ be weakly compatible pairs of self mappings of a complete metric space $(X, d)$ such that the following conditions hold:

1) $AX \subset TX$ and $BX \subset SX$,

2) one of $AX, BX, SX$ and $TX$ is closed,
3) for each $\varepsilon > 0$, there exists a $\delta > 0$ such that
\[ \varepsilon < M(x, y) < \varepsilon + \delta \implies d(Ax, By) \leq \varepsilon, \]

4) $x, y \in X$, $M(x, y) > 0$ implies $d(Ax, By) < M(x, y)$,

5) 
\[ d(Ax, By) < k \max \{ d(Sx, Ty), d(Sx, Ax), d(Ty, By), d(Sx, By), d(Ty, Ax) \} \]
for all $x, y \in X$, where $k \in [0, \frac{1}{3})$.

Then $A, B, S$ and $T$ have a unique common fixed point.

Other generalizations of Theorem 2.1 are proved in [5].

**Theorem 2.3.** Let $(A, S)$ and $(B, T)$ be weakly compatible pairs of self mappings of a complete metric space $(X, d)$ such that

1) $AX \subset TX$ and $BX \subset SX$,

2) one of $AX$, $BX$, $SX$ and $TX$ is closed,

3) given an $\varepsilon > 0$, there exists a $\delta > 0$ such that for all $x, y$ in $X$
\[ \varepsilon \leq M(x, y) < \varepsilon + \delta \implies d(Ax, By) < \varepsilon, \]

4) 
\[ d(Ax, By) \leq k \max \{ d(Sx, Ty), d(Sx, Ax), d(Ty, By), d(Sx, By), d(Ty, Ax) \} \]
for all $x, y \in X$, where $k \in [0, 1)$.

Then $A, B, S$ and $T$ have a unique common fixed point.

**Remark 2.1.** Because if $(X, d)$ is complete and $AX$, $BX$, $SX$, $TX$ are closed, then $AX$, $BX$, $SX$, $TX$ are complete subspaces of $X$, in Theorems 2.2, 2.3 the conditions that $X$ is complete and $AX$, $BX$, $SX$, $TX$ are closed should be replaced by the statement that one of $AX$, $BX$, $SX$, $TX$ is a complete subspace of $X$.

In [6], Branciari established the following result.
Theorem 2.4. Let \((X,d)\) be a complete metric space, \(c \in (0,1)\) and \(f : X \to X\) such that
\[
\int_0 d(fx, fy) h(t) dt \leq c \int_0 d(x, y) h(t) dt,
\]
whenever \(h : [0, \infty) \to [0, \infty)\) is a Lebesgue measurable mapping which is summable (i.e. with a finite integral) on each compact subset of \([0, \infty)\) such that for \(\varepsilon > 0\), \(\int h(t) dt > 0\). Then, \(f\) has a unique fixed point \(z\) such that, for each \(x \in X\), \(\lim_{n \to \infty} fx_n = z\).

Theorem 2.4 is extended to compatible, weakly compatible, occasional weakly compatible in [1], [15], [16], [20], [34] and in other papers.

Quite recently, Gairola and Rawat [7] proved a fixed point theorem for two pairs of maps satisfying a new contractive condition of integral type, using the concept of occasionally weakly compatible mappings, which generalize Theorem 2.1.

Theorem 2.5. Let \(A, B, S\) and \(T\) be self mappings of a metric space \((X,d)\) satisfying the following conditions:

1) \(AX \subset TX\) and \(BX \subset SX\),

2) given \(\varepsilon > 0\) there exists \(\delta > 0\) such that for all \(x, y \in X\),
\[
\int_0^M(x, y) h(t) dt < \varepsilon + \delta \quad \text{implies} \quad \int_0 h(t) dt \leq \varepsilon,
\]
where \(h(t)\) is as in Theorem 2.4 and \(\int_0 h(t) dt > 0\) implies
\[
\int_0 h(t) dt < \int_0 h(t) dt,
\]

3) \(\int_0 h(t) dt < k \left[ \int_0 d(Sx, Ty) h(t) dt + \int_0 d(Sx, Ax) h(t) dt + \int_0 d(Ty, By) h(t) dt + \int_0 d(Ty, Ax) h(t) dt \right].
\]

for all \(x, y \in X\) and \(k \in [0, \frac{1}{3})\).

