Generalizations of Lindelöf spaces via hereditary classes

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Abstract. In this paper by using hereditary classes [6], we define the notion of γ-Lindelöf modulo hereditary classes called γH-Lindelöf and obtain several properties of γH-Lindelöf spaces.

1 Introduction

Let \((X, \tau)\) be a topological space and \(\mathcal{P}(X)\) the power set of \(X\). In 1991, Ogata [13] introduced the notions of γ-operations and γ-open sets and investigated the associated topology \(\tau_\gamma\) and weak separation axioms \(\gamma-T_i\) (i = 0, 1/2, 1, 2). In 2011, Noiri [10] defined an operation on an \(m\)-structure with property \(B\) (the generalized topology in the sense of Lugojan [8]). The operation is defined as a function \(\gamma : \mathcal{M} \to \mathcal{P}(X)\) such that \(U \subseteq \gamma(U)\) for each \(U \in \mathcal{M}\) and is called a γ-operation on \(m\). Then, it turns out that the operation is an unified form of several operations (for example, semi-γ-operation [7], pre-γ-operation [4]) defined on the family of generalized open sets. Moreover, he obtained some characterizations of γ-compactness and suggested some generalizations of compact spaces by using recent modifications of open sets in a topological space.

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In this paper by using hereditary classes [6], we define the notion of \( \gamma \)-Lindelöf modulo hereditary classes called \( \gamma \mathcal{H} \)-Lindelöf and obtain several properties of \( \gamma \mathcal{H} \)-Lindelöf spaces. Recently papers [1, 2, 3] have introduced some new classes of sets via hereditary classes.

2 Preliminaries

First we state the following: in [11], a minimal structure \( m \) is defined as follows: \( m \) is called a minima structure if \( \emptyset, X \in m \). However, in this paper, we define as follows:

Definition 1 Let \( X \) be a nonempty set and \( \mathcal{P}(X) \) the power set of \( X \). A subfamily \( m \) of \( \mathcal{P}(X) \) is called a minimal structure (briefly \( m \)-structure) on \( X \) if \( m \) satisfies the following conditions:

1. \( \emptyset, X \in m \).
2. The union of any family of subsets belonging to \( m \) belongs to \( m \).

A set \( X \) with an \( m \)-structure is called an \( m \)-space and denoted by \((X, m)\). Each member of \( m \) is said to be \( m \)-open and the complement of an \( m \)-open set is said to be \( m \)-closed.

Definition 2 [9] Let \( X \) be a nonempty set and \( m \) an \( m \)-structure on \( X \). For a subset \( A \) of \( X \), the \( m \)-closure of \( A \) is defined as follows: \( \text{mcl}(A) = \bigcap \{ F : A \subseteq F, X \setminus F \in m \} \).

Lemma 1 [9] Let \( X \) be a nonempty set and \( m \) an \( m \)-structure on \( X \). For the \( m \)-closure, the following properties hold, where \( A \) and \( B \) are subsets of \( X \):

1. \( A \subseteq \text{mcl}(A) \),
2. \( \text{mcl}(\emptyset) = \emptyset, \text{mcl}(X) = X \),
3. If \( A \subseteq B \), then \( \text{mcl}(A) \subseteq \text{mcl}(B) \),
4. \( \text{mcl}(\text{mcl}(A)) = \text{mcl}(A) \).

Lemma 2 [14] Let \((X, m)\) be an \( m \)-space and \( A \) a subset of \( X \). Then \( x \in \text{mcl}(A) \) if and only if \( U \cap A \neq \emptyset \) for every \( U \in m \) containing \( x \).
Lemma 3 [15] Let \((X, m)\) be an \(m\)-space and \(A\) a subset of \(X\). Then, the following properties hold:

1. \(A\) is \(m\)-closed if and only if \(\text{mcl}(A) = A\),
2. \(\text{mcl}(A)\) is \(m\)-closed.

Definition 3 [10] Let \((X, m)\) be an \(m\)-space and \(\gamma\) an operation on \(m\). A subset \(A\) of \(X\) is said to be \(\gamma\)-open if for each \(x \in A\) there exists \(U \in m\) such that \(x \in U \subseteq \gamma(U) \subseteq A\). The complement of a \(\gamma\)-open set is said to be \(\gamma\)-closed. The family of all \(\gamma\)-open sets of \((X, m)\) is denoted by \(\gamma(X)\).

3 \(\gamma\mathcal{H}\)-Lindelöf spaces

First, we recall the definition of a hereditary class used in the sequel. A subfamily \(H\) of the power set \(P(X)\) is called a hereditary class on \(X\) [6] if it satisfies the following property: \(A \in H\) and \(B \subseteq A\) implies \(B \in H\).