If one of \(AX, BX, SX, TX\) is a complete subspace of \(X\), then:
a) $A$ and $S$ have a coincidence point,  
b) $B$ and $T$ have a coincidence point.

Moreover, if the pairs $(A, S)$ and $(B, T)$ are occasionally weakly compatible mappings, then $A$, $B$, $S$ and $T$ have a unique common fixed point.

**Definition 2.1.** An altering distance is a mapping $\psi : [0, \infty) \to [0, \infty)$ which satisfies

$($\psi_1$$): \psi(t)$ is increasing and continuous,

$($\psi_2$$): \psi(t) = 0$ if and only if $t = 0$.

Fixed point problem involving an altering distance have been studied in [14], [17], [32], [36], [37] and in other papers.

**Lemma 2.1.** The function $\psi(x) = \int_0^x h(t) \, dt$, where $h(t)$ is as in Theorem 2.4, is an altering distance.

**Proof.** By definitions of $\psi(t)$ and $h(t)$ it follows that $\psi(x)$ is increasing and $\psi(x) = 0$ if and only if $x = 0$. By Lemma 2.5 [20], $\psi(x)$ is continuous. \hfill $\square$

In [30] and [31] the study of fixed points for mappings satisfying implicit relations was initiated. A general fixed point theorem of Meir-Keeler type for noncontinuous weakly compatible mappings satisfying an implicit relation, which generalize (2.1) and others is proved in [33].

The purpose of this paper is to prove a general theorem of Meir-Keeler type using the notion of altering distance for occasionally weakly compatible mappings satisfying an implicit relation.

As an application, a problem of Meir-Keeler type satisfying a condition of integral type becomes a special case of Meir-Keeler type with an altering distance.

3 Implicit relation

Let $\mathcal{F}_{MK}$ be the set of all real continuous mappings $\phi(t_1, \ldots, t_6) : \mathbb{R}_+^6 \to \mathbb{R}$, increasing in $t_1$ satisfying the following conditions:

$($\phi_1$$): \phi(t, 0, 0, t, t, 0) \leq 0$ implies $t = 0$,

$($\phi_2$$): \phi(t, 0, t, 0, 0, t) \leq 0$ implies $t = 0$,

$($\phi_3$$): \phi(t, t, 0, 0, t, t) > 0$, \forall $t > 0$.

**Example 3.1.** $\phi(t_1, \ldots, t_6) = t_1 - k(t_2 + t_3 + t_4 + t_5 + t_6)$, where $k \in \left[0, \frac{1}{3}\right]$.

**Example 3.2.** $\phi(t_1, \ldots, t_6) = t_1 - at_2 - b(t_3 + t_4) - c(t_5 + t_6)$, where $a$, $b$, $c \geq 0$, $b + c < 1$ and $a + 2c < 1$. 
Example 3.3. \( \phi(t_1, ..., t_6) = t_1 - b(t_3 + t_4) - c \min\{t_5, t_6\} \), where \( b, c \geq 0 \), \( b < 1 \) and \( a + c < 1 \).

Example 3.4. \( \phi(t_1, ..., t_6) = t_1 - h \max\{t_2, t_3, t_4, t_5, t_6\} \), where \( h \in (0, 1) \).

Example 3.5. \( \phi(t_1, ..., t_6) = t_1 - h \max\{t_2, t_3, t_4, \frac{1}{3}(t_5 + t_6)\} \), where \( h \in (0, 1) \).

Example 3.6. \( \phi(t_1, ..., t_6) = t_1^2 - at_2^2 - t_3 t_4 - bt_5^2 - ct_6^2 \), where \( a, b, c \geq 0 \) and \( a + b + c < 1 \).

Example 3.7. \( \phi(t_1, ..., t_6) = t_3^k - k(t_3 + t_4 + t_5 + t_6) \), where \( k \in [0, 1) \).

Example 3.8. \( \phi(t_1, ..., t_6) = t_3^3 - \frac{t_3^2 \cdot t_4^2 + t_5^2 \cdot t_6^2}{1 + t_2 + t_3 + t_4} \).

Example 3.9. \( \phi(t_1, ..., t_6) = t_1 - \max\{ct_2, ct_3, ct_4, at_5 + bt_6\} \), where \( a, b, c \geq 0 \), \( c < 1 \) and \( a + b < 1 \).

Example 3.10. \( \phi(t_1, ..., t_6) = t_1 - \alpha \max\{t_2, t_3, t_4\} - (1 - \alpha)(at_5 + bt_6) \), where \( \alpha \in [0, 1) \), \( a, b \geq 0 \) and \( a + b < 1 \).

Example 3.11. \( \phi(t_1, ..., t_6) = t_1 - \max\left\{t_2, \frac{1}{2}(t_3 + t_4), \frac{1}{2}(t_5 + t_6)\right\} \), where \( k \in [0, 1) \).

Example 3.12. \( \phi(t_1, ..., t_6) = t_1 - \max\left\{k_1 t_2, \frac{k_2}{2}(t_3 + t_4), \frac{t_5 + t_6}{2}\right\} \), where \( k_1 \in [0, 1) \), \( k_2 \in [1, 2) \).

Example 3.13. \( \phi(t_1, ..., t_6) = t_1 - \max\left\{k_1(t_2 + t_3 + t_4), \frac{k_2}{2}(t_5 + t_6)\right\} \), where \( k_1 \in [0, 1) \), \( k_2 \in [0, 2) \).

Let \( \varphi : \mathbb{R}_+ \to \mathbb{R}_+ \) be a function satisfying the following conditions:

(\( \varphi_1 \)): \( \varphi \) is continuous,
(\( \varphi_2 \)): \( \varphi \) is nondecreasing on \( \mathbb{R}_+ \),
(\( \varphi_3 \)): \( 0 < \varphi(t) < t \) for \( t > 0 \).

Example 3.14. \( \phi(t_1, ..., t_6) = t_1 - \varphi \max\left\{t_2, t_3, t_4, t_5, \frac{t_6}{2}\right\} \).

Example 3.15. \( \phi(t_1, ..., t_6) = t_1 - \varphi \max\left\{t_2, \frac{t_3 + t_4}{2}, \frac{t_5 + t_6}{2}\right\} \).

Example 3.16. \( \phi(t_1, ..., t_6) = t_1 - \varphi \max\left\{t_2, t_3, t_4, \frac{k}{2}(t_5 + t_6)\right\} \), where \( k \in [0, 2) \).

Example 3.17. \( \phi(t_1, ..., t_6) = t_1 - \varphi \max\left\{t_2, t_3, t_4, \frac{t_5 + t_6}{2}\right\} \).
4 Main results

**Theorem 4.1.** Let $A$, $B$, $S$, $T$ be self mappings of a metric space $(X, d)$ satisfying the inequality:

\[
\phi(d(Ax, By)), \psi(d(Sx, Ty)), \psi(d(Ax, Sx)), \\
\psi(d(Ty, By)), \psi(d(Sx, By)), \psi(d(Ty, Ax)) \leq 0
\]

(4.1)

for all $x, y$ in $X$, where $\phi$ satisfies property $(\phi_3)$ and $\psi$ is an altering distance. If there exist $u, v \in X$ such that $Su = Au$ and $Tv = Bv$, then $A$ and $S$ have a unique point of coincidence and $B$ and $T$ have a unique point of coincidence.

**Proof.** First we prove that $Su = Tv$. If $Su \neq Tv$, using (4.1), we have, successively:

\[
\phi(d(Au, Bv)), \psi(d(Su, Tv)), \psi(d(Au, Su)), \\
\psi(d(Tv, Bv)), \psi(d(Su, Bv)), \psi(d(Tv, Au)) \leq 0,
\]

\[
\phi(d(Su, Tv)), \psi(d(Su, Tv)), 0, 0, \psi(d(Su, Tv)), \psi(d(Su, Tv)) \leq 0,
\]

a contradiction of $(\phi_3)$ if $d(Su, Tv) > 0$. Hence $\psi(d(Su, Tv)) = 0$, which implies that $Su = Tv$.

Assume that there exists a $p \in X$ such that $Ap = Sp$. Then by (4.1) we have successively:

\[
\phi(d(Ap, Bv)), \psi(d(Sp, Tv)), \psi(d(Sp, Ap)), \\
\psi(d(Tv, Bv)), \psi(d(Sp, Bv)), \psi(d(Tv, Ap)) \leq 0,
\]

\[
\phi(d(Sp, Tv)), \psi(d(Sp, Tv)), 0, 0, \psi(d(Sp, Tv)), \psi(d(Sp, Tv)) \leq 0,
\]

a contradiction of $(\phi_3)$ if $d(Sp, Tv) > 0$. Therefore $Sp = Tv$ and $z = Au = Su$ is the unique point of coincidence of $A$ and $S$. Similarly, $w = Tv = Bv$ is the unique point of coincidence of $B$ and $T$. \qed