Definition 4 Let \((X, m, H)\) be a hereditary \(m\)-space and \(\gamma\) an operation on \(m\), where \(H\) is a hereditary class on \(X\). Then \(m\)-space \((X, m)\) is said to be \(\gamma\mathcal{H}\)-Lindelöf (resp. \(\mathcal{H}\)-Lindelöf) if every cover \(\{U_\alpha : \alpha \in \Delta\}\) of \(X\) by \(m\)-open sets, there exists a countable subset \(\Delta_0\) of \(\Delta\) such that \(X \setminus \bigcup_{\alpha \in \Delta} \gamma(U_\alpha) \subseteq \mathcal{H}\) (resp. \(X \setminus \bigcup_{\alpha \in \Delta_0} U_\alpha \subseteq \mathcal{H}\)).

Theorem 1 Let \((X, m, \mathcal{H})\) be a hereditary \(m\)-space and \(\gamma\) an operation on \(m\), where \(\mathcal{H}\) is a hereditary class. Then the following properties are equivalent:

1. \((X, \gamma(X))\) is \(\mathcal{H}\)-Lindelöf;
2. For every family \(\{F_\alpha : \alpha \in \Delta\}\) of \(\gamma\)-closed sets such that \(\bigcap \{F_\alpha : \alpha \in \Delta_0\} \notin \mathcal{H}\) for every countable subfamily \(\Delta_0\) of \(\Delta\), \(\bigcap \{F_\alpha : \alpha \in \Delta\} \neq \emptyset\).

Proof. (1) \(\Rightarrow\) (2): Let \((X, \gamma(X))\) be \(\mathcal{H}\)-Lindelöf. Suppose that \(\bigcap \{F_\alpha : \alpha \in \Delta\} = \emptyset\), where \(F_\alpha\) is \(\gamma\)-closed set. Then \(X \setminus F_\alpha\) is \(\gamma\)-open for each \(\alpha \in \Delta\) and \(\bigcup_{\alpha \in \Delta} (X \setminus F_\alpha) = X \setminus \bigcap_{\alpha \in \Delta} F_\alpha = X\). By (1), there exists a countable subfamily \(\Delta_0\) of \(\Delta\) such that \(X \setminus \bigcup_{\alpha \in \Delta_0} (X \setminus F_\alpha) = \bigcap \{F_\alpha : \alpha \in \Delta_0\} \in \mathcal{H}\). This is a contradiction.

(2) \(\Rightarrow\) (1): Suppose that \((X, \gamma(X))\) is not \(\mathcal{H}\)-Lindelöf. There exists a cover \(\{U_\alpha : \alpha \in \Delta\}\) of \(X\) by \(\gamma\)-open sets such that \(X \setminus \bigcup \{U_\alpha : \alpha \in \Delta_0\} \notin \mathcal{H}\) for
every countable subset $\Delta_0$ of $\Delta$. Since $X \setminus U_\alpha$ is $\gamma$-closed for each $\alpha \in \Delta$ and
$\bigcap \{(X \setminus U_\alpha) : \alpha \in \Delta_0\} \not\in \mathcal{H}$ for every countable subset $\Delta_0$ of $\Delta$. By (2), we have
$\bigcap \{(X \setminus U_\alpha) : \alpha \in \Delta\} \not= \emptyset$. Therefore, $X \setminus \bigcup \{U_\alpha : \alpha \in \Delta\} \not= \emptyset$. This is contrary
that $\{U_\alpha : \alpha \in \Delta\}$ is a $\gamma$-open cover of $X$.

\[ \Box \]

Lemma 4 \[10\] Let $(X, m)$ be an $m$-space. For $\gamma(X)$, the following properties hold:

1. $\emptyset, X \in \gamma(X)$,
2. If $\Lambda_\alpha \in \gamma(X)$ for each $\alpha \in \Lambda$, then $\bigcup_{\alpha \in \Lambda} \Lambda_\alpha \in \gamma(X)$,
3. $\gamma(X) \subseteq m$.

Definition 5 \[10\] An $m$-space $(X, m)$ is said to be $\gamma$-regular if for each $x \in X$ and each $U \in m$ containing $x$, there exists $V \in m$ such that $x \in V \subseteq \gamma(V) \subseteq U$.

Lemma 5 \[10\] For an $m$-space $(X, m)$, the following properties are equivalent:

1. $m = \gamma(X)$;
2. $(X, m)$ is $\gamma$-regular;
3. For each $x \in X$ and each $U \in m$ containing $x$, there exists $W \in \gamma(X)$
   such that $x \in W \subseteq \gamma(W) \subseteq U$.

Theorem 2 Let $(X, m, \mathcal{H})$ be a hereditary $m$-space and $\gamma$ an operation on $m$,
where $\mathcal{H}$ is a hereditary class. The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) hold.
If $(X, m)$ is $\gamma$-regular, then the following properties are equivalent:

1. $(X, m)$ is $\mathcal{H}$-Lindelöf;
2. $(X, m)$ is $\gamma\mathcal{H}$-Lindelöf;
3. $(X, \gamma(X))$ is $\mathcal{H}$-Lindelöf;
4. $(X, \gamma(X))$ is $\gamma\mathcal{H}$-Lindelöf.