**Lemma 4.1.** Let $A$, $B$, $S$, $T$ be self mappings of a metric space $(X, d)$ such that $AX \subset TX$ and $BX \subset SX$ and $\psi$ is an altering distance. For each $\varepsilon > 0$ there exists a $\delta > 0$ such that

\[
\varepsilon < \psi(M(x, y)) < \varepsilon + \delta \text{ implies } \psi(d(Ax, By)) \leq \varepsilon, \\
\varepsilon > 0 \text{ implies } \psi(d(Ax, By)) < \psi(M(x, y)).
\]

(4.2)

(4.3)

For $x_0 \in X$ and $\{y_n\}$ defined by $y_{2n-1} = Tx_{2n-1} = Ax_{2n-2}$ and $y_{2n} = Sx_{2n} = Bx_{2n-1}$ for $n \in \mathbb{N}^*$ and $d_n = d(y_n, y_{n+1})$, then $\lim d_n = 0$. 

Proof. First we prove that if, for some \( k \in \mathbb{N}^* \), \( d_{k+1} > 0 \), then
\[
\psi(d_{k+1}) < \psi(d_k). \tag{4.4}
\]

a) Assume that \( d_{2k} > 0 \) for some \( k \in \mathbb{N}^* \). Then \( M(x_{2k}, x_{2k-1}) > 0 \)
otherwise \( Ax_{2k} = Bx_{2k-1} \), i.e. \( y_{2k} = y_{2k+1} \) so \( d_{2k} = 0 \), a contradiction.
Hence \( d_{2k} = d(Ax_{2k}, Bx_{2k-1}) < M(x_{2k}, x_{2k-1}) \) which implies by (4.3) that
\[
0 < \psi(d_{2k}) < \psi(M(x_{2k}, x_{2k-1})) \leq \psi(\max\{d_{2k-1}, d_{2k}\}) = \psi(d_{k-1}).
\]

b) If \( d_{2k+1} > 0 \) for some \( k \in \mathbb{N}^* \), using a similar argument as in a), one
may verify that \( \psi(d_{2k+1}) < \psi(d_{2k}) \).

c) Combining the results of a) and b) we may conclude that
\[
\psi(d_{k+1}) < \psi(d_k) \text{ for } k \in \mathbb{N}^*. \tag{4.5}
\]

Moreover, if for some \( k \in \mathbb{N}^* \), \( \psi(d_k) = 0 \), then \( d_k = 0 \) which implies
\( d_{k+1} = 0 \) because, if \( d_{k+1} > 0 \), then \( \psi(d_{k+1}) > 0 \) which implies by a) and b)
\( \psi(d_{k+1}) < \psi(d_k) = 0 \), a contradiction. Hence, for \( n \geq k \) we have \( y_n = y_k \)
and hence \( \lim d(y_n, y_{n+1}) = 0 \).

We prove that \( \lim d(y_n, y_{n+1}) = 0 \) for \( \psi(d_k) > 0 \).

By (4.5) it follows that \( \psi(d_n) \) is strictly decreasing, hence convergent to some \( \ell \in \mathbb{R}^+ \). Suppose that \( \ell > 0 \). Then by (4.2) for \( \varepsilon = \ell \), there exists a \( \delta > 0 \)
such that \( \ell < \psi(d_n) < \ell + \delta \) for \( n \geq k \). In particular \( \ell < \psi(M(x_{2k}, x_{2k+1})) \) \(
\ell + \delta \), since \( M(x_{2k}, x_{2k+1}) = \max\{d_{2k}, d_{2k+1}\} \leq \ell \). Hence \( \ell < \psi(d_{2k}) \leq \ell \),
a contradiction, and \( \ell = 0 \). Let \( a_n = \psi(d(y_n, y_{n+1})) \), \( n \geq 0 \). Then by the
continuity of \( \psi \) we obtain
\[
0 = \lim a_n = \lim \psi(d(y_n, y_{n+1})) = \psi(\lim d(y_n, y_{n+1})).
\]

Hence \( \lim d(y_n, y_{n+1}) = 0 \). \( \square \)

**Theorem 4.2.** Let \( A, B, S \) and \( T \) be self mappings of a metric space \((X, d)\)
satisfying the following conditions:

a) \( AX \subset TX \) and \( BX \subset SX \),

b) given an \( \varepsilon > 0 \) there exists a \( \delta > 0 \) such that for all \( x, y \in X \)
\[
\varepsilon < \psi(M(x, y)) < \varepsilon + \delta \text{ implies } \psi(d(Ax, By)) \leq \varepsilon,
\]

c) \( \psi(M(x, y)) > 0 \) implies \( \psi(d(Ax, By)) < \psi(M(x, y)) \),

d) the inequality (4.1) holds for all \( x, y \) in \( X \), where \( \phi \in \mathcal{F}_{MK} \) and \( \psi \) is an
altering distance.

If one of \( AX, BX, SX, TX \) is a complete subspace of \( X \), then:
Proof. First we prove that \( \{ y_n \} \) is a Cauchy sequence. Since by Lemma 4.1,
\[
\lim d(y_n, y_{n+1}) = 0
\]
it is sufficient to show that \( \{ y_{2n} \} \) is a Cauchy sequence. Suppose that \( \{ y_{2n} \} \) is not a Cauchy sequence. Then there exists an \( \varepsilon > 0 \) such that for even integer \( 2k \), there exists even integers \( 2m(k) \) and \( 2n(k) \) such that
\[
d(y_{2m(k)}, y_{2n(k)}) > \varepsilon \quad \text{with } 2m(k) > 2n(k) \geq 2k.
\]
For even integer \( 2k \), let \( 2m(k) \) be the least even integer exceeding \( 2n(k) \) such that \( d(y_{2n(k)}, y_{2m(k)}) < \varepsilon \).

As in Theorem 2.2 [12] we deduce that
\[
\lim d(y_{2m(k)}, y_{2m(k)}) = \varepsilon, \\
\lim d(y_{2n(k)}, y_{2m(k)} - 1) = \varepsilon, \\
\lim d(y_{2n(k) + 1}, y_{2m(k) - 1}) = \varepsilon.
\]

On the other hand we have successively
\[
d(y_{2m(k)}, y_{2n(k)}) \leq d_{2n(k)} + d(Ax_{2n(k)}, Bx_{2m(k)} - 1), \\
d(y_{2n(k)} + 1, y_{2m(k)}) \leq d(y_{2n(k)} + 1, y_{2m(k)}).
\]

Hence
\[
\psi(d(y_{2n(k)}, y_{2m(k)}) - d_{2n(k)}) \leq \psi(d(y_{2n(k) + 1}, y_{2m(k)})).
\]

Setting in (4.1) \( x = x_{2n(k)} \) and \( y = x_{2m(k) - 1} \) we obtain
\[
\phi(\psi(d(y_{2n(k)}, y_{2m(k)})), \psi(d(y_{2n(k)}, y_{2m(k)} - 1)), \psi(d(y_{2n(k)}, y_{2m(k) + 1})), \\
\psi(d(y_{2n(k) - 1}, y_{2m(k)})), \psi(d(y_{2n(k)}, y_{2m(k)})), \psi(d(y_{2n(k)} - 1, y_{2m(k)})) \leq 0,
\]
\[
\phi(\psi(d(y_{2n(k)}, y_{2m(k)}) - d_{2n(k)}), \psi(d(y_{2n(k)}, y_{2m(k)} - 1)), \psi(d(y_{2n(k)}, y_{2m(k) + 1})), \\
\psi(d(y_{2n(k) - 1}, y_{2m(k)})), \psi(d(y_{2n(k)}, y_{2m(k)})), \psi(d(y_{2n(k)} - 1, y_{2m(k)})) \leq 0.
\]

Letting \( n \) tend to infinity we obtain
\[
\phi(\psi(\varepsilon), \psi(\varepsilon), 0, 0, \psi(\varepsilon), \psi(\varepsilon)) \leq 0,
\]
a contradiction of \( (\phi_3) \). Hence, \( \{ y_{2n} \} \) is a Cauchy sequence. It follows that \( \{ y_n \} \) is a Cauchy sequence.

Assume that least one of \( AX \) or \( TX \) is a complete subspace of \( X \).