Proof. (1) $\Rightarrow$ (2): Let $(X, m)$ be $\mathcal{H}$-Lindelöf. For any cover $\{U_\alpha : \alpha \in \Delta\}$ of $X$
by $m$-open sets, there exists a countable subset $\Delta_0$ of $\Delta$ such that $X \setminus \bigcup \{\gamma(U_\alpha) : \alpha \in \Delta_0\} \subseteq X \setminus \bigcup \{U_\alpha : \alpha \in \Delta_0\} \in \mathcal{H}$. Therefore, $(X, m)$ is $\gamma\mathcal{H}$-Lindelöf.
(2) ⇒ (3): Let \((X, m)\) be a \(\gamma\)-\(H\)-Lindelöf and \(\{U_\alpha : \alpha \in \Delta\}\) a cover of \(X\) by \(\gamma\)-open sets. For each \(x \in X\) there exists \(\alpha(x) \in \Delta\) such that \(x \in U_{\alpha(x)}\). Since \(U_{\alpha(x)}\) is \(\gamma\)-open, there exists \(V_{\alpha(x)} \in m\) such that \(x \in V_{\alpha(x)} \subseteq \gamma(V_{\alpha(x)}) \subseteq U_{\alpha(x)}\). Since the family \(\{V_{\alpha(x)} : x \in X\}\) is a cover of \(X\) by \(m\)-open sets and \((X, m)\) is \(\gamma\)-\(H\)-Lindelöf, there exists a countable number of points, say, \(x_1, x_2, x_3, \ldots \in X\) such that \(X \setminus \bigcup_{i=1}^{\infty} \gamma(V_{\alpha(x_i)}) \in \mathcal{H}\) and hence \(X \setminus \bigcup_{i=1}^{\infty} U_{\alpha(x_i)} \in \mathcal{H}\). This shows that \((X, \gamma(X))\) is \(\mathcal{H}\)-Lindelöf.

(3) ⇒ (4): By Lemma 4, \(\gamma(X)\) is an \(m\)-structure and it follows that the same argument as (1) ⇒ (2) that \((X, \gamma(X))\) is \(\gamma\)-\(H\)-Lindelöf.

(4) ⇒ (1): Suppose that \((X, m)\) is \(\gamma\)-regular. Let \((X, \gamma(X))\) be \(\gamma\)-\(H\)-Lindelöf. Let \(\{U_\alpha : \alpha \in \Delta\}\) be any cover of \(X\) by \(m\)-open sets. For each \(x \in X\), there exists \(\alpha(x) \in \Delta\) such that \(x \in U_{\alpha(x)}\). Since \((X, m)\) is \(\gamma\)-regular, by Lemma 5 there exists \(V_{\alpha(x)} \in \gamma(X)\) such that \(x \in V_{\alpha(x)} \subseteq \gamma(V_{\alpha(x)}) \subseteq U_{\alpha(x)}\). Since \(\{V_{\alpha(x)} : x \in X\}\) is a cover of \(X\) by \(\gamma\)-open sets and \((X, \gamma(X))\) is \(\gamma\)-\(H\)-Lindelöf, there exist a countable number of points, say, \(x_1, x_2, x_3, \ldots \in X\) such that \(X \setminus \bigcup_{i=1}^{\infty} \gamma(V_{\alpha(x_i)}) \in \mathcal{H}\); and hence \(X \setminus \bigcup_{i=1}^{\infty} U_{\alpha(x_i)} \in \mathcal{H}\). This shows that \((X, m)\) is \(\mathcal{H}\)-Lindelöf.

**Definition 6** Let \((X, m, \mathcal{H})\) be a hereditary \(m\)-space. A subset \(A\) of \(X\) is said to be \(\gamma\)-\(H\)-Lindelöf relative to \(X\) if for every cover \(\{U_\alpha : \alpha \in \Delta\}\) of \(A\) by \(m\)-open sets of \(X\), there exists a countable subset \(\Delta_0\) of \(\Delta\) such that \(A \setminus \bigcup_{\alpha \in \Delta_0} \gamma(U_\alpha) \subseteq \mathcal{H}\).

**Theorem 3** Let \((X, m, \mathcal{H})\) be a hereditary \(m\)-space. If \(A\) is \(\gamma\)-closed and \(B\) is \(\gamma\)-\(H\)-Lindelöf relative to \(X\), then \(A \cap B\) is \(\gamma\)-\(H\)-Lindelöf relative to \(X\).