Since \( y_{2n+1} \in AX \subset TX \) and \( \{ y_{2n+1} \} \) is a Cauchy sequence, there exists a \( u \in TX \) such that \( \lim y_{2n+1} = u \). The sequence \( \{ y_n \} \) converges to \( u \) since it
is Cauchy and has the subsequence \( \{y_{2n+1}\} \) convergent to \( u \). Let \( v \in X \) such that \( u = Tv \). Setting \( x = x_{2n} \) and \( y = v \) in (4.1) we get

\[
\begin{align*}
\phi(\psi(d(Ax_{2n}, Bv)), & \psi(d(Sx_{2n}, Tv)), \psi(d(Sx_{2n}, Ax_{2n})), \\
\psi(d(Tv, Bv)), & \psi(d(Sx_{2n}, Bv)), \psi(d(Tv, Ax_{2n}))) \leq 0, \\
\phi(\psi(d(y_{2n+1}, Bv)), & \psi(d(y_{2n}, Tv)), \psi(d(y_{2n}, y_{2n+1})), \\
\psi(d(Tv, Bv)), & \psi(d(y_{2n}, Bv)), \psi(d(u, y_{2n+1}))) \leq 0.
\end{align*}
\]

Letting \( n \) tend to infinity in the above inequality we obtain

\[
\phi(\psi(d(u, Bv)), 0, 0, \psi(d(u, Bv)), \psi(d(u, Bv))), 0) \leq 0.
\]

From (\( \phi_1 \)), \( \psi(d(u, Bv)) = 0 \) which implies that \( d(u, Bv) = 0 \) i.e. \( u = Bv \). Hence \( u = Tv = Bv \) and \( v \) is a coincidence point of \( T \) and \( B \). Since \( u = Bv \in BX \subset SX \), there exists a \( w \in X \) such that \( u = Sw \). Using a similar argument as above we obtain \( u = Aw \). Hence, \( u = Tv = Bv = Sw = Aw \).

Indeed, setting \( x = w \) and \( y = x_{2n+1} \) in (4.1) we obtain

\[
\begin{align*}
\phi(\psi(d(Aw, y_{2n+2})), & \psi(d(u, y_{2n+2})), \psi(d(u, Aw)), \\
\psi(d(y_{2n+1}, y_{2n+2})), & \psi(d(u, y_{2n+2})), \psi(d(y_{2n+1}, Aw))) \leq 0,
\end{align*}
\]

and letting \( n \) tend to infinity we obtain

\[
\phi(\psi(d(Aw, u)), 0, \psi(d(u, Aw)), 0, 0, \psi(d(Aw, u))) \leq 0.
\]

By (\( \phi_1 \)) it follows that \( \psi(d(Aw, u)) = 0 \), hence \( u = Aw \).

By Theorem 4.1 \( u \) is the unique point of coincidence of \( A \), \( S \) and \( B \), \( T \).

If the pairs \( (A, S) \) and \( (B, T) \) are occasionally weakly compatible, then by Lemma 1.1 \( u \) is the unique common fixed point of \( A \), \( B \), \( S \) and \( T \).

Taking \( A = B \) and \( S = T \) in Theorem 4.2 we obtain

**Theorem 4.3.** Let \( A \) and \( S \) and \( T \) be self mappings of a metric space \((X, d)\) satisfying the following conditions:

1) \( AX \subset SX \),

2) \( \epsilon < \psi(M_1(x, y)) < \epsilon + \delta \) implies \( \psi(d(Ax, By)) \leq \epsilon \), where \( M_1(x, y) = \max\{d(Sx, Sy), d(Sx, Ax), d(Sy, Ty), d(Sx, Ay), d(Sy, Ax)\} \),

3) \( \psi(M_1(x, y)) > 0 \) implies \( \psi(d(Ax, By)) < \psi(M_1(x, y)) \),

4) \( \phi(\psi(d(Ax, Ay)), \psi(d(Sx, Sy)), \psi(d(Sx, Ax)), \psi(d(Sy, Ay)), \psi(d(Sx, Ay)), \psi(d(Sy, Ax))) \leq 0 \) for all \( x, y \) in \( X \), where \( \phi \in \mathcal{F}_{MK} \) and \( \psi \) is an altering distance.

If one of \( SX \) and \( AX \) is a complete subspace of \( X \), then:
5) \( A \) and \( S \) have a coincidence point.

Moreover, if the pair \((A, S)\) is occasionally weakly compatible, then \( A \) and \( S \) have a unique common fixed point.

For \( \psi(t) = t \) by Theorem 4.2 we obtain

**Theorem 4.4.** Let \( A, B, S \) and \( T \) be self mappings of a metric space \((X, d)\) satisfying the following conditions:

1) \( AX \subset SX \),

2) given an \( \varepsilon > 0 \), there exists a \( \delta > 0 \) such that for all \( x, y \) in \( X \), \( \varepsilon < M(x, y) < \varepsilon + \delta \) implies \( d(Ax, By) \leq \varepsilon \),

3) \( M(x, y) > 0 \) implies \( d(Ax, By) < M(x, y) \),

4) \( \phi(d(Ax, By), d(Sx, Ty), d(Sx, Ax), d(Ty, By), d(Sx, By), d(Ty, Ax)) \leq 0 \) for all \( x, y \) in \( X \) and \( \phi \in \mathcal{F}_{MK} \).

If one of \( AX, BX, SX, TX \) is a complete subspace of \( X \), then:

5) \( A \) and \( S \) have a coincidence point,

6) \( B \) and \( T \) have a coincidence point.

Moreover, if the pairs \((A, S)\) and \((B, T)\) are occasionally weakly compatible, then \( A, B, S \) and \( T \) have a unique common fixed point.

**Remark 4.1.** 1) By Example 3.1 and Theorem 4.4 we obtain a generalization of Theorem 2.2 and Theorem 2.1.

2) By Example 3.4 and Theorem 4.4 we obtain a generalization of Theorem 2.1 [5].

3) By Example 3.12 and Theorem 4.4 we obtain a generalization of the result from [27].

4) By Example 3.11 and Theorem 4.4 we obtain a generalization of the result from [28] for \( k \in [0, 1) \).

5) By Example 3.16 and Theorem 4.4 we obtain a generalization of Theorem 2.1 [26].

6) Theorem 4.4 is a generalization of Theorem 5 [33] for weakly compatible mappings satisfying an implicit relation.

7) By Examples 3.2, 3.3, 3.5 - 3.10, 3.13 - 3.15 we obtain new fixed point theorems of Meir - Keeler type.
5 Applications

Theorem 5.1. Let $A$, $B$, $S$ and $T$ be self mappings of a metric space $(X,d)$ satisfying conditions (a), (b), (c) of Theorem 4.2. Assume that there exists a $\phi \in \mathcal{F}_{MK}$ such that:

$$
\phi \left( \int_0^1 d(Ax,By) h(t) dt, \int_0^1 d(Sx,Ty) h(t) dt, \int_0^1 d(Sx,Ax) h(t) dt, \int_0^1 d(Ty,By) h(t) dt, \int_0^1 d(Ty,Ax) h(t) dt \right) \leq 0
$$

for all $x, y \in X$, which is inequality (4.1).

If one of $AX$, $BX$, $SX$, $TX$ is a complete subspace of $X$ then:

1) $A$ and $S$ have a coincidence point,

2) $B$ and $T$ have a coincidence point.

Moreover, if the pairs $(A,S)$ and $(B,T)$ are occasionally weakly compatible, then $A$, $B$, $S$ and $T$ have a unique common fixed point.

Proof. Let

$$
\psi(d(Ax,By)) = \int_0^1 h(t) dt, \psi(d(Sx,Ty)) = \int_0^1 h(t) dt,
$$

$$
\psi(d(Sx,Ax)) = \int_0^1 h(t) dt, \psi(d(Ty,By)) = \int_0^1 h(t) dt,
$$

$$
\psi(d(Sx,By)) = \int_0^1 h(t) dt, \psi(d(Ty,Ax)) = \int_0^1 h(t) dt,
$$

where $h(t)$ is as in Theorem 2.4.

By Lemma 2.1 $\psi(x) = \int_0^1 h(t) dt$ is an altering distance.

By (5.1) we obtain

$$
\phi(\psi(d(Ax,By)), \psi(d(Sx,Ty)), \psi(d(Sx,Ax)), \psi(d(Ty,By)), \psi(d(Sx,By)), \psi(d(Ty,Ax))) \leq 0
$$

for all $x, y \in X$, which is inequality (4.1).

Hence, the conditions of Theorem 4.2 are satisfied and the conclusion of Theorem 5.1 it follows from Theorem 4.2.

Remark 5.1. If $h(t) = 1$ we obtain Theorem 4.3.

Remark 5.2. By Theorem 5.1 and Example 3.1 we obtain Theorem 2.5.

Examples 3.2 - 3.17 are new results.
References


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