**Proof.** Let \(\{V_\alpha : \alpha \in \Delta\}\) be a cover of \(A \cap B\) by \(m\)-open subsets of \(X\). Then \(\{V_\alpha : \alpha \in \Delta\} \cup \{X \setminus A\}\) is a cover of \(B\) by \(m\)-open sets. Since \(X \setminus A\) is \(\gamma\)-open, for each \(x \in X \setminus A\), there exists an \(m\)-open set \(V_x\) such that \(x \in V_x \subseteq \gamma(V_x) \subseteq X \setminus A\). Thus \(\{V_\alpha : \alpha \in \Delta\} \cup \{V_x : x \in X \setminus A\}\) is a cover of \(B\) by \(m\)-open sets of \(X\) since \(B\) is \(\gamma\)-\(H\)-Lindelöf relative to \(X\), there exist a countable subset \(\Delta_0\) of \(\Delta\) and a countable points, say \(x_1, x_2, x_3, \ldots \in X \setminus A\) such that \(B \subseteq \bigcup_{\alpha \in \Delta_0} \gamma(V_\alpha) \cup \bigcup_{i=1}^{\infty} \gamma(V_{x_i}) \cup H_0 \subseteq \mathcal{H}\), where \(H_0 \in \mathcal{H}\). Hence \(A \cap B \subseteq \bigcup_{\alpha \in \Delta_0} \gamma(V_\alpha) \cap A \cup \bigcup_{i=1}^{\infty} \gamma(V_{x_i}) \cap A\) \cup (A \cap H_0) \subseteq \bigcup_{\alpha \in \Delta_0} \gamma(V_\alpha) \cup H_0\). Therefore, \(A \cap B \subseteq (\bigcup_{\alpha \in \Delta_0} \gamma(V_\alpha)) \subseteq H_0 \subseteq \mathcal{H}\). Hence \(A \cap B\) is \(\gamma\)-\(H\)-Lindelöf relative to \(X\).

**Corollary 1** If a hereditary \(m\)-space \((X, m, \mathcal{H})\) is \(\gamma\)-\(H\)-Lindelöf space, then every \(\gamma\)-closed subset of \(X\) is \(\gamma\)-\(H\)-Lindelöf relative to \(X\).
Proof. The proof is obvious by Theorem 3.

Lemma 6 [12] For a hereditary m-space \((X, m, \mathcal{H})\), the following properties hold:

1. \(m^*_H\) is an m-structure on \(X\) such that \(m^*_H\) has property B and \(m \subseteq m^*_H\).

2. \(\beta(m, \mathcal{H}) = \{U \setminus H : U \in m, H \in \mathcal{H}\}\) is a basis for \(m^*_H\) such that \(m \subseteq \beta(m, \mathcal{H})\).

Theorem 4 Let \((X, m, \mathcal{H})\) be a hereditary m-space. Then the following properties hold:

1. If \((X, m^*_H, \mathcal{H})\) is \(\mathcal{H}\)-Lindelöf, then \((X, m, \mathcal{H})\) is \(\mathcal{H}\)-Lindelöf.

2. If \((X, m, \mathcal{H})\) is \(\mathcal{H}\)-Lindelöf and \(\mathcal{H}\) is closed under countable union, then \((X, m^*_H, \mathcal{H})\) is \(\mathcal{H}\)-Lindelöf.

Proof. (1): The proof follows directly from the fact that every m-closed set is \(m^*_H\)-closed.

(2): Suppose that \(\mathcal{H}\) is closed under countable union and \(X\) is \(\mathcal{H}\)-Lindelöf. Let \(\{U_\alpha : \alpha \in \Delta\}\) be an \(m^*_H\)-open cover of \(X\), then for each \(x \in X\), \(x \in U_{\alpha(x)}(x)\) for some \(\alpha(x) \in \Delta\). By Lemma 6 there exist \(V_{\alpha(x)}(x) \subseteq m\) and \(H_{\alpha(x)}(x) \subseteq \mathcal{H}\) such that \(x \in V_{\alpha(x)}(x) \setminus H_{\alpha(x)}(x) \subseteq U_{\alpha(x)}(x)\). Since \(\{V_{\alpha(x)}(x) : \alpha(x) \in \Delta\}\) is a m-open cover of \(X\), there exists a countable subset \(\Delta_0\) of \(\Delta\) such that \(X \setminus \bigcup\{V_{\alpha(x)} : \alpha(x) \in \Delta_0\} = H \in \mathcal{H}\).

Since \(\mathcal{H}\) is closed under countable union, then \(\bigcup\{H_{\alpha(x)} : \alpha(x) \in \Delta_0\} \subseteq \mathcal{H}\). Hence, \(H \cup [\bigcup\{H_{\alpha(x)} : \alpha(x) \in \Delta_0\}] \subseteq \mathcal{H}\). Observe that \(X \setminus \bigcup\{U_\alpha : \alpha \in \Delta_0\} \subseteq H \cup [\bigcup\{H_{\alpha(x)} : \alpha(x) \in \Delta_0\}] \subseteq \mathcal{H}\). By the heredity property of \(\mathcal{H}\) we have \(X \setminus \bigcup\{U_\alpha : \alpha \in \Delta_0\} \subseteq \mathcal{H}\) and therefore, \((X, m^*_H, \mathcal{H})\) is \(\mathcal{H}\)-Lindelöf.

4 Strongly \(\mathcal{H}\)-Lindelöf spaces

Definition 7 A subset \(A\) of a hereditary m-space \((X, m, \mathcal{H})\) is said to be:

1. Strongly \(\mathcal{H}\)-Lindelöf relative to \(X\) if for every family \(\{V_\alpha : \alpha \in \Delta\}\) of m-open sets such that \(A \setminus \bigcup_{\alpha \in \Delta} V_\alpha \in \mathcal{H}\), there exists a countable subset \(\Delta_0\) of \(\Delta\) such that \(A \setminus \bigcup_{\alpha \in \Delta_0} V_\alpha \in \mathcal{H}\). If \(A = X\), then \((X, m, \mathcal{H})\) is said to be Strongly \(\mathcal{H}\)-Lindelöf.
2. Strongly \( \gamma \mathcal{H} \)-Lindelöf relative to \( X \) if for every family \( \{V_\alpha : \alpha \in \Delta\} \) of \( m \)-open sets such that \( A \setminus \bigcup_{\alpha \in \Delta} V_\alpha \in \mathcal{H} \), there exists a countable subset \( \Delta_0 \) of \( \Delta \) such that \( A \setminus \bigcup_{\alpha \in \Delta_0} \gamma(V_\alpha) \in \mathcal{H} \). If \( A = X \), then \( (X, m, \mathcal{H}) \) is said to be Strongly \( \gamma \mathcal{H} \)-Lindelöf.

**Theorem 5** Let \( (X, m, \mathcal{H}) \) be a hereditary \( m \)-space. Then the following properties hold:

1. If \( (X, m^*_H, \mathcal{H}) \) is Strongly \( \mathcal{H} \)-Lindelöf, then \( (X, m, \mathcal{H}) \) is Strongly \( \mathcal{H} \)-Lindelöf.

2. If \( (X, m, \mathcal{H}) \) is Strongly \( \mathcal{H} \)-Lindelöf and \( \mathcal{H} \) is closed under countable union, then \( (X, m^*_H, \mathcal{H}) \) is Strongly \( \mathcal{H} \)-Lindelöf.

**Theorem 6** Let \( (X, m, \mathcal{H}) \) be a hereditary \( m \)-space. Then the following properties are equivalent:

1. \( (X, m, \mathcal{H}) \) is Strongly \( \mathcal{H} \)-Lindelöf;

2. If \( \{F_\alpha : \alpha \in \Delta\} \) is a family of \( m \)-closed sets such that \( \bigcap \{F_\alpha : \alpha \in \Delta\} \in \mathcal{H} \), then there exists a countable subfamily \( \Delta_0 \) of \( \Delta \) such that \( \bigcap \{F_\alpha : \alpha \in \Delta_0\} \in \mathcal{H} \).

**Proof.** Suppose that \( (X, m, \mathcal{H}) \) is Strongly \( \mathcal{H} \)-Lindelöf. Let \( \{F_\alpha : \alpha \in \Delta\} \) be a family of \( m \)-closed sets such that \( \bigcap \{F_\alpha : \alpha \in \Delta\} \in \mathcal{H} \). Then \( \{X \setminus F_\alpha : \alpha \in \Delta\} \) is a family of \( m \)-open sets of \( X \). Let \( \mathcal{H} = \bigcap \{F_\alpha : \alpha \in \Delta\} \in \mathcal{H} \). Let \( X \setminus \mathcal{H} = X \setminus \bigcap \{F_\alpha : \alpha \in \Delta\} = \bigcup \{X \setminus F_\alpha : \alpha \in \Delta\} \). Since \( (X, m, \mathcal{H}) \) is Strongly \( \mathcal{H} \)-Lindelöf, there exists a countable \( \Delta_0 \) of \( \Delta \) such that \( X \setminus \bigcup \{X \setminus F_\alpha : \alpha \in \Delta_0\} \in \mathcal{H} \). This implies that \( \bigcap \{F_\alpha : \alpha \in \Delta\} \in \mathcal{H} \).

Conversely, let \( \{V_\alpha : \alpha \in \Delta\} \) be any family of \( m \)-open sets of \( X \) such that \( X \setminus \bigcup_{\alpha \in \Delta} V_\alpha \in \mathcal{H} \). Then \( \{X \setminus V_\alpha : \alpha \in \Delta\} \) is a family of \( m \)-closed sets of \( X \). By assumption we have \( \bigcap \{X \setminus V_\alpha : \alpha \in \Delta\} \in \mathcal{H} \) and there exists a countable subset \( \Delta_0 \) of \( \Delta \) such that \( \bigcap \{X \setminus V_\alpha : \alpha \in \Delta_0\} \in \mathcal{H} \). This implies that \( X \setminus \bigcup \{V_\alpha : \alpha \in \Delta_0\} \in \mathcal{H} \). This shows that \( (X, m, \mathcal{H}) \) is Strongly \( \mathcal{H} \)-Lindelöf. \( \square \)

**Definition 8** A subset \( A \) of a hereditary \( m \)-space \( (X, m, \mathcal{H}) \) is said to be \( m \mathcal{H}_\delta \)-closed if for every \( U \in m \) with \( A \setminus U \in \mathcal{H} \), \( \text{mcl}(A) \subseteq U \).

**Proposition 1** Let \( (X, m, \mathcal{H}) \) be a hereditary \( m \)-space. If \( (X, m, \mathcal{H}) \) is Strongly \( \mathcal{H} \)-Lindelöf and \( A \subseteq X \) is \( m \mathcal{H}_\delta \)-closed, then \( A \) is Strongly \( \mathcal{H} \)-Lindelöf relative to \( X \).
Proof. Let \( \{V_\alpha : \alpha \in \Delta\} \) be a family of \( m \)-open subsets of \( X \) such that \( \bigcup_{\alpha \in \Delta} V_\alpha \in \mathcal{H} \). Since \( A \) is \( m\mathcal{H}_g \)-closed, \( \text{mcl}(A) \subseteq \bigcup_{\alpha \in \Delta} V_\alpha \). Then \( \big(X \setminus \text{mcl}(A)\big) \cup \bigcup_{\alpha \in \Delta} V_\alpha \) is an \( m \)-open cover of \( X \) and so \( X \setminus \big(\big(X \setminus \text{mcl}(A)\big) \cup \bigcup_{\alpha \in \Delta} V_\alpha\big) \in \mathcal{H} \). Since \( X \) is Strongly \( \mathcal{H} \)-Lindelöf, there exists a countable subset \( \Delta_0 \) of \( \Delta \) such that \( X \setminus \big(\big(X \setminus \text{mcl}(A)\big) \cup \bigcup_{\alpha \in \Delta_0} V_\alpha\big) \in \mathcal{H} \). Thus, \( A \setminus \bigcup_{\alpha \in \Delta_0} V_\alpha \in \mathcal{H} \). Hence \( A \) is Strongly \( \mathcal{H} \)-Lindelöf relative to \( X \).

\[ \square \]

Theorem 7 Let \((X,m,\mathcal{H})\) be a hereditary \( m \)-space. Let \( A \) be an \( m\mathcal{H}_g \)-closed set such that \( A \subseteq B \subseteq \text{mcl}(A) \). Then \( A \) is Strongly \( \mathcal{H} \)-Lindelöf relative to \( X \) if and only if \( B \) is Strongly \( \mathcal{H} \)-Lindelöf relative to \( X \).

Proof.

Suppose that \( A \) is Strongly \( \mathcal{H} \)-Lindelöf relative to \( X \) and \( \{V_\alpha : \alpha \in \Delta\} \) is a family of \( m \)-open sets of \( X \) such that \( B \setminus \bigcup_{\alpha \in \Delta} V_\alpha \in \mathcal{H} \). By the heredity property, \( A \setminus \bigcup_{\alpha \in \Delta} V_\alpha \in \mathcal{H} \) and \( A \) is Strongly \( \mathcal{H} \)-Lindelöf relative to \( X \) and hence there exists a countable subset \( \Delta_0 \) of \( \Delta \) such that \( A \setminus \bigcup_{\alpha \in \Delta_0} V_\alpha \in \mathcal{H} \). Since \( A \) is \( m\mathcal{H}_g \)-closed, \( \text{mcl}(A) \subseteq \bigcup_{\alpha \in \Delta_0} V_\alpha \) and so \( \text{mcl}(A) \setminus \bigcup_{\alpha \in \Delta_0} V_\alpha \in \mathcal{H} \). This implies that \( B \setminus \bigcup_{\alpha \in \Delta_0} V_\alpha \in \mathcal{H} \).

Conversely, suppose that \( B \) is Strongly \( \mathcal{H} \)-Lindelöf relative to \( X \) and \( \{V_\alpha : \alpha \in \Delta\} \) is a family of \( m \)-open subsets of \( X \) such that \( A \setminus \bigcup_{\alpha \in \Delta} V_\alpha \in \mathcal{H} \). Given that \( A \) is \( m\mathcal{H}_g \)-closed, \( \text{mcl}(A) \setminus \bigcup_{\alpha \in \Delta} V_\alpha = \emptyset \in \mathcal{H} \) and this implies \( B \subseteq \bigcup_{\alpha \in \Delta} V_\alpha \). Since \( B \) is Strongly \( \mathcal{H} \)-Lindelöf relative to \( X \), there exists a countable subset \( \Delta_0 \) of \( \Delta \) such that \( B \setminus \bigcup_{\alpha \in \Delta_0} V_\alpha \in \mathcal{H} \). Hence \( A \setminus \bigcup_{\alpha \in \Delta_0} V_\alpha \in \mathcal{H} \).

\[ \square \]

5 \( (\gamma, \delta) \)-continuous functions

Definition 9 Let \((X,m)\) and \((Y,n)\) be minimal spaces and \( \gamma \) (resp. \( \delta \)) be an operation on \( m \) (resp. \( n \)). Then a function \( f : (X,m) \to (Y,n) \) is said to be \( (\gamma, \delta) \)-continuous if for each \( x \in X \) and each \( V \in n \) containing \( f(x) \), there exists \( U \in m \) containing \( x \) such that \( f(\gamma(U)) \subseteq \delta(V) \).

Lemma 7 Let \( f : X \to Y \) be a function.

1. If \( \mathcal{H} \) is a hereditary class on \( X \), then \( f(\mathcal{H}) \) is a hereditary class on \( Y \).

2. If \( \mathcal{H} \) is a hereditary class on \( Y \), then \( f^{-1}(\mathcal{H}) \) is a hereditary class on \( X \).

Proof. (1): This is due to Lemma 3.8 of [5].

(2): Let \( A \subseteq f^{-1}(\mathcal{H}) \), where \( \mathcal{H} \in \mathcal{H} \). Then \( f(A) \subseteq f(f^{-1}(\mathcal{H})) \subseteq \mathcal{H} \). Hence \( f(A) \in \mathcal{H} \) and \( A \subseteq f^{-1}(f(A)) \subseteq f^{-1}(\mathcal{H}) \) and hence \( A \in f^{-1}(\mathcal{H}) \).

\[ \square \]
Theorem 8 Let $(X, m)$ and $(Y, n)$ be minimal spaces and $\gamma$ (resp. $\delta$) be an operation on $m$ (resp. $n$) and $\mathcal{H}$ be a hereditary class on $X$. If $(X, m, \mathcal{H})$ is $\gamma\mathcal{H}$-Lindelöf and $f : (X, m, \mathcal{H}) \to (Y, n, f(\mathcal{H}))$ is a $(\gamma, \delta)$-continuous surjection, then $(Y, n, f(\mathcal{H}))$ is $\delta f(\mathcal{H})$-Lindelöf.

Proof. Let $\{V_\alpha : \alpha \in \Delta\}$ be any cover of $Y$ by $n$-open sets. For each $x \in X$, there exists $\alpha(x) \in \Delta$ such that $f(x) \in V_{\alpha(x)}$. Since $f$ is $(\gamma, \delta)$-continuous, there exists $U_{\alpha(x)} \in m$ containing $x$ such that $f(\gamma(U_{\alpha(x)})) \subseteq \delta(V_{\alpha(x)})$. Since $\{U_{\alpha(x)} : x \in X\}$ is a cover of $X$ by $m$-open sets and $(X, m, \mathcal{H})$ is $\gamma\mathcal{H}$-Lindelöf, there exists a countable points $x_1, x_2, x_3, \ldots \in X$ such that $X \setminus \bigcup_{i=1}^{\infty} \gamma(U_{\alpha(x_i)}) = H_0$, where $H_0 \in \mathcal{H}$. Therefore, we have $Y \subseteq f(\bigcup_{i=1}^{\infty} \gamma(U_{\alpha(x_i)})) \cup f(H_0) \subseteq \bigcup_{i=1}^{\infty} \delta(V_{\alpha(x_i)}) \cup f(H_0)$. Hence $(Y, n, f(\mathcal{H}))$ is $\delta f(\mathcal{H})$-Lindelöf. \qed

Definition 10 [11] A function $f : (X, m) \to (Y, n)$ is said to be $M$-closed if for each $m$-closed set $F$ of $X$, $f(F)$ is $n$-closed in $Y$.

Theorem 9 Let $f : (X, m) \to (Y, n, \mathcal{H})$ be an $M$-closed surjective function. If for every $y \in Y$, $f^{-1}(y)$ is $\text{Strongly } f^{-1}(\mathcal{H})$-Lindelöf in $X$, then $f^{-1}(A)$ is $\text{Strongly } f^{-1}(\mathcal{H})$-Lindelöf relative to $X$ whenever $A$ is $\text{Strongly } \mathcal{H}$-Lindelöf relative to $Y$ and $A \setminus \bigcup U \in \mathcal{H}$ for every $U \in n$.

Proof. Let $\{V_\alpha : \alpha \in \Delta\}$ be a family of $m$-open subsets of $X$ such that $f^{-1}(A) \setminus \bigcup(U_{\alpha} : \alpha \in \Delta) \in f^{-1}(\mathcal{H})$. For each $y \in A$ there exists a countable subset $\Delta_0(y)$ of $A$ such that $f^{-1}(y) \setminus \bigcup(V_\alpha : \alpha \in \Delta_0(y)) \in f^{-1}(\mathcal{H})$. Let $V_y = \bigcup(V_\alpha : \alpha \in \Delta_0(y))$. Each $V_y$ is an $m$-open set in $(X, m)$ and $f^{-1}(y) \setminus V_y \in f^{-1}(\mathcal{H})$.

Now each set $f(X \setminus V_y)$ is $n$-open in $Y$ and hence, $U(y) = Y - f(X \setminus V_y)$ is $n$-open in $Y$. Note that $f^{-1}(\bigcup(y)) \subseteq V_y$. Thus, $\{U(y) : y \in A\}$ is a family of $n$-open subsets of $Y$ such that $A \setminus \bigcup(U(y) : y \in A) \in \mathcal{H}$. Since $A$ is $\text{Strongly } \mathcal{H}$-Lindelöf relative to $Y$, there exists a countable subset $\{U_{y_i} : i \in \mathbb{N}\}$ such that $A \setminus \bigcup(U_{y_i} : i \in \mathbb{N}) \in \mathcal{H}$ and hence $f^{-1}[A \setminus \bigcup(U_{y_i} : i \in \mathbb{N})] = f^{-1}(A) \setminus \bigcup(f^{-1}(U_{y_i})) : i \in \mathbb{N}] \subseteq f^{-1}(\mathcal{H})$. Since $f^{-1}(A) \setminus \bigcup(U_{y_i} : i \in \mathbb{N}) \subseteq f^{-1}(A) \setminus \bigcup(f^{-1}(U_{y_i})) : i \in \mathbb{N}$, then $f^{-1}(A) \setminus \bigcup(U_{y_i} : i \in \mathbb{N}) = f^{-1}(A) \setminus \bigcup(V_\alpha : \alpha \in \Delta_0(y_i), i \in \mathbb{N}) \in f^{-1}(\mathcal{H})$. Hence, $f^{-1}(A)$ is $\text{Strongly } f^{-1}(\mathcal{H})$-Lindelöf relative to $X$. \qed

A subset $K$ of an $m$-space is said to be $m$-compact [14] if every cover of $K$ by $m$-open sets of $X$ has a finite subcover.

Theorem 10 Let $f : (X, m) \to (Y, n, \mathcal{H})$ be an $M$-closed surjective function. If for every $y \in Y$, $f^{-1}(y)$ is $m$-compact in $X$, then $f^{-1}(A)$ is $f^{-1}(\mathcal{H})$-Lindelöf relative to $X$ whenever $A$ is $\mathcal{H}$-Lindelöf relative to $Y$. 

Proof. Let \( \{V_\alpha : \alpha \in \Delta \} \) be a cover of \( f^{-1}(A) \) by \( m \)-open sets of \( X \). For each \( y \in A \) there exists a finite subset \( \Delta_0(y) \) of \( \Delta \) such that \( f^{-1}(y) \subseteq \bigcup \{V_\alpha : \alpha \in \Delta_0(y)\} \). Let \( V_y = \bigcup \{V_\alpha : \alpha \in \Delta_0(y)\} \). Each \( V_y \) is an \( m \)-open set in \( (X, m) \) and \( f^{-1}(y) \subseteq V_y \). Since \( f \) is \( M \)-closed, by Theorem 3.1 of [11] there exists an \( n \)-open set \( U_y \) containing \( y \) such that \( f^{-1}(U_y) \subseteq V_y \). The collection \( \{U_y : y \in A\} \) is a cover of \( A \) by \( n \)-open sets of \( Y \). Hence, there exists a countable subcollection \( \{U_{y(i)} : i \in \mathbb{N}\} \) such that \( A \setminus \bigcup \{U_{y(i)} : i \in \mathbb{N}\} \in \mathcal{H} \).

Then \( f^{-1}(A \setminus \bigcup \{U_{y(i)} : i \in \mathbb{N}\}) = f^{-1}(A) \setminus \bigcup \{f^{-1}(U_{y(i)}) : i \in \mathbb{N}\} \in f^{-1}(\mathcal{H}) \). Since \( f^{-1}(A) \setminus \bigcup \{V_{y(i)} : i \in \mathbb{N}\} \subseteq f^{-1}(A) \setminus \bigcup \{f^{-1}(U_{y(i)}) : i \in \mathbb{N}\} \), then \( f^{-1}(A) \setminus \bigcup \{V_{y(i)} : i \in \mathbb{N}\} \in f^{-1}(\mathcal{H}) \). Thus, \( f^{-1}(A) \) is \( f^{-1}(\mathcal{H}) \)-Lindelöf relative to \( X \). □

References


